Undecidability of Interval Temporal Logics with the Overlap Modality

Davide Bresolin\textsuperscript{1}, Dario Della Monica\textsuperscript{2}, Valentin Goranko\textsuperscript{3}, Angelo Montanari\textsuperscript{2}, Guido Sciavicco\textsuperscript{4}
\textsuperscript{1}University of Verona, Verona, Italy, \textsuperscript{2}University of Udine, Udine, Italy
\textsuperscript{3}Technical University of Denmark, Denmark, \textsuperscript{4}University of Murcia, Murcia, Spain
\textsuperscript{1}davide.bresolin@univr.it, \textsuperscript{2}\{dario.dellamonica|angelo.montanari\}@dimi.uniud.it, \textsuperscript{3}vfgo@imm.dtu.dk, \textsuperscript{4}guido@um.es

Abstract

We investigate fragments of Halpern-Shoham’s interval logic HS involving the modal operators for the relations of left or right overlap of intervals. We prove that most of these fragments are undecidable, by employing a non-trivial reduction from the octant tiling problem.

1. Introduction

Interval temporal logics are based on temporal structures over linearly (or partially) ordered domains, where time intervals, rather than time instants, are the primitive ontological entities. A systematic analysis of the variety of relations between intervals on linear orders was first accomplished by Allen [1], who explored the use of interval reasoning in systems for time management and planning. The problem of representing and reasoning about time intervals arises naturally in various other fields of computer science, artificial intelligence, and temporal databases, such as theories of action and change, natural language processing, and constraint satisfaction problems. Temporal logics with interval-based semantics have also been proposed as a useful formalism for the specification and verification of hardware [16] and of real-time systems [8].

Interval temporal logics feature modal operators that correspond to various relations between intervals, in particular the thirteen different binary interval relations on linear orders, known as Allen’s relations [1]. In [11], Halpern and Shoham introduce a modal logic for reasoning about interval structures, nowadays known as HS, with modal operators corresponding to Allen’s interval relations. This logic turns out to be undecidable under very weak assumptions on the class of interval structures [11]. In particular, undecidability holds for any class of interval structures over linear orders that contains at least one linear order with an infinite ascending (or descending) chain, thus including the natural time flows $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. The complex and generally bad computational behavior of interval temporal logics is essentially due to the fact that formulas are evaluated over pairs of points and translate into binary relations. In a few cases, decidability has been recovered by imposing severe restrictions on the set of modalities and/or on the interval-based semantics, which essentially reduce the logic to a point-based one. For a long time, the sweeping undecidability results of Halpern and Shoham have discouraged attempts for practical applications and further research on interval logics. A renewed interest in the area has been recently stimulated by the discovery of some interesting decidable fragments of HS [4, 5, 6, 7]. The classification of decidable and undecidable fragments of HS has thus become one of the major topics of the current research agenda in interval temporal logics. The current state of affairs in that area has recently been summarized in [3], where the main techniques so far exploited for proving decidability and undecidability have been presented. Such results have so far been obtained for fragments of HS involving all, but one, pairs of modal operators, with respect to various classes of linear orders. The only so far unexplored case is the one of fragments involving the operators $\langle O \rangle$ and/or its transpose $\langle O \rangle^t$, which respectively capture the Overlap relation and its inverse. Such logics have received almost no attention in the literature (the only exception we are aware of is a number of representation theorems for interval structures containing such a relation, which have recently been obtained in [9]).

In this paper, we show that most extensions of the interval logic of Overlap are undecidable, thus making a further step toward the complete classification of the (un)decidability of all HS fragments. The proofs employ a non-trivial reduction from the octant tiling problem. The idea of using tiling problems to prove undecidability of interval logics and many-dimensional logics goes back to [15] and it has been subsequently applied to the compass logic [14], to various product logics [10], to modal spatial logics.
of topological relations [13], and to other fragments of HS [3, 6]. Most of these results, however, apply to (relatively) more expressive logical languages than those considered in the present paper. Moreover, the techniques for the encoding of tiling problems used so far do not transfer to our cases in any obvious way.

The paper is structured as follows. In Section 2 we introduce syntax and semantics of the logics of the Overlap relation. In Section 3, we provide a detailed account of the undecidability proofs for the fragments AO, A0, AO, and A O. In Section 4, we summarize the undecidability results for other extensions. In the conclusion, we provide an assessment of the work and outline future research directions.

2. The logics of the Overlap relation

Let D = (D, <) be a linearly ordered set. An interval over D can be defined as an ordered pair [a, b], where a, b ∈ D and a < b, thus excluding intervals with coincident endpoints (strict semantics). As an alternative, one can define an interval over D as a pair [a, b], with a, b ∈ D and a ≤ b (non-strict semantics). Hereafter, we confine our attention to strict semantics; as we will show later, the non-strict case can be easily reduced to the strict one.

