A General Tableau Method for Propositional Interval Temporal Logics

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Abstract. Logics for time intervals provide a natural framework for representing and reasoning about timing properties in various areas of computer science. However, while various tableau methods have been developed for linear and branching time point-based temporal logics, not much work has been done on tableau methods for interval-based temporal logics. In this paper, we introduce a new, very expressive propositional interval temporal logic, called (Non-Strict) Branching CDT (BCDT$^+$) which extends most of the propositional interval temporal logics proposed in the literature. Then, we provide BCDT$^+$ with a generic tableau method which combines features of explicit tableau methods for modal logics with constraint label management and the classical tableau method for first-order logic, and we prove its soundness and completeness.

1 Introduction

Logics for time intervals provide a natural framework for representing and reasoning about timing properties in various areas of computer science. However, while various tableau methods have been developed for linear and branching time point-based temporal logics [Wol85,Eme90,SGL97,CMP99], not much work has been done on tableau methods for interval-based temporal logics. One reason for this disparity is that operators of interval temporal logics are in many respects more difficult to deal with. As an example, there exist straightforward inductive definitions of the main operators of point-based temporal logics, such as the *future* operator and the *until* operator, while inductive definitions of basic interval modalities (consider, for instance, the one for the *chop* operator given in [BT03]) turn out to be more complex.

Various propositional and first-order interval temporal logics have been proposed in the literature. In this paper we focus our attention on propositional ones. There are two different natural semantics for interval logics, namely, a **strict** one, which excludes point-intervals, and a **non-strict** one, which includes them. The most studied propositional interval logics are Halpern and Shoham's Modal Logic of Time Intervals (HS) [HS91], Venema's CDT logic [Ven91], Moszkowski's Propositional Interval Temporal Logic (PITL) [Mos83], and Goranko, Montanari, and Sciavicco's family of Propositional Neighborhood Logics $\mathcal{PNL}$ [GMS03].

HS features four basic operators: $\langle B \rangle$ (*begin*) and $\langle E \rangle$ (*end*), and their transposes $\langle \overline{B} \rangle$ and $\langle \overline{E} \rangle$. Given a formula ϕ and an interval $[d_0, d_1]$, $\langle B \rangle \phi$ holds at $[d_0, d_1]$ if ϕ holds at $[d_0, d_2]$, for some $d_2 < d_1$, and $\langle E \rangle \phi$ holds at $[d_0, d_1]$ if ϕ holds at $[d_2, d_1]$, for some $d_2 > d_0$. A number of other temporal operators can be defined by means of the basic ones. In particular, it is possible to define the *strict after* operator $\langle A \rangle$ (and its transpose $\langle \overline{A} \rangle$) such that $\langle A \rangle \phi$ holds at $[d_0, d_1]$ if ϕ holds at $[d_1, d_2]$ for some $d_2 > d_1$; the *non-strict after* operator $\Diamond_r$ (and its transpose $\Diamond_l$) such that $\Diamond_r \phi$ holds at $[d_0, d_1]$ if ϕ holds at $[d_1, d_2]$ for some $d_2 \geq d_1$; and the *subinterval* operator $\langle D \rangle$ such that $\langle D \rangle \phi$ holds at a given interval $[d_0, d_1]$ if ϕ holds at a proper subinterval of $[d_0, d_1]$.

CDT has three binary operators C (*chop*), D, and T, which correspond to the ternary interval relations occurring when an extra point is added in one of the three possible distinct positions with respect to the two endpoints of the current interval (*before, between*, and *after*), plus a modal constant π which holds at a given interval if and only if it is a point-interval. PITL provides two modalities, namely, $\bigcirc$ (*next*) and C (the specialization of the *chop* operator for discrete structures). In PITL an interval is defined as a finite or infinite sequence of states. Given two formulas ϕ, ψ and an interval $s_0, \ldots, s_n$, $\bigcirc \phi$ holds over $s_0, \ldots, s_n$ if and only if ϕ holds over $s_1, \ldots, s_n$, while $\phi C \psi$ holds over $s_0, \ldots, s_n$ if and only if there exists i, with $0 \leq i \leq n$, such that ϕ holds over $s_0, \ldots, s_i$ and ψ holds over $s_i, \ldots, s_n$. Finally, propositional neighborhood logics in $\mathcal{PNL}$ feature two modalities for right and left interval neighborhoods, namely, $\langle A \rangle$ and $\langle \overline{A} \rangle$ in the *strict* semantics ($\mathcal{PNL}^-$ logics), and $\Diamond_r$ and $\Diamond_l$ in the *non-strict* semantics ($\mathcal{PNL}^+$ logics).

The main contributions of the paper are:

(i) Introduction of a new propositional interval logic, called (Non-Strict) Branching CDT (BCDT^+ for short), which features the same operators as CDT, but it is interpreted over partially ordered domains with linear intervals, and it is therefore expressive enough to include as subsystems or specializations all the above-described interval logics.

(ii) Development of an original sound and complete tableau method for BCDT^+, which combines features of tableau methods for modal logics with constraint label management and the classical tableau method for first-order logic. The proposed method can be adapted to variations and subsystems of BCDT^+, thus providing a general tableau method for propositional interval logics.

We conclude this introduction with a brief comparison between the tableaux method proposed here and other existing methods for point-based and interval-based modal and temporal logics (see [Wol85,Eme90,KLWZ02]). As a preliminary remark, we note that most tableau methods for modal and temporal logics are terminating tableaux for *decidable* logics, and thus they yield decision procedures. Tableau methods for modal and (point-based) temporal logics can be classified as *explicit* or *implicit* (see [DGHP99]). Unlike implicit tableaux, explicit ones maintain the accessibility relation by means of some sort of external device. In implicit tableaux [Fit83,Rau83], the accessibility relation is built-in into the rules. In particular, in linear and branching time point-based temporal logics the tableau represents a model of the satisfiable formulas (a time-line or a tree, respectively). The non-standard finite model property can then be exploited to show that the resulting tableau methods are actually decision procedures (they do not lead to infinite computations). Explicit tableau methods have been developed for several modal logics. They capture the accessibility relation by means of labeled formulas, and they provide suitable notions of closed branches and tableaux. Whenever the logic is decidable, its properties can be exploited to turn the tableau method into a decision procedure. In this respect, the tableau method for BCDT^+, while sharing basic features with explicit tableaux for modal logics, comes closer to the classical, possibly non-terminating tableau method for first-order logic, which only provides a semi-decision procedure for non-satisfiability. It also presents some similarities with the explicit tableau method developed for the *guarded fragment* of first-order logic (see [GHO02]).

To the best of our knowledge, there exist very few other tableau methods for interval temporal logics (and duration calculi) in the literature. A tableau-based decision procedure for an extension of Local QPITL (a decidable fragment of PITL extended with quantification over propositional variables, which has been obtained by imposing a suitable *locality* constraint), which, besides the *chop* operator C and the modal constant π, has a projection operator *proj*, has been proposed by Kono [Kon95] and later refined by Bowman and Thompson [BT03]. They introduce a normal form for the formulas of the resulting logic that allows them to exploit a classical tableau method, devoid of any mechanism for constraint label management. In [CSF00], Chetcuti-Sperandio and Fariñas del Cerro focus on a decidable fragment of Duration Calculus (DC) which encompasses a proper subset of DC operators, namely, $\wedge$, $\vee$, and C. The tableau construction for the resulting logic combines application of the rules of classical tableaux with that of a suitable constraint resolution algorithm and it essentially depends on the assumption of *bounded* variability of the state variables. Finally, tableau systems for the propositional and first-order Linear Temporal Logic, which employ a mechanism for labeling formulas with temporal constraints somewhat similar to ours, have been developed respectively in [SGL97] and [CMP99]. The main differences between these tableau methods and ours are: (i) they are specifically designed to deal with integer time structures (i.e., linear and discrete) while ours is quite generic; (ii) the LTL is essentially point-based, and intervals only play a secondary role in it (viz., a formula it true on an interval if and only if it is true at every point in it), while in our systems intervals are primary semantic objects on which the truth definitions are entirely based; (iii) the closedness of a tableau in the cited papers is defined in terms of unsatisfiability of the associated set of temporal constraints, while in our system it is entirely syntactic.

