Complete Axiomatization and Decidability of Alternating-time Temporal Logic

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Abstract

Alternating-time Temporal Logic (**ATL**), introduced by Alur, Henzinger and Kupferman, is a logical formalism for the specification and verification of open systems involving multiple autonomous players (agents, components). In particular, this logic allows for the explicit expression of coalition abilities in such systems, modelled as infinite transition games between the coalition and its complement.

Formally, **ATL** is a non-normal multi-modal extension of **CTL** (regarded as a one-player fragment of **ATL**) with temporal operators indexed by coalitions of players, and thus expressing selective quantification over those paths which can be effected as outcomes of infinite transition games between the coalition and its complement.

We present a sound and complete axiomatization of the logic **ATL**, based on Pauly's axiomatization of his Coalition Logic, augmented with axioms and rules for fixed point formulae characterizing the temporal operators. The completeness proof is by construction of a bounded branching tree model for each **ATL**-consistent formula. These models can be folded into finite models, thus rendering the finite model property for **ATL**.

We also describe an automata-based decision procedure for **ATL** by translating the satisfiability problem to the nonemptiness problem for alternating automata on infinite trees. When considering formulae over a fixed finite set of players the decidability problem is shown to be EXPTIME-complete.

Key words: alternating-time temporal logic, open systems, concurrent game structures, satisfiability, alternating tree automata, complete axiomatization

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1 Introduction

The Computation Tree Logic (**CTL**) [4] is a temporal logic for non-terminating or reactive systems, interpreted over branching-time models, where the different possible paths (futures, computations) in a transition system are explicitly considered. The temporal operators in **CTL** allow the expression of 'next time', 'always' and 'until' temporal conditions and are combined with existential and universal path quantifiers. Thus, for example, a **CTL** formula $\exists\bigcirc p \wedge \forall\Box q$ holds at a state where proposition p holds at some immediate successor, and proposition q holds at the current and all future states along all possible paths.

Alternating-time Temporal Logic (**ATL**), introduced in [1], is a non-normal, multi-modal extension of **CTL** for modelling open, multi-agent systems (or multi-player games). The temporal operators are parametrised by sets of players, thus expressing selective quantification over those paths which can be effected as outcomes of infinite transition games between the coalition and its complement, and so allowing for the explicit modelling of coalition abilities in multi-player models. Thus, **ATL** is a natural language for expressing existence of strategies and co-strategies of coalitions. **CTL** can be regarded as a one-player particular case of **ATL** modelling, for example, a single operating system.

Independently, Pauly introduced and studied in [9–11] the closely related Coalition Logic and Extended Coalition Logic which are essentially fragments of **ATL**.

Originally (see [1]) **ATL** was interpreted over Alternating Transition Systems (ATS), later generalized in [2] to Concurrent Game Structures (CGS) which are labelled transition systems where every transition may be viewed as a strategic game that determines the successor state. In game-theoretic terms, Concurrent Game Structures correspond to Multi-player Game Models (see [11,10,9]) for infinite extensive games with simultaneous moves. Each of these semantics turns out equivalent to a more abstract one introduced by Pauly, based on multi-modal monotone neighbourhood Kripke models, called Coalition Effectivity Models. The relationship between various semantics for **ATL** is further explored in [6]. In this paper we use the semantics for **ATL** based on Concurrent Game Structures.

Syntactically, **ATL** expands on the language of **CTL** by allowing path quantifiers to explicitly contain coalitions (subsets of players), thus defining a path quantifier $\langle\langle A\rangle\rangle$ for each coalition. Formulae containing such coalition quantifiers are interpreted as "coalition A has a strategy to ensure . . . ". For example, an **ATL** formula $\langle\langle A\rangle\rangle\bigcirc p \wedge \langle\langle B\rangle\rangle\Box q$ holds at a state exactly when the coalition

A has a strategy to ensure that proposition p holds at the immediate successor state, and coalition B has a strategy to ensure that proposition q holds at the current and all future states along all possible paths.