In general, the language of a propositional interval logic with unary modalities consists of a set AP of propositional letters, any complete set of boolean operators (such as ∨ and ¬), and a set of unary modal operators (X1), . . . , (Xk), each of them associated with a specific binary relation over intervals ( decidability issues for binary modal operators have been addressed in [12]). Formulas are defined by the grammar:

ϕ ::= p | ¬ϕ | ϕ ∨ ϕ | (X1)ϕ | . . . | (Xk)ϕ,

The semantics is given in terms of interval models M = ⟨⟨D⟩, V ⟩, where ⟨⟨D⟩⟩ is the set of all intervals over D and the valuation function V : AP → 2⟨⟨D⟩⟩ assigns to every p ∈ AP the set of intervals V(p) over which it holds. The truth of a formula over a given interval [a, b] in a model M is defined by structural induction on formulas:

- M, [a, b] ⊨ p iff [a, b] ∈ V(p), for all p ∈ AP;
- M, [a, b] ⊨ ¬ψ iff it is not the case that M, [a, b] ⊨ ψ;
- M, [a, b] ⊨ ϕ ∨ ϕ iff M, [a, b] ⊨ ϕ or M, [a, b] ⊨ ψ;
- M, [a, b] ⊨ (X1)ψ iff there exists an interval [c, d] such that [a, b] RX1 [c, d] and M, [c, d] ⊨ ψ,

where RX1 is the binary interval relation corresponding to the unary modal operator (X1).

In the following, we will focus our attention on a specific family of HS fragments and, for that purpose, we will assume all operators listed in Fig. 1 and their transposes to be primitive in the language, where by a transpose of a unary modal operator (X) we mean the modal operator (X) for the inverse of X. It is worth pointing out that the semantics of the operators slightly differs from the original one [11], which do not perfectly match the semantics of Allen’s relations. As a general notation rule, we will denote any fragment of HS with the set of its operators; for example, the HS fragment featuring the operators (A) and (O) will be denoted by AO. Also, we will denote by X1 X2 the set consisting of the fragments X1X2, X1X2, X2X1, and X2X2.

The family of fragments of HS we consider here includes A*O*, B*O*, E*O*, and D*O*. We will prove that all these logics are undecidable. We will give the details of the undecidability proof for the logics A*O*; the proofs for the other logics in the class are quite similar. Besides, we have been able to prove the undecidability of the logic O* with respect to any class of discrete linear orders. For lack of space, proof details for this case are omitted, but they will be included in an extended forthcoming version. All results are readily transferable to the case of non-strict semantics, using the formula (O) ⊤ to impose the requirement that every interval we deal with has an internal point.

3. Undecidability of the logics A*O*

In this section, we show that the logics AO, A0, A0, and A O are undecidable. In fact, we will prove the undecidability of the satisfiability problem for AO and then we will show that the proof can be tailored to deal with the other cases. For these proofs we will use a reduction from the tiling problem for the second octant O of the integer plane Z × Z.

3.1. The tiling problem for O

The tiling problem for O is the problem of establishing whether a given finite set of tile types T = {t1, . . . , tk} can tile O = {(i, j) : i, j ∈ N ∧ 0 ≤ i ≤ j}. For every tile type ti ∈ T, let right(ti), left(ti), up(t), and down(ti) be the colors of the corresponding sides of ti. To solve the problem, one must find a function f : O → T such that

right(f(n, m)) = left(f(n + 1, m)), with n < m,
and up(f(n, m)) = down(f(n, m + 1)).

Using König’s lemma one can prove that a tiling system tiles an octant if and only if it tiles arbitrarily large squares if and only if it tiles N × N if and only if it tiles Z × Z. The undecidability of the former thus immediately follows from that of the latter [2].

3.2. Generic reduction of the tiling problem for O to satisfiability in interval logics

Hereafter, we assume that AP contains some special propositional letters: {u, ld, tile, ∗, b, f} and others that will
be introduced in due course. For every propositional letter \( q \), we denote by the expression \( q\text{-interval} \) an interval satisfying \( q \). We will provide a reduction of the tiling problem for \( \mathcal{O} \) to the satisfiability problem for any fragment \( \mathcal{F} \) of HS considered here in any class of interval models containing at least one model \( M \) with an unbounded-to-the-right (for the logics \( \mathcal{AO} \) and \( \overline{\mathcal{AO}} \)) or unbounded-to-the-left (for the logics \( \overline{\mathcal{AO}} \) and \( \overline{\mathcal{O}} \)) sequence of points. The reduction is based on the following main steps. First, we set our framework by forcing the existence of a unique infinite chain of \( u\)-intervals (\( u\text{-chain} \), for short) on the linear order, which covers an initial segment of the domain. Such \( u\)-intervals will be used as cells (‘unit-intervals’) to arrange the tiling. Next, we define a chain of \( Id\)-intervals (\( Id\text{-chain} \), for short), each of them representing a row of the octant. An \( Id\)-interval is composed by a sequence of \( u\)-intervals; each \( u\)-interval is used either to represent a part of the plane or to separate two \( Id\)-intervals. In the former case it is labelled with the propositional letter \( tile \) in the latter case it is labelled with the propositional letter \( * \). Then, we define two relations that connect each tile with its above neighbor and right neighbor in the octant, respectively. By using these relations, we force the \( j \)-th \( Id\)-interval to contain exactly \( j \) tile intervals. Finally, we introduce a set of propositional letters \( T = \{ t_1, t_2, \ldots, t_k \} \) corresponding to the set of tile types \( T = \{ t_1, t_2, \ldots, t_k \} \) and we construct a formula \( \Phi_T \) belonging to the fragment \( \mathcal{F} \) which is satisfiable if and only if there exists a proper tiling of the octant \( \mathcal{O} \) by \( T \), i.e., one that satisfies the color constraints on vertically- and horizontally-adjacent tiles.