2 Non-Strict Branching CDT (BCDT^+)

In this section we give syntax and semantics of BCDT^+ and discuss its expressive power. To this end, we introduce some preliminary notions. Let $\mathbb{D}$ be a set of time points, called **domain**, and $<$ be a partial order on it. A (non-strict) **interval** on

$\mathbb{D}$ is an ordered pair $[d_0, d_1]$ such that $d_0, d_1 \in \mathbb{D}$, and $d_0 \leq d_1$. When $d_0 < d_1$ we say that the interval is **proper** or **strict**; when $d_0 = d_1$ it is a **point-interval**.

As in [HS91], we assume intervals to be *linear*, that is, for every interval $[d_0, d_1]$ and every pair of points d, d' belonging to it, namely, $d_0 \leq d \leq d_1$ and $d_0 \leq d' \leq d_1$, $d < d'$ or $d' < d$ or $d = d'$. Such an assumption keeps the temporal setting still very general, while making it fitting our intuition about the nature of time [HS91]. A pair $\langle \mathbb{D}, < \rangle$ is called an **interval structure**. An interval structure is: **linear** if every two points are comparable; **discrete** if every point with a successor/predecessor has an immediate successor/predecessor along every path starting from/ending at it; **dense** if for every pair of comparable (under $<$) points there exists another point in between; **unbounded** if and only if there are no points without successors (resp., predecessors); **Dedekind complete** if every non-empty and upward bounded set of points has a least upper bound. An element $d \in \mathbb{D}$ such that there are no elements $d' \in \mathbb{D}$ with $d < d'$ (resp., $d' < d$) is called **minimal** (resp., **maximal**) element.

Here we assume the non-strict semantics, but we add the modal constant π (as in [Ven91]) that is satisfied by point-intervals only, and hence enables one to distinguish between point-intervals and proper ones.

2.1 BCDT$^+$ Syntax and Semantics

The language $\mathbf{L}^+$ for BCDT$^+$ consists of a set of propositional variables $\mathcal{AP}$, the logical connectives $\neg$ and $\wedge$, the modalities C, D, and T, and the modal constant π. The other logical connectives, as well as the logical constants $\top$ (*true*) and $\bot$ (*false*), can be defined in the usual way. BCDT$^+$ **formulas**, denoted by $\phi, \psi, \ldots$, are recursively defined as follows (where $p \in \mathcal{AP}$):

$$\phi = \pi \mid p \mid \neg\phi \mid \phi \wedge \psi \mid \phi C \psi \mid \phi D \psi \mid \phi T \psi.$$

The semantics of BCDT$^+$ is given in terms of **non-strict models**, i.e., based on non-strict interval structures, equipped with a **valuation function** for propositional variables. The valuation function is a mapping $V : \mathbb{I}(\mathbb{D})^+ \mapsto 2^{\mathcal{AP}}$, where $\mathbb{I}(\mathbb{D})^+$ is the set of all intervals in $\mathbb{D}$, such that, for any $p \in \mathcal{AP}$, p is **true** over $[d_0, d_1]$ if and only if $p \in V([d_0, d_1])$. **Truth** over an interval $[d_0, d_1]$ in a model $\mathbf{M}^+$ is defined by induction on the structure of formulas:

1. $\mathbf{M}^+, [d_0, d_1] \Vdash \pi$ iff $d_0 = d_1$;
2. $\mathbf{M}^+, [d_0, d_1] \Vdash p$ iff $p \in V([d_0, d_1])$, for all $p \in \mathcal{AP}$;
3. $\mathbf{M}^+, [d_0, d_1] \Vdash \neg\psi$ iff it is not the case that $\mathbf{M}^+, [d_0, d_1] \Vdash \psi$;
4. $\mathbf{M}^+, [d_0, d_1] \Vdash \phi \wedge \psi$ iff $\mathbf{M}^+, [d_0, d_1] \Vdash \phi$ and $\mathbf{M}^+, [d_0, d_1] \Vdash \psi$;
5. $\mathbf{M}^+, [d_0, d_1] \Vdash \phi C \psi$ iff there exists $d_2 \in \mathbb{D}$ such that $d_0 \leq d_2 \leq d_1$, $\mathbf{M}^+, [d_0, d_2] \Vdash \phi$, and $\mathbf{M}^+, [d_2, d_1] \Vdash \psi$;
6. $\mathbf{M}^+, [d_0, d_1] \Vdash \phi D \psi$ iff there exists $d_2 \in \mathbb{D}$ such that $d_2 \leq d_0$, $\mathbf{M}^+, [d_2, d_0] \Vdash \phi$, and $\mathbf{M}^+, [d_2, d_1] \Vdash \psi$;
7. $\mathbf{M}^+, [d_0, d_1] \Vdash \phi T \psi$ iff there exists $d_2 \in \mathbb{D}$ such that $d_1 \leq d_2$, $\mathbf{M}^+, [d_1, d_2] \Vdash \phi$, and $\mathbf{M}^+, [d_0, d_2] \Vdash \psi$.

Satisfiability and validity of BCDT$^+$ formulas are defined in the usual way.

2.2 Expressive Power of BCDT$^+$

Let us compare the expressive power of BCDT$^+$ with that of the above-described propositional interval logics. We say that a logic $\mathbf{L_1}$ is **at least as expressive** as a logic $\mathbf{L_2}$ if for every $\mathbf{L_2}$ formula there exists an equivalent $\mathbf{L_1}$ formula, and that $\mathbf{L_1}$ is (strictly) **more expressive** than $\mathbf{L_2}$ if and only if $\mathbf{L_1}$ is at least as expressive as $\mathbf{L_2}$, but not vice versa.

We first note that both CDT and $\mathcal{PNL}^+$ logics are interpreted over linear structures, and that the operators of $\mathcal{PNL}^+$ logics can be expressed in CDT by means of the formulas $\Diamond_r \phi := \phi T \top$ and $\Diamond_l \phi := \phi D \top$. Furthermore, it is well known that CDT does not semantically include HS in its full generality, since the latter allows the interval structure to be branching, while the former does not. On the other hand, HS is not more expressive than CDT, because it cannot express the *chop* operator (see [MV97]).

BCDT$^+$ generalizes Venema's CDT (and thus propositional neighborhood logics in $\mathcal{PNL}^+$) by allowing the interval structure to be non-linear, for as long as all intervals in it are linear (as in HS). Furthermore, it is strictly more expressive than HS and PITL. HS operators can be defined in BCDT$^+$ as follows: $\langle B \rangle \phi := \phi C \neg \pi$, $\langle \overline{B} \rangle \phi := \neg \pi T \phi$, $\langle E \rangle \phi := \neg \pi C \phi$, and $\langle \overline{E} \rangle \phi := \neg \pi D \phi$. Besides, the strict neighborhood operators $\langle A \rangle$ and $\langle \overline{A} \rangle$ can be defined in BCDT$^+$ by using π as follows: $\langle A \rangle \phi := (\phi \wedge \neg \pi) T \top$, and $\langle \overline{A} \rangle \phi := (\phi \wedge \neg \pi) D \top$.

By exploiting such derived operators, all conditions on the interval structure mentioned in the preliminaries can be easily expressed in BCDT$^+$. In particular, linearity can be expressed in BCDT$^+$ by means of the following formula:

$$\langle A \rangle p \rightarrow [A](p \vee \langle B \rangle p \vee \langle \overline{B} \rangle p) \wedge \langle \overline{A} \rangle p \rightarrow [\overline{A}](p \vee \langle E \rangle p \vee \langle \overline{E} \rangle p),$$

while discreteness of linear interval structures can be imposed by means of the formula:

$$\pi \vee l1 \vee (\langle B \rangle l1 \wedge \langle E \rangle l1),$$

where $l1$ stands for $\langle B \rangle \top \wedge [B][B] \bot$, together with the dual one.