In this paper we address the fundamental logical questions of a complete axiomatization and decidability of satisfiability for **ATL**.

The *axiomatic system* that we present for **ATL** is composed of local axioms and rules that are essentially the axioms provided by [9] to axiomatize the Coalition Logic. We add to that system axioms and rules to describe the fixed point characterizations of the temporal operators $\Box$ and $\mathcal{U}$.

To prove that the axiomatic system is complete we construct a CGS model for each **ATL**-consistent formula. We define an extended closure (similar to the Fisher-Ladner closure in **PDL**) for each **ATL**-consistent formula. For the combination of an **ATL**-consistent subset of the closure and an eventuality in the closure, we define a tree-component that witnesses realization of the eventuality and is labelled by sets of formulae in a locally consistent way. This family of tree components can then be combined into an infinite tree model, or folded into a cyclic finite model. From the construction we also derive a *bounded-branching tree model property* for **ATL**: every formula that is satisfiable in some CGS is satisfiable in a bounded-branching tree model.

The techniques and results used in the completeness proof have been strongly influenced by three sources which gave us valuable insights: [9], providing the basic concepts and results related to coalition logics; [4], where the use of fragments from pseudo-Hintikka structures was introduced to build models for **CTL**; and [5] where a more traditional completeness proof for **CTL** is presented.

The next part of the paper describes an *automata-theoretic* approach to derive an effective decision procedure for the satisfiability problem for **ATL**. A succinct description of the application of automata-theoretic techniques to the satisfiability and model checking problems for a variety of temporal logics, including **CTL**, **CTL*** and the μ-Calculus is provided in e.g. [7]. The gist of the automata-theoretic satisfiability technique is as follows: we translate the satisfiability problem for some formula in the logic to the nonemptiness problem for an associated automaton which is supposed to accept precisely those structures (typically words or trees) that represent models of the formula in the logic. Standard automata-theoretic results that solve the nonemptiness problem for the automaton can then be applied to provide the desired satisfiability result.

To check the satisfiability of **ATL** formulae we provide a translation to *Alternating Büchi Tree Automata* that accept sets of infinite trees as a language. The tree language accepted by the automaton represents those models in the logic, CGSs in this case, where the formula under consideration is satisfied. Checking nonemptiness for Alternating Büchi Tree Automata can be done in exponential time, thus giving an upper bound for the time complexity of the satisfiability problem of **ATL** when considering the language over a fixed finite set of players. This bound is in fact the optimal complexity since the decidability problem for **CTL** is EXPTIME-complete and **CTL** embeds as a fragment into **ATL**.

The paper is structured as follows:

- Section 2 defines **ATL** with the Concurrent Game Structure semantics.
- Section 3 discusses normal forms, fixed points and tree models for **ATL**.
- Section 4 provides a sound and complete axiomatic system for **ATL**.
- Section 5 describes the automata-theoretic decision procedure for **ATL**.
- Section 6 contains brief concluding remarks.

2 Alternating-time Temporal Logic

In this section we introduce Alternating-time Temporal Logic interpreted over Concurrent Game Structures, as defined in [2].

2.1 Concurrent Game Structures

Concurrent Game Structures are generalizations of transitions systems, where each transition is determined by the outcome of a strategic game between the players.

Definition 1 (Concurrent Game Structure) *A* Concurrent Game Structure *(CGS) is a tuple $\mathcal{S} = \langle n, Q, \Pi, \pi, d, \delta\rangle$ with:*