In the rest of this section we will illustrate this technique by constructing the formula \( \Phi_T \) for the fragment \( \mathcal{AO} \).

### 3.3. Definition of \( u\text{-chain} \) and \( Id\text{-chain} \)

Given an interval \([a, b]\), we define \( \mathcal{G}_{[a, b]} \) as the set of intervals that contains the interval \([a, b]\) and all the intervals starting after \( a \) and ending after \( b \). Moreover, we define the operator \( [G] \) (always in the future) as follows:

\[
\]

\( [G]p \) holds over the interval \([a, b]\) iff \( p \) holds over each interval in \( \mathcal{G}_{[a, b]} \).

Let \([a, b]\) be the interval over which we evaluate formulas (we may think of it as the interval to the right of which the \( u\)-chain starts). From now on, when we talk about an interval or a set of intervals, we will implicitly refer to intervals belonging to \( \mathcal{G}_{[a, b]} \).

We start the encoding by constructing a \( u\)-chain of unit intervals:

\[
\neg u \land (A) u \land (O) \supset \neg (O) u \tag{1}
\]

\[
[G](u \rightarrow (A) u) \land [G](u \rightarrow (O) T) \tag{2}
\]

\[
[G](u \rightarrow \neg (O) u) \tag{3}
\]

\[
[G](u \iff \ast \lor tile) \land (tile \rightarrow \neg \ast) \tag{4}
\]

and a \( Id\)-chain that encodes the levels of the octant:

\[
[G](A) \ast \iff (A) Id \tag{5}
\]

\[
[G](Id \rightarrow (A) \ast) \tag{6}
\]

\[
[G](A) \ast \iff \neg (O) Id \tag{7}
\]

\[
\neg Id \land \neg (O) Id \land (A)(\ast \land (A)(tile \land (A)(tile \land (A)(tile))) \land \ldots \land \ast) \tag{8}
\]

\[\ast \land \ldots \land \ast \tag{9}\]

**Lemma 1.** If \( M, [a, b] \models (9) \), then there exists a sequence of points \( b = b_0 \prec b_1 \prec \ldots \prec b_{k_1} = b_0' \prec b_1' \prec \ldots < b_{k_2} = b_0' \prec \ldots \) such that for each \( j \geq 0 \) we have:

a) \( M, [b_j, b_{j+1}] \models u \) for each \( 0 \leq i < k_j \) and no other interval \([c, d] \in \mathcal{G}_{[a, b]} \) satisfies \( u \), unless \( c > b_j \) for each \( i, j > 0 \);

b) \( M, [b_j, b_{j+1}] \models Id \) and no other interval \([c, d] \in \mathcal{G}_{[a, b]} \) satisfies \( Id \), unless \( c > b_j \) for each \( i, j > 0 \);

c) \( M, [b_j, b_{j+1}] \models \ast \), and no other interval \([c, d] \in \mathcal{G}_{[a, b]} \) satisfies \( \ast \), unless \( c > b_j \) for each \( i, j > 0 \);

d) \( M, [b_j, b_{j+1}] \models tile \) for each \( 1 \leq i < k_j \), and no other interval \([c, d] \in \mathcal{G}_{[a, b]} \) satisfies \( tile \), unless \( c > b_j \) for each \( i, j > 0 \);

e) \( k_1 = 2 \) and \( k_l > 2 \) for each \( l > 1 \).

**Proof.** a) The existence of such a sequence is guaranteed by (1) and the left conjunct of (2). Now, suppose, for contradiction, that there exists an interval \([c, d] \in \mathcal{G}_{[a, b]} \)

![Figure 1. The semantics of basic interval modalities.](image)
satisfying \( u \) such that \( c \leq b_j^i \) for some \( i, j > 0 \), and 
\([c, d] \neq [b_j^i, b_j^{i+1}]\) for each \( i \geq 0 \) and for each \( j > 0 \). We
distinguish the following cases:

- if \([c, d] = [a, b] \) or \( a < c < b < d \), then we have a
contradiction with (1);
- if \( c = b_j^i \) for some \( i \geq 0, j > 0 \), then we have
\( d \neq b_j^{i+1} \). By (1), there exists a point \( a' \) such
that \( a < a' < b \). If \( d < b_j^{i+1} \), \([a', d]\) meets the u-
interval starting at \( d \), by the first conjunct of (2), and
overlaps the u-interval \([b_j^i, b_j^{i+1}]\), contradicting
(3). Otherwise, if \( d > b_j^{i+1} \), then \([a', b_j^{i+1}]\) meets
the u-interval starting at \( b_j^{i+1} \) and overlaps the u-
interval \([c, d]\), contradicting (3);
- if \( c \neq b_j^i \) for each \( i \geq 0, j > 0 \), then we have
\( b_j^i < c < b_j^{i+1} \) for some \( i \geq 0 \) and \( j > 0 \). In
this case, the interval \([a', c]\) meets the u-interval
starting at \( c \) and overlaps the u-interval \([b_j^i, b_j^{i+1}]\),
contradicting (3).