As for the PITL operators, C is an operator of BCDT$^+$, while $\bigcirc$ can be defined over (linear) discrete structures as follows: $\bigcirc \phi := l1 C \phi$.

The undecidability of BCDT$^+$ with respect to a number of interval structures immediately follows from results in [HS91], while finding meaningful decidable fragments of BCDT$^+$ is an interesting open problem.

3 A tableau method for BCDT$^+$

In this section we devise a tableau method for BCDT$^+$. That method can be adapted to its strict version BCDT$^-$, and can be accordingly restricted to CDT, HS, PITL, and $\mathcal{PNL}$ logics.

First, some basic terminology. A **finite tree** is a finite directed connected graph in which every node, apart from one (the **root**), has exactly one incoming arc. A **successor** of a node $\mathbf{n}$ is a node $\mathbf{n}'$ such that there is an edge from $\mathbf{n}$ to

$\mathbf{n}'$. A **leaf** is a node with no successors; a **path** is a sequence of nodes $\mathbf{n_0}, \ldots, \mathbf{n_k}$ such that, for all $i = 0, \ldots, k-1$, $\mathbf{n_{i+1}}$ is a successor of $\mathbf{n_i}$; a **branch** is a path from the root to a leaf. The **height** of a node $\mathbf{n}$ is the maximum length (number of edge) of a path from $\mathbf{n}$ to a leaf. If $\mathbf{n}, \mathbf{n}'$ belong to the same branch and the height of $\mathbf{n}$ is less than or equal to the height of $\mathbf{n}'$, we write $\mathbf{n} \prec \mathbf{n}'$.

Let $\langle \mathbb{C}, < \rangle$ be a finite partial order. A **labeled formula**, with label in $\mathbb{C}$, is a pair $(\phi, [c_i, c_j])$, where $\phi \in \mathrm{BCDT}^+$ and $[c_i, c_j] \in \mathbb{I}(\mathbb{C})^+$.

For a node $\mathbf{n}$ in a tree, the **decoration** $\nu(\mathbf{n})$ is a triple $((\phi, [c_i, c_j]), \mathbb{C}, u_{\mathbf{n}})$, where $\langle \mathbb{C}, < \rangle$ is a finite partial order, $(\phi, [c_i, c_j])$ is a labeled formula, with label in $\mathbb{C}$, and $u_{\mathbf{n}}$ is a **local flag function** which associates the values 0 or 1 with every branch B containing $\mathbf{n}$. Intuitively, the value 0 for a node n with respect to a branch B means that n can be expanded on B. For the sake of simplicity, we will often assume the interval $[c_i, c_j]$ to consist of the elements $c_i < c_{i+1} < \cdots < c_j$, and sometimes, with a little abuse of notation, we will write $\mathbb{C} = \{c_i < c_k, c_m < c_j, \ldots\}$. A **decorated tree** is a tree in which every node has a decoration $\nu(\mathbf{n})$. For every decorated tree, we define a **global flag function** u acting on pairs *(node, branch through that node)* as $u(\mathbf{n}, B) = u_{\mathbf{n}}(B)$. Sometimes, for convenience, we will include in the decoration of the nodes the global flag function instead of the local ones. For any branch B in a decorated tree, we denote by $\mathbb{C}_B$ the ordered set in the decoration of the leaf B, and for any node $\mathbf{n}$ in a decorated tree, we denote by $\Phi(\mathbf{n})$ the formula in its decoration. If B is a branch, then $B \cdot \mathbf{n}$ denotes the result of the expansion of B with the node $\mathbf{n}$ (addition of an edge connecting the leaf of B to $\mathbf{n}$). Similarly, $B \cdot \mathbf{n_1} \mid \ldots \mid \mathbf{n_k}$ denotes the result of the expansion of B with k immediate successor nodes $\mathbf{n_1}, \ldots, \mathbf{n_k}$ (which produces k branches extending B). A tableau for BCDT^+ will be defined as a special decorated tree. We note again that $\mathbb{C}$ remains finite throughout the construction of the tableau.

Definition 1. *Given a decorated tree $\mathcal{T}$, a branch B in $\mathcal{T}$, and a node $\mathbf{n} \in B$ such that $\nu(\mathbf{n}) = ((\phi, [c_i, c_j]), \mathbb{C}, u)$, with $u(\mathbf{n}, B) = 0$, the* ***branch-expansion rule*** *for B and $\mathbf{n}$ is defined as follows (in all the considered cases, $u(\mathbf{n}', B') = 0$ for all new pairs $(\mathbf{n}', B')$ of nodes and branches).*

- *If $\phi = \neg\neg\psi$, then expand the branch to $B \cdot \mathbf{n_0}$, with $\nu(\mathbf{n_0}) = ((\psi, [c_i, c_j]), \mathbb{C}_B, u)$.*
- *If $\phi = \psi_0 \wedge \psi_1$, then expand the branch to $B \cdot \mathbf{n_0} \cdot \mathbf{n_1}$, with $\nu(\mathbf{n_0}) = ((\psi_0, [c_i, c_j]), \mathbb{C}_B, u)$ and $\nu(\mathbf{n_1}) = ((\psi_1, [c_i, c_j]), \mathbb{C}_B, u)$.*
- *If $\phi = \neg(\psi_0 \wedge \psi_1)$, then expand the branch to $B \cdot \mathbf{n_0} | \mathbf{n_1}$, with $\nu(\mathbf{n_0}) = ((\neg\psi_0, [c_i, c_j]), \mathbb{C}_B, u)$ and $\nu(\mathbf{n_1}) = ((\neg\psi_1, [c_i, c_j]), \mathbb{C}_B, u)$.*
- *If $\phi = \neg(\psi_0 C \psi_1)$ and c is the least element of $\mathbb{C}_B$, with $c_i \leq c \leq c_j$, which has not been used yet to expand the node $\mathbf{n}$ on B, then expand the branch to $B \cdot \mathbf{n_0} | \mathbf{n_1}$, with $\nu(\mathbf{n_0}) = ((\neg\psi_0, [c_i, c]), \mathbb{C}_B, u)$ and $\nu(\mathbf{n_1}) = ((\neg\psi_1, [c, c_j]), \mathbb{C}_B, u)$.*
- *If $\phi = \neg(\psi_0 D \psi_1)$, c is a minimal element of $\mathbb{C}_B$ such that $c \leq c_i$, and there exists $c' \in [c, c_i]$ which has not been used yet to expand the node $\mathbf{n}$ on B, then take the least such $c' \in [c, c_i]$ and expand the branch to $B \cdot \mathbf{n_0} | \mathbf{n_1}$, with $\nu(\mathbf{n_0}) = ((\neg\psi_0, [c', c_i]), \mathbb{C}_B, u)$ and $\nu(\mathbf{n_1}) = ((\neg\psi_1, [c', c_j]), \mathbb{C}_B, u)$.*