- *A natural number $n \geq 1$ of* players *(also called* agents*). We identify the players with the numbers $1, \ldots, n$ and denote by Σ the set $\{1, \ldots, n\}$ of players.*
- *A set Q of* states*.*
- *A finite set Π of* atomic propositions *(also called* observables*).*
- *For each state $q \in Q$, a set $\pi(q) \subseteq \Pi$ of atomic propositions true at q. The function π is called the* labelling *(or* observation*) function.*
- *For each player $a \in \Sigma$ and each state $q \in Q$, a natural number $d_a(q) \geq 1$ of moves available at state q to player a. We identify the moves of player a at state q with the numbers $1, \ldots, d_a(q)$. For each state $q \in Q$, a* move vector *at q is a tuple $\langle j_1, \ldots, j_k\rangle$ such that $1 \leq j_a \leq d_a(q)$ for each player a. Given a*

state $q \in Q$, we write $D(q)$ for the set $\{1, \ldots, d_1(q)\} \times \cdots \times \{1, \ldots, d_n(q)\} \subseteq \mathbb{N}^n$ of move vectors. The function D is called the move function.
- *For each state $q \in Q$ and each move vector $\langle j_1, \ldots, j_n \rangle \in D(q)$, a state $\delta(q, j_1, \ldots, j_n) \in Q$ that results from the state q if each player $a \in \{1, \ldots, n\}$ chooses move j_a. The function δ is called the* transition function.

Remark 2 *In this definition we allow the set Q of states in a CGS to be infinite, whereas the definition in [2] is restricted to a finite number of states. The main motivation for this change is to allow a tree model unrolling of the transition system to still be a proper CGS. By proving a finite model property for the logic **ATL**, the completeness and satisfiability results transfer to finitary CGSs as well.*

For two states q and q', we say that q' is a *successor* of q if there is a move vector $\langle j_1, \ldots, j_n \rangle \in D(q)$ such that $q' = \delta(q, j_1, \ldots, j_n)$. Thus, q' is a successor of q iff whenever the game is in state q, the players can choose moves so that q' is the next state. A *computation* of $\mathcal{S}$ is an infinite sequence $\lambda = q_0, q_1, q_2, \ldots$ of states such that for all positions $i \geq 0$, the state q_{i+1} is a successor of the state q_i. We refer to a computation starting at state q as a *q-computation.* For a computation λ and a position $i \geq 0$, we use $\lambda[i]$, $\lambda[0, i]$, and $\lambda[i, \infty]$ to denote respectively the i^{th} state of λ, the finite prefix $q_0, q_1, \ldots, q_i$ of λ, and the infinite suffix $q_i, q_{i+1}, \ldots$ of λ.

*2.2 Syntax and informal semantics of **ATL***

ATL is defined with respect to a nonempty set Π of atomic propositions and a finite set $\Sigma = \{1, \ldots, n\}$ of players.

Remark 3 *Our restriction to a fixed finite set of players follows the definitions in [2]. This restriction has no relevance to the proof of completeness, but the proof we present for the complexity of the automata-based decision procedure depends on this assumption, as discussed in a further remark at the end of section 5.*

Syntactically, **ATL** is a multimodal version of the computation tree logic **CTL** (see [4]), associating with each set of players $A \subseteq \Sigma$ the following modal operators:

- $\langle\langle A \rangle\rangle \bigcirc \varphi$ meaning *'The coalition A can force in the next move an outcome satisfying φ'*,
- $\langle\langle A \rangle\rangle \Box \varphi$ meaning *'The coalition A can maintain forever outcomes satisfying φ'*, and
- $\langle\langle A \rangle\rangle \varphi_1 \mathcal{U} \varphi_2$ meaning *'The coalition A can eventually force an outcome satisfying φ_2 while meanwhile maintaining the truth of φ_1'*.

Definition 4 *An* ***ATL*** *formula is one of the following:*

(1) p *for propositions* $p \in \Pi$.
(2) $\neg\varphi$ *where* φ *is an* ***ATL*** *formula.*
(3) $\varphi_1 \vee \varphi_2$ *where* φ_1, φ_2 *are* ***ATL*** *formulae.*
(4) $\langle\langle A\rangle\rangle\bigcirc\varphi$, $\langle\langle A\rangle\rangle\Box\varphi$, $\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$ *where* $A \subseteq \Sigma$ *and* φ, φ_1, φ_2 *are* ***ATL*** *formulae.*

The *temporal operators* are $\bigcirc$ (*nexttime*), $\Box$ (*always*) and $\mathcal{U}$ (*until*), and the operator $\langle\langle\ \rangle\rangle$ is called a *path quantifier*.