b) The existence of a \( l \)-chain is guaranteed by (5), (6), and
(8). Now suppose, for contradiction, that there exists
an interval \([c, d] \in \mathcal{G}_{[a, b]}\) satisfying \( l \)-d such that \( c \leq b_j^i \)
for some \( i, j > 0 \), and \([c, d] \neq [b_j^i, b_j^{i+1}] \) for each \( j > 0 \).
Notice that \( c \) and \( d \) start a \( \ast \)-interval by (5) and (6),
respectively. We must consider the following cases:

- if \([c, d] = [a, b] \) or \( a < c < b < d \), then we have a
contradiction with (8);
- if \( c = b_j^i \) for some \( j > 0 \), then we have \( d \neq b_j^{i+1} \).
If \( d < b_j^{i+1} \), \([a', d]\) meets the \( \ast \)-interval starting
at \( d \) and overlaps the \( l \)-interval \([b_j^i, b_j^{i+1}]\), contra-
dicting (7). Otherwise, if \( d > b_j^{i+1} \), then \([a', b_j^{i+1}]\)
meets the \( \ast \)-interval starting at \( b_j^{i+1} \) and overlaps
the \( l \)-interval \([c, d]\), contradicting (7);
- if \( c \neq b_j^i \) for each \( j > 0 \), then we have \( b_j^i < c < b_j^{i+1} \)
for some \( i \geq 0 \) and \( j > 0 \). In this case, the interval
\([a', c]\) meets the \( \ast \)-interval starting at \( c \) and
overlaps the \( l \)-interval \([b_j^i, b_j^{i+1}]\), contradicting
(7).

c) The first u-interval of each ld-interval \((b_j^i, b_j^{i+1})\) is a
\( \ast \)-interval by (5). Now suppose, for contradiction, that
there exists an interval \([c, d] \in \mathcal{G}_{[a, b]}\) satisfying \( \ast \) such
that \( c \leq b_j^i \) for some \( i, j > 0 \), and \([c, d] \neq [b_j^i, b_j^{i+1}] \)
for each \( j > 0 \). By point a) of this lemma and by
(4), we have that \([c, d] = [b_j^i, b_j^{i+1}] \) for some \( i, j > 0 \).
Since the interval \([a', c]\) meets the \( \ast \)-interval \([b_j^i, b_j^{i+1}]\)
and overlaps the ld-interval \([b_j^i, b_j^{i+1}]\), we have a contra-
diction with (7).

d) By point c) of this lemma and (4), we can conclude that
for each \( i, j > 0, b_j^i, b_j^{i+1} \) satisfies tile. Moreover, by
point c), (4), and point a), we have that no other interval
\([c, d] \in \mathcal{G}_{a, b}\) is a tile-interval, unless \( c \geq b_j^i \) for each
\( i, j > 0 \).

e) \( k_1 = 2 \) and \( k_2 > 2 \) for each \( l > 1 \) immediately follows
from (8).

3.4. The above-neighbor relation

We focus now on the ‘above-neighbor’ relation, whose
encoding is shown in Fig. 2. Intuitively, the above-neighbor
relation connects each tile-interval with its vertical neighbor
in the octant (e.g., \( t_{2.2} \) with \( t_{2.3} \) in Fig. 2).

We distinguish between backward and forward ld-intervals,
which alternate, by labeling each u-interval either with \( b \),
if it belongs to a backward ld-interval, or with \( f \), if it
belongs to a forward one (formulas from (10) to (12)).
Intuitively, we have that the tile-intervals are placed in ascend-
ing order in forward ld-intervals and in descending order in
backward ld-intervals. In particular, this means that the left-
most tile-interval of a backward ld-interval represents the
last tile of that level (and not the first one) in the octant.
Let \( \alpha, \beta \in \{b, f\} \), with \( \alpha \neq \beta \):

\[ (A)b \wedge [G]\langle (u \rightarrow b \lor f) \wedge (b \rightarrow \neg f) \rangle \]  
\[ (G)[u \wedge \alpha \wedge \neg \langle A \rangle \alpha \rightarrow \langle A \rangle \alpha] \]  
\[ (G)[u \wedge \alpha \wedge \langle A \rangle \alpha \rightarrow \langle A \rangle \beta] \]  
\[ (10) \wedge \ldots \wedge (12) \]

Lemma 2. If \( \mathcal{M}, [a, b] \models (9) \wedge (13) \), then there exists
a sequence of points like that defined in Lemma 1 such that
\( \mathcal{M}, [b_j^i, b_j^{i+1}] \models b \) if and only if \( j \) is an odd number
and \( \mathcal{M}, [b_j^i, b_j^{i+1}] \models f \) if and only if \( j \) is an even number.
Furthermore, we have that no other interval \([c, d] \in \mathcal{G}_{[a, b]}\) satisfies
\( b \) or \( f \), unless \( c > b_j^i \) for each \( i, j > 0 \).

Because of the alternation between forward and back-
ward ld-intervals, the encoding of the above-neighbor rela-
tion is not simple. For example, in Fig. 2b, consider the 3rd
and the 4th ld’s: the 1st tile of the 3rd ld \( (t_{3.3}) \) is connected
with the next-to-last tile of the 4th ld \( (t_{3.4}) \), the 2nd tile of
the 3rd ld \( (t_{2.3}) \) is connected with the third from last tile of
the 4th ld \( (t_{3.4}) \), and so on. Notice that, in the forward (resp.,
backward) ld-intervals, the last (resp., first) tile-interval has
no tile-intervals connected above with it, to constrain each
level of the octant to have exactly one tile more than the
previous one (these tile-intervals are labeled with last).