- *If* $\phi = \neg(\psi_0 T \psi_1)$, *c is a maximal element of* $\mathbb{C}_B$ *such that* $c_j \leq c$, *and there exists* $c' \in [c_j, c]$ *which has not been used yet to expand the node* $\mathbf{n}$ *on* B, *then take the greatest such* $c' \in [c_j, c]$ *and expand the branch to* $B \cdot \mathbf{n_0} | \mathbf{n_1}$, *so that* $\nu(\mathbf{n_0}) = ((\neg\psi_0, [c_j, c']), \mathbb{C}_B, u)$ *and* $\nu(\mathbf{n_1}) = ((\neg\psi_1, [c_i, c']), \mathbb{C}_B, u)$.
- *If* $\phi = (\psi_0 C \psi_1)$, *then expand the branch to* $B \cdot (\mathbf{n_i} \cdot \mathbf{m_i}) | \ldots | (\mathbf{n_j} \cdot \mathbf{m_j}) | (\mathbf{n'_i} \cdot \mathbf{m'_i}) | \ldots | (\mathbf{n'_{j-1}} \cdot \mathbf{m'_{j-1}})$, *where:*
  1. *for all* $c_k \in [c_i, c_j]$, $\nu(\mathbf{n_k}) = ((\psi_0, [c_i, c_k]), \mathbb{C}_B, u)$ *and* $\nu(\mathbf{m_k}) = ((\psi_1, [c_k, c_j]), \mathbb{C}_B, u)$;
  2. *for all* $i \leq k \leq j-1$, *let* $\mathbb{C}_k$ *be the interval structure obtained by inserting a new element* c *between* c_k *and* c_{k+1} *in* $[c_i, c_j]$, $\nu(\mathbf{n'_k}) = ((\psi_0, [c_i, c]), \mathbb{C}_k, u)$, *and* $\nu(\mathbf{m'_k}) = ((\psi_1, [c, c_j]), \mathbb{C}_k, u)$.
- *If* $\phi = (\psi_0 D \psi_1)$, *then repeatedly expand the current branch, once for each minimal element* c *(where* $[c, c_i] = \{c = c_0 < c_1 < \cdots c_i\}$*), by adding the decorated sub-tree* $(\mathbf{n_0} \cdot \mathbf{m_0}) | \ldots | (\mathbf{n_i} \cdot \mathbf{m_i}) | (\mathbf{n'_1} \cdot \mathbf{m'_1}) | \ldots | (\mathbf{n'_i} \cdot \mathbf{m'_i}) | (\mathbf{n''_0} \cdot \mathbf{m''_0}) | \ldots | (\mathbf{n''_i} \cdot \mathbf{m''_i})$ *to its leaf, where:*
  1. *for all* c_k *such that* $c_k \in [c, c_i]$, $\nu(\mathbf{n_k}) = ((\psi_0, [c_k, c_j]), \mathbb{C}_B, u)$ *and* $\nu(\mathbf{m_k}) = ((\psi_1, [c_k, c_i]), \mathbb{C}_B, u)$;
  2. *for all* $0 < k \leq i$, *let* $\mathbb{C}_k$ *be the interval structure obtained by inserting a new element* c' *immediately before* c_k *in* $[c, c_i]$, *and* $\nu(\mathbf{n'_k}) = ((\psi_0, [c', c_i]), \mathbb{C}_k, u)$ *and* $\nu(\mathbf{m'_k}) = ((\psi_1, [c', c_j]), \mathbb{C}_k, u)$;
  3. *for all* $0 \leq k \leq i$, *let* $\mathbb{C}_k$ *be the interval structure obtained by inserting a new element* c' *in* $\mathbb{C}_B$, *with* $c' < c_k$, *which is incomparable with all existing predecessors of* c_k, $\nu(\mathbf{n''_k}) = ((\psi_0, [c', c_i]), \mathbb{C}_k, u)$, *and* $\nu(\mathbf{m''_k}) = ((\psi_1, [c', c_j]), \mathbb{C}_k, u)$.
- *If* $\phi = (\psi_0 T \psi_1)$, *then repeatedly expand the current branch, once for each maximal element* c *(where* $[c_j, c] = \{c_j < c_{j+1} < \cdots c_n = c\}$*), by adding the decorated sub-tree* $(\mathbf{n_j} \cdot \mathbf{m_j}) | \ldots | (\mathbf{n_n} \cdot \mathbf{m_n}) | (\mathbf{n'_j} \cdot \mathbf{m'_j}) | \ldots | (\mathbf{n'_{n-1}} \cdot \mathbf{m'_{n-1}}) | (\mathbf{n''_j} \cdot \mathbf{m''_j}) | \ldots | (\mathbf{n''_n} \cdot \mathbf{m''_n})$ *to its leaf, where:*
  1. *for all* c_k *such that* $c_k \in [c_j, c]$, $\nu(\mathbf{n_k}) = ((\psi_0, [c_j, c_k]), \mathbb{C}_B, u)$ *and* $\nu(\mathbf{m_k}) = ((\psi_1, [c_i, c_k]), \mathbb{C}_B, u)$;
  2. *for all* $j \leq k < n$, *let* $\mathbb{C}_k$ *be the interval structure obtained by inserting a new element* c' *immediately after* c_k *in* $[c_j, c]$, *and* $\nu(\mathbf{n'_k}) = ((\psi_0, [c_j, c']), \mathbb{C}_k, u)$ *and* $\nu(\mathbf{m'_k}) = ((\psi_1, [c_i, c']), \mathbb{C}_k, u)$;
  3. *for all* $j \leq k \leq n$, *let* $\mathbb{C}_k$ *be the interval structure obtained by inserting a new element* c' *in* $\mathbb{C}_B$, *with* $c_k < c'$, *which is incomparable with all existing successors of* c_k, $\nu(\mathbf{n''_k}) = ((\psi_0, [c_j, c']), \mathbb{C}_k, u)$, *and* $\nu(\mathbf{m''_k}) = ((\psi_1, [c_i, c']), \mathbb{C}_k, u)$.

Finally, for any node $\mathbf{m}$ $(\neq \mathbf{n})$ *in* B *and any branch* B' *extending* B, *let* $u(\mathbf{m}, B')$ *be equal to* $u(\mathbf{m}, B)$, *and for any branch* B' *extending* B, $u(\mathbf{n}, B') = 1$, *unless* $\phi = \neg(\psi_0 C \psi_1)$, $\phi = \neg(\psi_0 D \psi_1)$, *or* $\phi = \neg(\psi_0 T \psi_1)$ *(in such cases* $u(\mathbf{n}, B') = 0$*).*

Let us briefly explain the expansion rules for $\psi_0 C \psi_1$ and $\neg(\psi_0 C \psi_1)$ (similar considerations hold for the other temporal operators). The rule for the (existential) formula $\psi_0 C \psi_1$ deals with the two possible cases: either there exists c_k in $\mathbb{C}_B$ such that $c_i \leq c_k \leq c_j$ and ψ_0 holds over $[c_i, c_k]$ and ψ_1 holds over $[c_k, c_j]$

or such an element c_k must be added. The (universal) formula $\neg(\psi_0 C \psi_1)$ states that, for all $c_i \leq c \leq c_j$, ψ_0 does not hold over $[c_j, c]$ or ψ_1 does not hold over $[c, c_j]$. As a matter of fact, the expansion rule imposes such a condition for a single element c in $\mathbb{C}_B$ (the least element which has not been used yet), and it does not change the flag (which remains equal to 0). In this way, all elements will be eventually taken into consideration, including those elements in between c_i and c_j that will be added to $\mathbb{C}_B$ in some subsequent steps of the tableau construction.

Let us define now the notions of open and closed branch. We say that a node $\mathbf{n}$ in a decorated tree $\mathcal{T}$ is **available on a branch** B to which it belongs if and only if $u(\mathbf{n}, B) = 0$. The branch-expansion rule is **applicable** to a node $\mathbf{n}$ on a branch B if the node is available on B and the application of the rule generates at least one successor node with a new labeled formula. This second condition is needed to avoid looping of the application of the rule on formulas $\neg(\psi_0 C \psi_1), \neg(\psi_0 D \psi_1)$, and $\neg(\psi_0 T \psi_1)$.

Definition 2. *A branch B is **closed** if some of the following conditions holds:*

(i) there are two nodes $\mathbf{n}, \mathbf{n}' \in B$ such that $\nu(\mathbf{n}) = ((\psi, [c_i, c_j]), \mathbb{C}, u)$ and $\nu(\mathbf{n}')$ $= ((\neg\psi, [c_i, c_j]), \mathbb{C}', u)$ for some formula ψ and $c_i, c_j \in \mathbb{C} \cap \mathbb{C}'$;

(ii) there is a node $\mathbf{n}$ such that $\nu(\mathbf{n}) = ((\pi, [c_i, c_j]), \mathbb{C}, u)$ and $c_i \neq c_j$; or

(iii) there is a node $\mathbf{n}$ such that $\nu(\mathbf{n}) = ((\neg\pi, [c_i, c_j]), \mathbb{C}, u)$ and $c_i = c_j$.