We define $\top = p \vee \neg p$ for some fixed $p \in \Pi$, $\bot = \neg\top$ and $\varphi_1 \wedge \varphi_2 = \neg(\neg\varphi_1 \vee \neg\varphi_2)$, as usual.

Note that the operator $\langle\langle A\rangle\rangle\Diamond\varphi$, meaning *'The coalition A can eventually force an outcome satisfying φ'* is definable as $\langle\langle A\rangle\rangle\top\mathcal{U}\varphi$.

*2.3 Semantics of **ATL** based on CGSs*

In this section we provide formal semantics for **ATL** based on CGSs, as in [2]. We will interpret **ATL** formulae over the states of a given CGS $\mathcal{S}$ that has the same atomic propositions Π and set of players $\Sigma = \{1, \ldots, n\}$.

The concept of a *strategy* is introduced in [2] as follows:

Definition 5 *Consider a game structure* $\mathcal{S} = \langle n, Q, \Pi, \pi, d, \delta\rangle$. *A* strategy *for a player* $a \in \Sigma$ *is a mapping* $f_a : Q^+ \to \mathbb{N}$ *that maps every nonempty finite state sequence* λ *to a natural number such that if the last state of* λ *is* q, *then* $1 \leq f_a(\lambda) \leq d_a(q)$.

Thus, the strategy f_a determines for every finite prefix λ of a computation a move $f_a(\lambda)$ for player a. Each strategy f_a induces a set of computations that player a can enforce.

Given a state $q \in Q$ and a set $A \subseteq \Sigma$ of players, an A-strategy $F_A = \{f_a \mid a \in A\}$ is a set of strategies, one for each player in A. We define the outcomes of F_A from q to be the set $\mathrm{out}(q, F_A)$ of all q-computations that the players in A can enforce when they follow the strategies in F_A; that is, a computation $\lambda = q_0, q_1, \ldots$ is in $\mathrm{out}(q, F_A)$ if $q_0 = q$ and for all positions $i \geq 0$, there is a move vector $\langle j_i, \ldots, j_n\rangle \in D(q_i)$ such that:
(1) $j_a = f_a(\lambda[0, i])$ for all players $a \in A$, and
(2) $\delta(q_i, j_i, \ldots, j_n) = q_{i+1}$.

The following notion is inspired by [12]:

Definition 6 (A-move) *For a state $q \in Q$ and a subset $A \subseteq \Sigma$ of players, with $|A| = m$ we define an A-move σ as a tuple $(\sigma_a)_{a \in A}$ such that $1 \leq \sigma_a \leq d_a(q)$. We denote by $D_A(q)$ the set of all A-moves at state q. A state q' is* consistent *with an A-move $\sigma \in D_A(q)$ when there is a move vector $\langle j_i, \ldots, j_k \rangle \in D(q)$ such that (1) $j_a = \sigma_a$ for all $a \in A$, and (2) $\delta(q, j_i, \ldots, j_k) = q'$. We denote by $\mathrm{out}(\sigma)$ the set of states consistent with σ. Intuitively, when the system is in state q, for every A-move $\sigma \in D_A(q)$ of the players in A, no matter what the other players do, the next state of the system will be in $\mathrm{out}(\sigma)$.*

Note that $D_\Sigma(q)$ is the set $D(q)$. The combination of an A-move $\sigma \in D_A(q)$ and a $\Sigma \backslash A$-move $\sigma' \in D_{\Sigma \backslash A}(q)$ defines a unique q-successor $q' = \delta(q, j_1, \ldots, j_k)$ where, $j_i = \sigma_i$ for $i \in A$, and $j_i = \sigma'_i$ for $i \in \Sigma \backslash A$. The state q' is the unique q-successor consistent with both σ and σ'.