We define the above-neighbor relation as follows. If
\([b_j^i, b_j^{i+1}] \) is a tile-interval belonging to a forward (resp.,
backward) ld-interval, then we say that it is above-
connected with the tile-interval \([b_j^{i+1}, b_j^{i+1} + i] \) (resp.,
\([b_j^{i+1}, b_j^{i+1} + i] \)). We capture this situation
by labelling with up_rel the interval \([b_j^i, b_j^{i+1} + i] \)
(resp., \([b_j^i + i, b_j^{i+1} + i] \)). Moreover, we distinguish
between up_rel-intervals starting from a forward and a back-
ward ld-interval and, for each one of these cases, be-
tween those starting from an odd and an even tile-interval.
Lemma 3. If \( M, [a, b] \models (9) \land (13) \land (19) \), then there exists a sequence of points like that defined in Lemma 1 such that, for each \( i \geq 0, j > 0 \), the following properties hold:

a) \([b_j', b_j'']\) satisfies \( \uprel \) if and only if it satisfies exactly one between \( \uprel^b \) and \( \uprel^f \) and \([b_j', b_j'']\) satisfies \( \uprel^b \) (resp., \( \uprel^f \)) if and only if it satisfies exactly one between \( \uprel^b \) and \( \uprel^b \) (resp., between \( \uprel^f \) and \( \uprel^b \));

b) for each \( \alpha, \beta \in \{b, f\} \) and \( \gamma, \delta \in \{o, e\} \), if \([b_j', b_j'']\) satisfies \( \uprel^\gamma \), then there is no other interval starting at \( b_j' \) satisfying \( \uprel^\delta \) such that \( \uprel^\gamma \neq \uprel^\delta \);

c) each \( \uprel^\gamma \) interval (resp., \( \uprel^\gamma \) interval) does not overlap any other \( \uprel^\gamma \) interval (resp., \( \uprel^\gamma \) interval);

d) if \([b_j', b_j'']\) satisfies \( \uprel^\gamma \) (resp., \( \uprel^\gamma \), \( \uprel^\gamma \)), \( \uprel^\gamma \), then \([b_j', b_j'']\) satisfies tile and there exists an \( \uprel^\gamma \) interval (resp., \( \uprel^\gamma \) interval, \( \uprel^\gamma \) interval) starting at \( b_j'' \).

Figure 2. The above-neighbor relation encoded in the fragment \( \text{AO} \).

To this end, we use a new propositional letter, namely, \( \uprel^\gamma \) (resp., \( \uprel^\gamma \), \( \uprel^\gamma \), \( \uprel^\gamma \)) to label \( \uprel \) intervals starting from an odd \( \uprel \) interval (resp., even \( \uprel \) interval, odd/forward, even/forward). Moreover, to ease the reading of the formulas, we group \( \uprel^\gamma \) and \( \uprel^\gamma \) in \( \uprel^\gamma \) (\( \uprel^\gamma \leftrightarrow \uprel^\gamma \uprel^\gamma \)), similarly for \( \uprel^\gamma \). Finally, \( \uprel \) is exactly one among \( \uprel^\gamma \) and \( \uprel^\gamma \) (\( \uprel \leftrightarrow \uprel^\gamma \uprel^\gamma \)). From the above conditions, it follows that the \( \uprel \) intervals between any pair of consecutive \( \uprel \) intervals are placed one strictly contained in the other.

Let \( \alpha, \beta \in \{b, f\} \) and \( \gamma, \delta \in \{o, e\} \), with \( \alpha \neq \beta \) and \( \gamma \neq \delta \):

\[
\begin{align*}
[G](\uprel \leftrightarrow \uprel^b \lor \uprel^f) & \quad (14) \\
[G](\uprel^\gamma \leftrightarrow \uprel^\gamma \uprel^\gamma) & \quad (15) \\
[G](\langle A \rangle \uprel^\gamma \leftrightarrow \neg \langle A \rangle \uprel^\gamma \land \neg \langle A \rangle \uprel^\gamma) & \quad (16) \\
[G](\uprel^\gamma \leftrightarrow \neg \langle O \rangle \uprel^\gamma) & \quad (17) \\
[G](\uprel^\gamma \rightarrow \langle A \rangle (\uprel^\gamma \uprel^\gamma)) & \quad (18) \\
\quad \land \ldots \land (18) & \quad (19)
\end{align*}
\]

Lemma 4. If \( M, [a, b] \models (9) \land (13) \land (19) \land (35) \), then there exists a sequence of points like that defined in Lemma 1 such that, for each \( i \geq 0, j > 0 \), the following properties hold:
that the following properties hold:

- for each up_rel-interval $[c, d]$, there exist $c', d'$ such that $[c', c]$ and $[d, d']$ are tile-intervals and if $[c, d]$ satisfies up_rel\(^b\) (resp., up_rel\(^f\)), then $[c', c]$ satisfies b (resp., f) and $[d, d']$ satisfies f (resp., b);

- (strict alternation property) for each tile-interval $[b^\prime_j, b^{i+1}_j]$, with $i < k_j - 1$, such that there exists a up_rel\(^b\)-interval (resp., up_rel\(^f\)-interval, up_rel\(^o\)-interval) starting at $b^{i+1}_j$, there exists a up_rel\(^b\)-interval (resp., up_rel\(^f\)-interval, up_rel\(^o\)-interval) starting at $b^{i+2}_j$;

- for every tile-interval $[b^\prime_j, b^{i+1}_j]$ satisfying last, there is no up_rel-interval ending at $b^{i+1}_j$;

- for each up_rel-interval $[b^\prime_j, b^{i+1}_j]$, with $1 < i \leq k_j$, we have that $j' = j + 1$.