*If none of the above conditions hold, the branch is **open**.*

Definition 3. *The **branch-expansion strategy** for a branch B in a decorated tree $\mathcal{T}$ is defined as follows:*

1. *Apply the branch-expansion rule to a branch B only if it is open;*
2. *If B is open, apply the branch-expansion rule to the closest to the root available node in B for which the branch-expansion rule is applicable.*

Definition 4. *A **tableau** for a given formula $\phi \in$BCDT$^+$ is any finite decorated tree $\mathcal{T}$ obtained by expanding the three-node decorated tree built up from an empty-decoration root and two leaves with decorations $((\phi, [c_b, c_e]), \{c_b < c_e\}, u)$ and $((\phi, [c_b, c_b]), \{c_b\}, u)$, where the value of u is 0, through successive applications of the branch-expansion strategy to the existing branches.*

It is easy to show that if $\phi \in$ BCDT$^+$, $\mathcal{T}$ is a tableau for ϕ, $\mathbf{n} \in \mathcal{T}$, and $\mathbb{C}$ is the ordered set in the decoration of $\mathbf{n}$, then $\langle \mathbb{C}, < \rangle$ is an interval structure.

Definition 5. *A tableau for* BCDT$^+$ *is **closed** if and only if every branch in it is closed, otherwise it is **open**.*

As an example, consider the unsatisfiable BCDT$^+$ formula $\phi = pT\psi$, where $\psi = \neg(\top Cp)$. Here we show some steps of the construction of a closed tableau for that formula.

The initial tableau is:

$$\begin{array}{ccc} & \mathbf{root} & \\ \swarrow & & \searrow \\ \mathbf{n_0} : ((pT\psi, [c_0, c_1]), \{c_0 < c_1\}, 0) & & \mathbf{n_1} : ((pT\psi, [c_0, c_0]), \{c_0\}, 0) \end{array}$$

We suppose that the flag function is correctly updated during the construction. According to the branch-expansion strategy, by expanding $\mathbf{n_0}$ we obtain:

$$\begin{array}{ccc} & \mathbf{n_0} & \\ \swarrow & & \searrow \\ \mathbf{n_2} : ((\psi, [c_0, c_1]), \{c_0 < c_1\}, 0) & & \mathbf{n_4} : ((\psi, [c_0, c_2]), \{c_0 < c_1 < c_2\}, 0) \\ | & & | \\ \mathbf{n_3} : ((p, [c_1, c_1]), \{c_0 < c_1\}, 0) & & \mathbf{n_5} : ((p, [c_1, c_2]), \{c_0 < c_1 < c_2\}, 0) \end{array}$$

The node $\mathbf{n_2}$ is expanded by an application of a $\neg C$ rule, attaching the decorated sub-tree

$$\begin{array}{ccc} & \mathbf{n_3} & \\ \swarrow & & \searrow \\ \mathbf{n_6} : (\bot, [c_0, c_0], \{c_0 < c_1\}, 0) & & \mathbf{n_7} : (\neg p, [c_0, c_1], \{c_0 < c_1\}, 0) \end{array}$$

to $\mathbf{n_3}$ and the following one to each of the leaves $\mathbf{n_6}$ and $\mathbf{n_7}$:

$$\begin{array}{ccc} & \mathbf{n_6}/\mathbf{n_7} & \\ \swarrow & & \searrow \\ \mathbf{n_8}/\mathbf{n_{10}} : (\bot, [c_0, c_1], \{c_0 < c_1\}, 0) & & \mathbf{n_9}/\mathbf{n_{11}} : (\neg p, [c_1, c_1], \{c_0 < c_1\}, 0) \end{array}$$

It is straightforward to check that all branches are closed. The remaining branches can be obtained in a similar way, and they are closed as well.

3.1 Soundness and Completeness

Definition 6. *Given a set S of labeled formulas with labels in an interval structure $\langle \mathbb{C}, < \rangle$, we say that S is* ***satisfiable over*** $\mathbb{C}$ *if there exists a non-strict model $\mathbf{M}^+ = \langle \mathbb{D}, V \rangle$ such that $\langle \mathbb{D}, < \rangle$ is an extension of $\langle \mathbb{C}, < \rangle$, $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$ for all $(\psi, [c_i, c_j]) \in S$.*

If S contains only one labeled formula, the notion of satisfiability of a (labeled) formula over $\mathbb{C}$ is equivalent to the notion of satisfiability given in Section 2.

Theorem 1 (Soundness). *If $\phi \in$BCDT$^+$ and a tableau $\mathcal{T}$ for ϕ is closed, then ϕ is not satisfiable.*

Proof. We will prove by induction on the height h of a node $\mathbf{n}$ in the tableau $\mathcal{T}$ the following claim: *if every branch including* $\mathbf{n}$ *is closed, then the set* $S(\mathbf{n})$ *of all labeled formulas in the decorations of the nodes between* $\mathbf{n}$ *and the root is not satisfiable over* $\mathbb{C}$*, where* $\mathbb{C}$ *is the interval structure in the decoration of* $\mathbf{n}$.

If $h = 0$, then $\mathbf{n}$ is a leaf and the unique branch B containing $\mathbf{n}$ is closed. Then, either $S(\mathbf{n})$ contains both the labeled formulas $(\psi, [c_k, c_l])$ and $(\neg\psi, [c_k, c_l])$ for some BCDT$^+$-formula ψ and $c_k, c_l \in \mathbb{C}$, or the labeled formula $(\pi, [c_k, c_l])$ and $c_k \neq c_l$, or the labeled formula $(\neg\pi, [c_k, c_l])$ and $c_k = c_l$. Take any model $\mathbf{M}^+ = \langle \mathbb{D}, V\rangle$ where $\langle \mathbb{D}, <\rangle$ is an extension of $\langle \mathbb{C}, <\rangle$. In the first case, clearly $\mathbf{M}^+, [c_k, c_l] \Vdash \psi$ if and only if $\mathbf{M}^+, [c_k, c_l] \not\Vdash \neg\psi$. In the second (resp., third) case, $\mathbf{M}^+, [c_k, c_l] \Vdash \pi$ (resp., $\neg\pi$) if and only if $c_k = c_l$ (resp., $c_k \neq c_l$). Hence, $S(\mathbf{n})$ is not satisfiable over $\mathbb{C}$.

Suppose $h > 0$. Then either $\mathbf{n}$ has been generated as one of the successors, *but not the last one*, when applying the branch-expansion rule in $\wedge$, C, D, T, $\neg C, \neg D$, or $\neg T$ cases, or the branch-expansion rule has been applied to some labeled formula $(\psi, [c_i, c_j]) \in S(\mathbf{n}) - \{\Phi(\mathbf{n})\}$ to extend the branch at $\mathbf{n}$. We deal with the latter case. The former can be dealt with in the same way. Let $\mathbb{C} = \{c_1, \ldots, c_n\}$, be the interval structure from the decoration of $\mathbf{n}$. Notice that every branch passing through any successor of $\mathbf{n}$ must be closed, so the inductive hypothesis applies to all successors of $\mathbf{n}$. We consider the possible cases for the branch-expansion rule applied at $\mathbf{n}$:

- Let $\psi = \neg\neg\xi$. Then there exists $\mathbf{n_0}$ such that $\nu(\mathbf{n_0}) = ((\xi, [c_i, c_j]), \mathbb{C}, u)$ and $\mathbf{n_0}$ is a successor of $\mathbf{n}$. Since every branch containing $\mathbf{n}$ is closed, then every branch containing $\mathbf{n_0}$ is closed. By the inductive hypothesis, $S(\mathbf{n_0})$ is not satisfiable over $\mathbb{C}$ (since $\mathbf{n_0} \prec \mathbf{n}$). Since ξ_0 and $\neg\neg\xi_0$ are equivalent, $S(\mathbf{n})$ cannot be satisfiable over $\mathbb{C}$.
- Let $\psi = \xi_0 \wedge \xi_1$. Then there are two nodes $\mathbf{n_0} \in B$ and $\mathbf{n_1} \in B$ such that $\nu(\mathbf{n_0}) = ((\xi_0, [c_i, c_j]), \mathbb{C}, u)$, $\nu(\mathbf{n_1}) = ((\xi_1, [c_i, c_j]), \mathbb{C}, u)$, and, without loss of generality, $\mathbf{n_0}$ is the successor of $\mathbf{n}$ and $\mathbf{n_1}$ is the successor of $\mathbf{n_0}$. Since every branch containing $\mathbf{n}$ is closed, then every branch containing $\mathbf{n_1}$ is closed. By the inductive hypothesis, $S(\mathbf{n_1})$ is not satisfiable over $\mathbb{C}$ since $\mathbf{n_1} \prec \mathbf{n}$. Since every model over $\mathbb{C}$ satisfying $S(\mathbf{n})$ must, in particular, satisfy $(\xi_0 \wedge \xi_1, [c_i, c_j])$, and hence $(\xi_0, [c_i, c_j])$ and $(\xi_1, [c_i, c_j])$, it follows that $S(\mathbf{n})$, $S(\mathbf{n_0})$, and $S(\mathbf{n_1})$ are equi-satisfiable over $\mathbb{C}$. Therefore, $S(\mathbf{n})$ is not satisfiable over $\mathbb{C}$.
- Let $\psi = \neg(\xi_1 \wedge \xi_2)$. Then there exist two successor nodes $\mathbf{n_0}$ and $\mathbf{n_1}$ of $\mathbf{n}$ such that $\nu(\mathbf{n_0}) = ((\xi_0, [c_i, c_j]), \mathbb{C}, u_0)$, $\nu(\mathbf{n_1}) = ((\xi_1, [c_i, c_j]), \mathbb{C}, u_1)$, $\mathbf{n_0}, \mathbf{n_1} \prec \mathbf{n}$. Since every branch containing $\mathbf{n}$ is closed, then every branch containing $\mathbf{n_0}$ and every branch containing $\mathbf{n_1}$ is closed. By the inductive hypothesis $S(\mathbf{n_0})$ and $S(\mathbf{n_1})$ are not satisfiable over $\mathbb{C}$. Since every model over $\mathbb{C}$ satisfying $S(\mathbf{n})$ must also satisfy $(\xi_0, [c_i, c_j])$ or $(\xi_1, [c_i, c_j])$, it follows that $S(\mathbf{n})$ cannot be satisfiable over $\mathbb{C}$.
- Let $\psi = \neg(\xi_0 C \xi_0)$. Suppose that $S(\mathbf{n})$ is satisfiable over $\mathbb{C}$. Then, since $(\neg(\xi_0 C \xi_1), [c_i, c_j]) \in S(\mathbf{n})$, there is a model $\mathbf{M}^+ = \langle \mathbb{D}, V\rangle$ such that $\langle \mathbb{D}, <\rangle$ is an extension of $\langle \mathbb{C}, <\rangle$ and $\mathbf{M}^+, [c_i, c_j] \Vdash \neg(\xi_0 C \xi_1)$. So, for every c_k such

that $c_i \leq c_k \leq c_j$, we have that $\mathbf{M}^+, [c_i, c_k] \Vdash \neg\xi_0$ or $\mathbf{M}^+, [c_k, c_j] \Vdash \neg\xi_1$. By construction, the two immediate successors of $\mathbf{n}$ are $\mathbf{n_1}$ and $\mathbf{n_2}$ such that, for an element c_k with $c_i \leq c_k \leq c_j$, $(\neg\xi_0, [c_i, c_k])$ is in the decoration of $\mathbf{n_0}$ and $(\neg\xi_1, [c_k, c_j])$ is in the decoration of $\mathbf{n_1}$. By inductive hypothesis, since $\mathbf{n_1}, \mathbf{n_2} \prec \mathbf{n}$, $S(\mathbf{n_1})$ and $S(\mathbf{n_2})$ are not satisfiable over $\mathbb{C}$. Thus, such a model $\mathbf{M}^+$ cannot exist, and $S(\mathbf{n})$ is not satisfiable over $\mathbb{C}$.

– The cases $\psi = \neg(\xi_0 D \xi_1)$ and $\psi = \neg(\xi_0 T \xi_1)$ are analogous.

– Let $\psi = \xi_0 C \xi_1$. Assuming that $S(\mathbf{n})$ is satisfiable over $\mathbb{C}$, there is a model $\mathbf{M}^+ = \langle \mathbb{D}, V \rangle$, where $\langle \mathbb{D}, < \rangle$ is an extension of $\langle \mathbb{C}, < \rangle$, such that $\mathbf{M}^+, [c_i, c_j] \Vdash \theta$ for all $(\theta, [c_i, c_j]) \in S(\mathbf{n})$. In particular, $\mathbf{M}^+, [c_i, d] \Vdash \xi_0$ and $\mathbf{M}^+, [d, c_j] \Vdash \xi_1$ for some $c_i \leq d \leq c_j$. Consider two cases:

  1. If $d \in \mathbb{C}$, then $d = c_m$ for some $c_i \leq c_m \leq c_j$. But among the successors of $\mathbf{n}$ there are two nodes $\mathbf{n_m}, \mathbf{m_m}$ where $\nu(\mathbf{n_m}) = ((\xi_0, [c_i, c_m]), \mathbb{C}, u)$ and $\nu(\mathbf{m_m}) = ((\xi_1, [c_m, c_j]), \mathbb{C}, u)$, and since $\mathbf{n_m}, \mathbf{m_m} \prec \mathbf{n}$ (without loss of generality, suppose $\mathbf{n_m} \prec \mathbf{m_m}$), by the inductive hypothesis $S(\mathbf{n_m}) = S(\mathbf{n}) \cup \{(\xi_0, [c_i, c_m]), (\xi_1, [c_m, c_j])\}$ is not satisfiable over $\mathbb{C}$, which is a contradiction;
  2. If $d \notin \mathbb{C}$, then there is an m such that $i \leq m \leq j-1$ and $c_m < d < c_{m+1}$. Hence, there are two successors $\mathbf{n'_m}, \mathbf{m'_m}$ of $\mathbf{n}$ such that $\nu(\mathbf{n'_m}) = ((\xi_0, [c_i, d]), \mathbb{C} \cup \{d\}, u)$, $\nu(\mathbf{m'_m}) = ((\xi_1, [d, c_j]), \mathbb{C} \cup \{d\}, u)$, and since $\mathbf{n'_m}, \mathbf{m'_m} \prec \mathbf{n}$ (without loss of generality, suppose $\mathbf{n'_m} \prec \mathbf{m'_m}$), by the inductive hypothesis $S(\mathbf{n'_m}) = S(\mathbf{n}) \cup \{(\xi_0, [c_i, d]), (\xi_1, [d, c_j])\}$ is not satisfiable over $\mathbb{C} \cup \{d\}$ which, again, is a contradiction.

  Thus, in either case $S(\mathbf{n})$ is not satisfiable over $\mathbb{C}$.

– Let $\psi = \xi_0 D \xi_1$. Assuming that $S(\mathbf{n})$ is satisfiable over $\mathbb{C}$, there is a model $\mathbf{M}^+ = \langle \mathbb{D}, V \rangle$, where $\langle \mathbb{D}, < \rangle$ is an extension of $\langle \mathbb{C}, < \rangle$, such that $\mathbf{M}^+, [c_i, c_j] \Vdash \theta$ for all $(\theta, [c_i, c_j]) \in S(\mathbf{n})$. In particular, $\mathbf{M}^+, [d, c_i] \Vdash \xi_0$ and $\mathbf{M}^+, [d, c_j] \Vdash \xi_1$ for some $d \leq c_i$. Consider 3 cases:

  1. If $d \in \mathbb{C}$, then $d = c_m$ for some $c_m \leq c_i$. But between the successors of $\mathbf{n}$ there are two nodes $\mathbf{n_m}, \mathbf{m_m}$ where $\nu(\mathbf{n_m}) = ((\xi_0, [c_m, c_i]), \mathbb{C}, u)$ and $\nu(\mathbf{m_m}) = ((\xi_1, [c_m, c_j]), \mathbb{C}, u)$, and since $\mathbf{n_m}, \mathbf{m_m} \prec \mathbf{n}$ (without loss of generality, suppose $\mathbf{n_m} \prec \mathbf{m_m}$), by the inductive hypothesis $S(\mathbf{n_m}) = S(\mathbf{n}) \cup \{(\xi_0, [c_m, c_i]), (\xi_1, [c_m, c_j])\}$ is not satisfiable over $\mathbb{C}$, which is a contradiction.
  2. If $d \notin \mathbb{C}$ and there is a minimal element $c \in \mathbb{C}$ and an index m such that $c_m, c_{m+1} \in [c, c_i]$ and $c_m < d < c_{m+1}$, then there are two successors $\mathbf{n'_m}, \mathbf{m'_m}$ of $\mathbf{n}$ such that $\nu(\mathbf{n'_m}) = ((\xi_0, [c_i, d]), \mathbb{C} \cup \{d\}, u)$ and $\nu(\mathbf{m'_m}) = ((\xi_1, [d, c_j]), \mathbb{C} \cup \{d\}, u)$, and since $\mathbf{n'_m}, \mathbf{m'_m} \prec \mathbf{n}$ (without loss of generality, suppose $\mathbf{n'_m} \prec \mathbf{m'_m}$), by the inductive hypothesis $S(\mathbf{n'_m}) = S(\mathbf{n}) \cup \{(\xi_0, [c_i, d]), (\xi_1, [d, c_j])\}$ is not satisfiable over $\mathbb{C} \cup \{d\}$ which, again, is a contradiction.
  3. If $d \notin \mathbb{C}$ and there is an index m such that $c_{m+1} \in [c, c_i]$, $d < c_{m+1}$, and d is not comparable with all predecessors of c_{m+1}, then, again, there are two successor nodes $\mathbf{n''_m}, \mathbf{m''_m}$ of $\mathbf{n}$ such that $\nu(\mathbf{n''_m}) = ((\xi_0, [c_i, d]), \mathbb{C} \cup \{d\}, u)$ and $\nu(\mathbf{m''_m}) = ((\xi_1, [d, c_j]), \mathbb{C} \cup \{d\}, u)$, and since $\mathbf{n''_m}$, $\mathbf{m''_m} \prec \mathbf{n}$

(without loss of generality, suppose $\mathbf{n}''_{\mathbf{m}} \prec \mathbf{m}''_{\mathbf{m}}$), by the inductive hypothesis $S(\mathbf{n}''_{\mathbf{m}}) = S(\mathbf{n}) \cup \{(\xi_0, [c_i, d]), (\xi_1, [d, c_j])\}$ is not satisfiable over $\mathbb{C} \cup \{d\}$ which, again, is a contradiction.

Thus, in either case $S(\mathbf{n})$ is not satisfiable over $\mathbb{C}$.

– The case of $\psi = \xi_0 T \xi_1$ is similar.

□

Definition 7. *If $\mathcal{T}_0$ is the three-node tableau built up from a root with void decoration and two leaves decorated respectively by $((\phi, [c_b, c_e]), \{c_b < c_e\}, 0)$ and $((\phi, [c_b, c_b]), \{c_b\}, 0)$ for a given* BCDT$^+$*-formula ϕ, the **limit tableau** $\overline{\mathcal{T}}$ for ϕ is the (possibly infinite) decorated tree obtained as follows. First, for all i, $\mathcal{T}_{i+1}$ is the tableau obtained by the simultaneous application of the branch-expansion strategy to every branch in $\mathcal{T}_i$. Then, we ignore all flags from the decorations of the nodes in every $\mathcal{T}_i$. Thus, we obtain a chain by inclusion of decorated trees: $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \ldots$, and we define $\overline{\mathcal{T}} = \bigcup_{i=0}^{\infty} \mathcal{T}_i$.*

Notice that the chain above may stabilize at some $\mathcal{T}_i$ if it closes, or if the branch-expansion rule is not applicable to any of its branches. If $\overline{\mathcal{T}}$ is a limit tableau, we associate with each branch B in $\overline{\mathcal{T}}$ the interval structure $\mathbb{C}_B = \bigcup_{i=0}^{\infty} \mathbb{C}_{B_i}$, where, for all i, $\mathbb{C}_{B_i}$ is the interval structure from the decoration of the leaf of the (sub-)branch B_i of B in $\mathcal{T}_i$. The definitions of closed and open branches readily apply to $\overline{\mathcal{T}}$.

Definition 8. *A branch in a (limit) tableau is **saturated** if there are no nodes on that branch to which the branch-expansion rule is applicable on the branch. A (limit) tableau is **saturated** if every open branch in it is saturated.*

Now we will show that the set of all labeled formulas on an open branch in a limit tableau has the saturation properties of a Hintikka set in first-order logic.

Lemma 1. *Every limit tableau is saturated.*

Proof. Given a node $\mathbf{n}$ in a limit tableau $\overline{\mathcal{T}}$, we denote by $d(\mathbf{n})$ the distance (number of edges) between $\mathbf{n}$ and the root of $\overline{\mathcal{T}}$. Now, given a branch B in $\overline{\mathcal{T}}$, we will prove by induction on $d(\mathbf{n})$ that after every step of the expansion of that branch at which the branch-expansion rule becomes applicable to $\mathbf{n}$ (because $\mathbf{n}$ has just been introduced, or because a new point has been introduced in the interval structure on B) that rule is subsequently applied on B to that node.

Suppose the inductive hypothesis holds for all nodes with distance to the root less than l. Let $d(\mathbf{n}) = l$ and the branch-expansion rule has become applicable to $\mathbf{n}$. If there are no nodes between the root (incl. the root) and $\mathbf{n}$ (excl. $\mathbf{n}$) to which the branch-expansion rule is applicable at that moment, the next application of the branch-expansion rule on B is to $\mathbf{n}$. Otherwise, consider the closest-to-$\mathbf{n}$-node $\mathbf{n}^*$ between the root and $\mathbf{n}$ to which the branch-expansion rule is applicable or will become applicable on B at least once thereafter. (Such a node exists because there are only finitely many nodes between $\mathbf{n}$ and the root.) Since $d(\mathbf{n}^*) < d(\mathbf{n})$, by the inductive hypothesis the branch-expansion rule has been subsequently applied to $\mathbf{n}^*$. Then the next application of the branch-expansion rule on B must have been to $\mathbf{n}$ and that completes the induction. Now, assuming that a branch in a limit tableau is not saturated, consider the closest to the root node $\mathbf{n}$ on that branch B to which the branch-expansion rule is applicable on that branch. If $\Phi(\mathbf{n})$ is none of the cases $\neg C, \neg D$, and $\neg T$, then the branch-expansion rule has become applicable to $\mathbf{n}$ at the step when $\mathbf{n}$ is introduced, and by the claim above, it has been subsequently applied, at which moment the node has become unavailable thereafter, which contradicts the assumption. Suppose that $\Phi(\mathbf{n}) = \neg(\psi_0 C \psi_1)$. Then an application of the rule on B would create two successors with labels $(\neg\psi_0, [c_i, c])$ and $(\neg\psi_1, [c, c_j])$, at least one of them new on B. But c_i, c_j, c have already been introduced at some (finite) step of the construction of B and at the first step when the three of them, as well as $\mathbf{n}$, have appeared on the branch, the branch-expansion rule has become applicable to $\mathbf{n}$, hence is has been subsequently applied on B and that application must have introduced the labels $(\psi_0, [c_i, c])$ and $(\psi_1, [c, c_j])$ on B, which again contradicts the assumption. The same holds if $\Phi(\mathbf{n}) = \neg(\psi_0 D \psi_1)$ or $\Phi(\mathbf{n}) = \neg(\psi_0 D \psi_1)$. $\square$

Corollary 1. *Let ϕ be a* BCDT$^+$*-formula and $\overline{\mathcal{T}}$ be the limit tableau for ϕ. For every open branch B in $\overline{\mathcal{T}}$, the following closure properties hold.*