Proof.  a) Let $[c, d]$ be a up_rel-interval. By (18), we have that there exists $d'$ such that $[d, d']$ is a tile-interval and by (22), (23), and Lemma 1, there exists $c'$ such that $[c', c]$ is a tile-interval. Now, suppose that $[c, d]$ satisfies up_rel\(^b\) (the other case is symmetric) and that $[c', c]$ satisfies f. Then, (24) is contradicted. Similarly, if $[d, d']$ satisfies b, then (25) is contradicted.

b) Straightforward, by (27);

c) Straightforward, by (34);

d) Let $[b^\prime_j, b^{i+1}_j]$ be a up_rel-interval, with $1 < i \leq k_j$, and suppose, for contradiction, that $j' \neq j + 1$. Suppose that $[b^\prime_j, b^{i+1}_j]$ is a up_rel\(^b\)-interval (the other case is symmetric). By point a) of this lemma, we have that $[b^{i-1}_j, b^\prime_j]$ satisfies b and that $[b^{i+1}_j, b^{i+1}_j]$ satisfies f. Two cases are possible:

(i) if $j' = j$, then $[b^{i-1}_j, b^\prime_j]$ and $[b^{i+1}_j, b^{i+1}_j]$ belong to the same ld-interval. By Lemma 2, they must be both labelled with b or f, against the hypothesis;

(ii) if $j' > j + 1$, then consider a tile-interval $[b^{j+1}_j, b^{j+1}_j]$ belonging to the $(j + 1)$-th ld-interval. By Lemma 2, we have that $[b^{i-1}_j, b^{i+1}_j]$ satisfies f (since $[b^{i-1}_j, b^{i+1}_j]$ satisfies b) and, by (21) and (24), we have that there is a up_rel\(^f\)-interval starting at $b^{j+1}_j$ and ending at some point $b^{j'}_{j+2}$ for some $j'' > j + 1$ (by point (i)). Consider the *-interval $[b^{j+2}_j, b^{j+2}_j]$. By the right conjunct of (2), there exists a point b' such that $b^{j+2}_j < b' < b^{j+2}_j$. Thus, the interval $[b, b']$ overlaps the *-interval $[b^{j+2}_j, b^{j+2}_j]$, the up_rel\(^b\)-interval $[b^{i+1}_j, b^{i+1}_j]$ and the up_rel\(^b\)-interval $[b^{i+1}_j, b^{i+1}_j]$, contradicting (26).

Hence, the only possibility is $j' = j + 1$.

Lemma 5. Each tile-interval $[b^\prime_j, b^{i+1}_j]$ is above-connected with exactly one tile-interval and if $[b^\prime_j, b^{i+1}_j]$ does not satisfy last, then there exists exactly one tile-interval which is above-connected with it.

Proof. First of all, we observe that each tile-interval is above-connected with at least one tile, by (21) and by Lemma 4, item (a). Now suppose, for contradiction, that there exists a tile-interval $[b^{i-1}_j, b^{i+1}_j]$ not satisfying last and such that there is no tile-interval above-connected with it. If it is the rightmost interval of the j-th ld-interval not satisfying last (base case) and it satisfies f (resp., b), then we have that $i = k_j - 2$ (resp., $i = k_j - 1$) and (32) (resp., (31)) guarantees the existence of a up_rel-interval ending at $b^{i+1}_j$, leading to a contradiction. Now, suppose that there is a up_rel-interval ending at $b^{i+1}_j$ and starting at some point $b^{i-1}_j$ (inductive case). Without loss of generality, suppose that $[b^{i-1}_j, b^{i+1}_j]$ satisfies up_rel\(^o\). Then, by Lemma 3, item (d), there exists a up_rel\(^o\)-interval starting at $b^{i+2}_j$ and, by the strict alternation property (Lemma 4, item (b)), there exists a up_rel\(^b\)-interval starting at $b^{i+1}_j$. Let c be a point such that $b^{i-1}_j < c < b^{i+1}_j$ (the existence of such a point is guaranteed by the right conjunct of (2)). Similarly, let d be a point such that $b^{i-1}_j < d < b^{i+1}_j$. We show that, by applying (33) to any interval ending in c and starting before than $b^{i+1}_j$, say $[c, c']$, we get a contradiction. Indeed, $[c', c]$ satisfies $(O)(u \land (A)up_rel\(^o\))$ and it meets $[c, d]$, which satisfies the following formulas:

- $(O)up_rel\(^b\): [b^{i-1}_j, b^{i+1}_j]$ satisfies up_rel\(^b\);

- $(O)(u \land (A)(u \land \neg last \land (A)up_rel\(^o\))):$ the interval $[b^{i-1}_j, b^{i+1}_j]$ satisfies u and meets the u-interval $[b^{i+1}_j, b^{i+1}_j]$, which does not satisfy last (by hypothesis) and whose right endpoint starts a up_rel\(^o\)-interval.

We show that $[c, d]$ does not satisfy the formula $(O)up_rel\(^o\)$, getting a contradiction with (33). Suppose that there exists an interval $[e, f]$ satisfying up_rel\(^o\) and such that $c < e < d < f$. We distinguish the following cases:

- if $f > b^{i+1}_j$ and $e > b^{i+1}_j$, then the up_rel\(^o\)-interval $[b^{i-1}_j, b^{i+1}_j]$ overlaps the up_rel\(^o\)-interval $[e, f]$, contradicting Lemma 3, item (c);

- if $f > b^{i+1}_j$ and $e = b^{i+1}_j$, then there are a up_rel\(^o\)- and a up_rel\(^f\)-interval starting at $b^{i+1}_j$, contradicting Lemma 3, item (b);

- if $f = b^{i+1}_j$, then there are a up_rel\(^o\)- and a up_rel\(^f\)-interval ending at $b^{i+1}_j$ and, by Lemma 3, item (d), there are a up_rel\(^o\)- and a up_rel\(^o\)-interval starting at $b^{i+1}_j$, contradicting Lemma 3, item (b);

- finally, if $f = b^{i+1}_j$, we have a contradiction with the hypothesis.

Thus, there exists no such an interval, contradicting (33).

This proves that each tile-interval is above-connected with at least one tile-interval and if it does not satisfy last, then there exists at least one tile-interval above-connected with it. Now, we show that such connections are unique.
Suppose, for contradiction, that for some $[b'_j, b'_j + 1]$ and $[b''_j, b''_j + 1]$, with $b'_j + 1 < b''_j + 1$ (the case $b''_j + 1 > b''_j + 1$ is symmetric), we have that both $[b'_j, b'_j + 1]$ and $[b''_j, b''_j + 1]$ are $\uprel$-intervals. By Lemma 3 and Lemma 4, we have that they both satisfy the same propositional letter among $\uprel_{e}$, $\uprel_{e}$, $\uprel_{e}$, and $\uprel_{e}$, say $\uprel_{e}$ (the other cases are symmetric). Then both $b'_j + 1$ and $b''_j + 1$ start a $\uprel_{e}$-interval by Lemma 3, item (d). By the strict alternation property, a $\uprel_{e}$-interval starts at the point $b'_j + 1$. Since $[b'_j + 1, b''_j + 1]$ is not the rightmost tile of the $(j+1)$-th $\uprel_{e}$-interval, then, as we have already shown, there exists a point $c$ such that $[c, b''_j + 1]$ is a $\uprel_{e}$-interval. By Lemma 3, items (d) and (b), we have that $[c, b''_j + 1]$ is a $\uprel_{e}$-interval. We show that the existence of such an interval leads to a contradiction:

- if $c < b'_j$, then the $\uprel_{e}$-interval $[c, b'_j + 1]$ overlaps the $\uprel_{e}$-interval $[b'_j, b'_j + 1]$, contradicting Lemma 3, item (c);
- if $c = b'_j$, then $b'_j$ starts both a $\uprel_{e}$- and a $\uprel_{e}$-interval, contradicting Lemma 3, item (b);
- if $c > b'_j$, then the $\uprel_{e}$-interval $[b'_j, b'_j + 1]$ overlaps the $\uprel_{e}$-interval $[c, b''_j + 1]$, contradicting Lemma 3, item (c).

In a similar way, we can prove that it cannot happen that two distinct $\uprel$-intervals end at the same point. \[\square\]

### 3.5. The right-neighbor relation

Intuitively, the right-neighbor relation connects each tile-duration with its horizontal neighbor in the octant, if any (e.g., $t_{2,3}$ with $t_{3,3}$ in Fig. 2).

Again, we must distinguish between forward and backward $\uprel$-intervals: a tile-duration belonging to a forward (resp., backward) $\uprel$-interval is right connected with the tile-duration immediately on its right (resp., left), if any. For example, in Fig. 2, the $2$nd tile-duration of the $4$th $\uprel$-interval ($t_{2,4}$) is right connected with the tile-duration immediately on its right ($t_{3,4}$); since the $4$th $\uprel$-interval is a forward one, while the $2$nd tile-duration of the $3$rd $\uprel$-interval ($t_{2,3}$) is right connected with the tile-duration immediately on its left ($t_{3,3}$), since the $3$rd $\uprel$-interval is a backward one.

As a consequence, we define the right-neighbor relation as follows. If $[b'_j, b'_j + 1]$ is a tile-duration belonging to a forward (resp., backward) $\uprel$-interval, with $i \neq k_j - 1$ (resp., $i \neq 1$), then we say that it is right connected with the tile-duration $[b''_j, b''_j + 1]$ (resp., $[b''_j, b''_j + 2]$).