- *If there is a node $\mathbf{n} \in B$ such that $\nu(\mathbf{n}) = ((\neg\neg\psi, [c_i, c_j]), \mathbb{C}, u)$, then there is a node $\mathbf{n_0} \in B$ such that $\nu(\mathbf{n_0}) = ((\psi, [c_i, c_j]), \mathbb{C}, u_0)$.*
- *If there is a node $\mathbf{n} \in B$ such that $\nu(\mathbf{n}) = ((\psi_0 \wedge \psi_1, [c_i, c_j]), \mathbb{C}, u)$, then there is a node $\mathbf{n_0} \in B$ such that $\nu(\mathbf{n_0}) = ((\psi_0, [c_i, c_j]), \mathbb{C}, u_0)$ and a node $\mathbf{n_1} \in B$ such that $\nu(\mathbf{n_1}) = ((\psi_1, [c_i, c_j]), \mathbb{C}, u_1)$.*
- *If there is a node $\mathbf{n} \in B$ such that $\nu(\mathbf{n}) = ((\neg(\psi_0 \wedge \psi_1), [c_i, c_j]), \mathbb{C}, u)$, then there is a node $\mathbf{n_0} \in B$ such that $\nu(\mathbf{n_0}) = ((\neg\psi_0, [c_i, c_j]), \mathbb{C}, u_0)$ or a node $\mathbf{n_1} \in B$ such that $\nu(\mathbf{n_1}) = ((\neg\psi_1, [c_i, c_j]), \mathbb{C}, u_1)$.*
- *If there is a node $\mathbf{n} \in B$ such that $\nu(\mathbf{n}) = ((\psi_0 C \psi_1, [c_i, c_j]), \mathbb{C}, u)$, then, for some $c \in \mathbb{C}_B$ such that $c_i \leq c \leq c_j$ there are two nodes $\mathbf{n}', \mathbf{m}' \in B$ such that $\nu(\mathbf{n}') = ((\psi_0, [c_i, c]), \mathbb{C}', u')$ and $\nu(\mathbf{m}') = ((\psi_1, [c, c_j]), \mathbb{C}', u')$.*
- *Similarly for every node $\mathbf{n}$ with $\Phi(\mathbf{n}) = \psi_0 D \psi_1$ or $\Phi(\mathbf{n}) = \psi_0 T \psi_1$.*
- *If there is a node $\mathbf{n} \in B$ such that $\nu(\mathbf{n}) = ((\neg(\psi_0 C \psi_1), [c_i, c_j]), \mathbb{C}, u)$, then for all $c \in \mathbb{C}_B$ such that $c_i \leq c \leq c_j$, there is a node $\mathbf{n}' \in B$ such that $\nu(\mathbf{n}') = ((\neg\psi_0, [c_i, c]), \mathbb{C}', u')$ or a node $\mathbf{m}' \in B$ such that $\nu(\mathbf{m}') = ((\neg\psi_1, [c, c_j]), \mathbb{C}', u')$.*
- *Similarly for every node $\mathbf{n}$ with $\Phi(\mathbf{n}) = \neg(\psi_0 D \psi_1)$ or $\Phi(\mathbf{m}) = \neg(\psi_0 T \psi_1)$.*

Lemma 2. *If the limit tableau for some formula $\phi \in$*BCDT$^+$ *is closed, then some finite tableau for ϕ is closed.*

Proof. Suppose the limit tableau for ϕ is closed. Then every branch closes at some finite step of the construction and then remains finite. Since the branch-expansion rule always produces finitely many successors, every finite tableau is finitely branching, and hence so is the limit tableau. Then, by König's lemma, the limit tableau, being a finitely branching tree with no infinite branches, must be finite, hence its construction stabilizes at some finite stage. At that stage a closed tableau for ϕ is constructed. $\square$

Theorem 2 (completeness). *Let $\phi \in$BCDT$^+$ be a valid formula. Then there is a closed tableau for $\neg\phi$.*

Proof. We will show that the limit tableau $\overline{\mathcal{T}}$ for $\neg\phi$ is closed, whence the claim follows by the previous lemma.

By contraposition, suppose that $\overline{\mathcal{T}}$ has an open branch B. Let $\mathbb{C}_B$ be the interval structure associated with B and $S(B)$ be the set of all labeled formulas on B. Consider the model $\mathbf{M}^+ = \langle \mathbb{C}_B, V\rangle$ where, for every $[c_i, c_j] \in \mathbb{I}(\mathbb{C}_B)^+$ and $p \in \mathcal{AP}$, $p \in V([c_i, c_j])$ iff $(p, [c_i, c_j]) \in \Phi(B)$. We show by induction on ψ that, for every $(\psi, [c_i, c_j]) \in S(B)$, $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$.

We reason by induction on the complexity of ψ:

- Let $\psi = \pi$ (resp., $\psi = \neg\pi$). Since $(\pi, [c_i, c_j]) \in S(B)$ (resp., $(\neg\pi, [c_i, c_j]) \in S(B)$) and B is open, then $c_i \neq c_j$ (resp., $c_i = c_j$). Hence $\mathbf{M}^+, [c_i, c_j] \Vdash \pi$ (resp., $\mathbf{M}^+, [c_i, c_j] \Vdash \neg\pi$).
- Let $\psi = p$ or $\psi = \neg p$ where $p \in \mathcal{AP}$. Then the claim follows by definition, because if $(\neg p, [c_i, c_j]) \in S(B)$ then $(p, [c_i, c_j]) \notin S(B)$ since B is open.
- Let $\psi = \neg\neg\xi$. Then by Corollary 1, $(\xi, [c_i, c_j]) \in S(B)$, and by inductive hypothesis $\mathbf{M}^+, [c_i, c_j] \Vdash \xi$. So $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$.
- Let $\psi = \xi_0 \wedge \xi_1$. Then by Corollary 1, $(\xi_0, [c_i, c_j]) \in S(B)$ and $(\xi_1, [c_i, c_j]) \in S(B)$. By inductive hypothesis, $\mathbf{M}^+, [c_i, c_j] \Vdash \xi_0$ and $\mathbf{M}^+, [c_i, c_j] \Vdash \xi_1$, so $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$.
- Let $\psi = \neg(\xi_0 \wedge \xi_1)$. Then by Corollary 1, $(\neg\xi_0, [c_i, c_j]) \in S(B)$ or $(\neg\xi_1, [c_i, c_j]) \in S(B)$. By inductive hypothesis $\mathbf{M}^+, [c_i, c_j] \Vdash \neg\xi_0$ or $\mathbf{M}^+, [c_i, c_j] \Vdash \neg\xi_1$, so $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$.
- Let $\psi = \xi_0 C \xi_1$. Then by Corollary 1, $(\xi_0, [c_i, c]) \in S(B)$ and $(\xi_1, [c, c_i]) \in S(B)$ for some $c \in \mathbb{C}_B$ such that $c_i \leq c \leq c_j$. Thus, by inductive hypothesis, $\mathbf{M}^+, [c_i, c] \Vdash \xi_0$ and $\mathbf{M}^+, [c, c_j] \Vdash \xi_1$, and thus $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$.
- Similarly for $\psi = \xi_0 D \xi_1$ and $\psi = \xi_0 T \xi_1$.
- Let $\psi = \neg(\xi_0 C \xi_1)$. Then by Corollary 1, for all $c \in \mathbb{C}_B$ such that $c_i \leq c \leq c_j$, $(\neg\xi_0, [c_i, c]) \in S(B)$ and $(\neg\xi_1, [c, c_j]) \in S(B)$. Hence, by the inductive hypothesis, $\mathbf{M}^+, [c_i, c] \Vdash \neg\xi_0$ and $\mathbf{M}^+, [c, c_j] \Vdash \neg\xi_1$, for all $c_i \leq c \leq c_j$. Thus, $\mathbf{M}^+, [c_i, c_j] \Vdash \psi$.
- Similarly for $\psi = \neg(\xi_0 D \xi_1)$ and $\psi = \neg(\xi_0 T \xi_1)$.

This completes the induction. In particular, we obtain that $\neg\phi$ is satisfied in $\mathbf{M}^+$, which is in contradiction with the assumption that ϕ is valid. $\square$

Concluding remark: The main natural continuation of this work would be to identify cases (fragments of the logic, or classes of interval structures) when the tableau will terminate and therefore provide a decision procedure.

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