**Lemma 6** (Commutativity property). If $M, [a, b] \models (9) \land (13) \land (19) \land (35)$, then there exists a sequence of points like that defined in Lemma 1 such that the following commutativity property holds: given two tile-intervals $[c, d]$ and $[e, f]$, if there exists a tile-duration $[d_1, e_1]$, such that $[c, d]$ is right connected with $[d_1, e_1]$ and $[d_1, e_1]$ is above-connected with $[e, f]$, then there exists also a tile-duration $[d_2, e_2]$ such that $[c, d]$ is above-connected with $[d_2, e_2]$ and $[d_2, e_2]$ is right connected with $[e, f]$.

### 3.6. Tiling the plane

The following formulas constrain each tile-duration (and no other interval) to be tiled by exactly one tile ((36) and (37)) and constrain the tiles that are right or above-connected to respect the color constraints (from (38) to (40)):

\[
|G|((\bigvee_{i=1}^{k} t_{i}) \leftrightarrow \text{tile}) \tag{36}
\]

\[
|G| \bigwedge_{i,j=1,i\neq j} \neg(t_{i} \land t_{j}) \tag{37}
\]

\[
|G|(\text{tile} \rightarrow \bigvee_{up(t_{i})=down(t_{j})} (t_{i} \land (A)(\uprel_{e} \land (A)t_{j}))) \tag{38}
\]

\[
|G|(\text{tile} \rightarrow \bigvee_{right(t_{i})=left(t_{j})} (t_{i} \land b \land (A)t_{j})) \tag{39}
\]

\[
|G|(\text{tile} \rightarrow \bigvee_{left(t_{i})=right(t_{j})} (t_{i} \land b \land (A)t_{j})) \tag{40}
\]

(36) \land \ldots \land (40) \tag{41}

Given the set of tiles $T = \{t_{1}, t_{2}, \ldots, t_{k}\}$, let $\Phi_T$ be the formula:

\[(9) \land (13) \land (19) \land (35) \land (41)\]

**Lemma 7.** Given any finite set of tile types $T = \{t_{1}, t_{2}, \ldots, t_{k}\}$, the formula $\Phi_T$ is satisfiable if and only if $T$ can tile the second octant $O$.

Since the above construction can be carried out on any linear ordering containing an infinite ascending chain of points, such as, for instance, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, the following theorem holds.

**Theorem 1.** The satisfiability problem for the logic $AO$ is undecidable over any class of linear orders that contains at least one linear order with an infinite ascending sequence.

### 3.7. Undecidability of $\overline{AO}$, $\overline{\overline{AO}}$, and $\overline{\overline{\overline{AO}}}$

To prove the undecidability of the logic $\overline{AO}$ we can exploit the same construction we use for $AO$, provided that we change the formulas containing the operators $(O)$ or $[O]$ as follows. In formulas (1), (2), (17), and (26), we replace all occurrences of the operator $(O)$ with $(\overline{O})$. Besides, we
replace formulas (3), (7), and (33) by the following ones:

\[ G\alpha(u \rightarrow \neg(G\alpha\langle A\rangle\!u)) \quad (42) \]
\[ G\alpha(Id \rightarrow \neg(G\alpha\langle A\rangle\!*)\quad (43) \]
\[ G\alpha(\text{tile} \land \neg\text{last} \rightarrow [G\alpha\langle (\langle T\rangle\!u \land \neg(T)\!Id \rightarrow (T)\!up\!rel))\quad (44) \]

**Theorem 2.** The satisfiability problem for the logic $AO$ is undecidable over any class of linear orders that contains at least one linear order with an infinite ascending sequence.

The previous reductions can easily be extended to the logics $\overline{A}O$ and $\overline{A}\overline{O}$, provided that the class of models contains at least one linear order with an infinite descending sequence of points, such as, for instance, $\mathbb{Z}^{\leq 0}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

**Theorem 3.** The satisfiability problem for the logics $\overline{A}O$ and $\overline{A}\overline{O}$ is undecidable over any class of linear orders that contains at least one linear order with an infinite descending sequence.

### 4. Other undecidable logics

Using analogous constructions we have proved the undecidability of the logics $B^*O^*$, $E^*O^*$, and $D^*O^*$. The obtained results are summarized by the following theorem.

**Theorem 4.** The satisfiability problem for the logics $BO$, $BO$, $EO$, $DO$, and $DO$ (resp., $BO$, $BO$, $EO$, $DO$, $DO$, and $DO$) is undecidable over any class of linear orders that contains at least one linear order with an infinite ascending (resp., descending) sequence.

We have also constructed a similar reduction for the logic $SO$ interpreted on discrete linear orders, as formally stated by the following theorem.

**Theorem 5.** The satisfiability problem for the logic $SO$ is undecidable over any class of discrete linear orders that contains at least one linear order with an infinite ascending or descending sequence.

The details of these constructions will be included in a forthcoming extended version.

### 5. Conclusions and future work

In this paper, we have shown that most extensions of the logics $O$ and $\overline{O}$ are undecidable. The undecidability proof for the various logics has essentially the same structure, based on a suitable reduction from the octant tiling problem. The only extensions for which the decision problem remains open are those of the form $L^*O^*$. The most interesting related open problem, however, is that concerning the decidability status of the logic $O$ (and respectively $\overline{O}$). They are the only one-modality fragments of HS for which we do not have yet any positive decidability result on important classes of linear orders.

### References