The definition of an A-move leads us to introduce the following equivalent notion of a strategy:

Definition 7 (A-strategy) *For a set of players A, we can describe an A-strategy F_A as a mapping $F_A : Q^+ \to \bigcup\{D_A(q) \mid q \in Q\}$ such that for all $\lambda \in Q^*$ and for all $q \in Q$, we have $F_A(\lambda \cdot q) \in D_A(q)$.*

Then a q-computation $\lambda = q_0, q_1, \ldots$, with $q_0 = q$, is *consistent* with F_A, written $\lambda \in \mathrm{out}(q, F_A)$, if for all positions $i \geq 0$ the state q_{i+1} is a successor of q_i satisfying $q_{i+1} \in F_A(\lambda[0, i])$, thus if q_{i+1} is consistent with $F_A(\lambda[0, i])$. The set of players A can ensure that a computation followed by the system is consistent with an A-strategy F_A, no matter what moves are made by the players in $\Sigma \backslash A$.

Definition 8 (Standard ATL Semantic Interpretation) *We write $\mathcal{S}, q \models \varphi$ to indicate that the formula φ holds at state q of a CGS $\mathcal{S}$. When $\mathcal{S}$ is clear from the context, we write $q \models \varphi$. The relation $\models$ is defined, for all states q of $\mathcal{S}$, inductively as follows:*

- *For $p \in \Pi$ we have $q \models p$ iff $p \in \pi(q)$.*
- *$q \models \neg\varphi$ iff $q \not\models \varphi$.*
- *$q \models \varphi_1 \vee \varphi_2$ iff $q \models \varphi_1$ or $q \models \varphi_2$.*
- *$q \models \langle\langle A \rangle\rangle \bigcirc \varphi$ iff there exists an A-strategy, F_A, such that for each computation $\lambda \in \mathrm{out}(q, F_A)$ we have $\lambda[1] \models \varphi$; equivalently, iff there exists an A-move $\sigma \in D_A(q)$ such that for all $q' \in out(\sigma)$ we have $q' \models \varphi$.*
- *$q \models \langle\langle A \rangle\rangle \Box \varphi$ iff there exists an A-strategy F_A such that for each computation $\lambda \in \mathrm{out}(q, F_A)$ and all positions $i \geq 0$, we have $\lambda[i] \models \varphi$.*
- *$q \models \langle\langle A \rangle\rangle \varphi_1 \mathcal{U} \varphi_2$ iff there exists an A-strategy F_A such that for each computation $\lambda \in \mathrm{out}(q, F_A)$ there exists a position $i \geq 0$ such that $\lambda[i] \models \varphi_2$ and for all positions $0 \leq j < i$ we have $\lambda[j] \models \varphi_1$.*

3 Normal Forms, Fixed Points and Tree Models for ATL

In this section we define some additional concepts that will simplify the succeeding proofs. The main aim is to provide a consistent way of handling negated formulae.

We define a normal form syntax for **ATL**. Using fixed point characterizations of temporal operators, we then describe the **ATL** semantics in terms of *co-moves* and *co-strategies*, which may be considered dual to moves and strategies respectively. We also describe how certain labelled trees may be used as models for **ATL**.

3.1 Normal Form Syntax

In many cases we will prefer to work with formulae where the negations only occur directly in front of propositions or path quantifiers. That leads to the following definition.

Definition 9 (Normal Form of φ, $\sim\varphi$) *The normal form of an* ***ATL*** *formula φ is obtained by applying the equivalences $\neg(\varphi_1 \wedge \varphi_2) \equiv \neg\varphi_1 \vee \neg\varphi_2$ and $\neg(\varphi_1 \vee \varphi_2) \equiv \neg\varphi_1 \wedge \neg\varphi_2$ repeatedly until the only negations in the formula occur directly in front of propositions or path quantifiers, thus of the form $\neg p$, $\neg\langle\langle A\rangle\rangle\bigcirc\varphi$, $\neg\langle\langle A\rangle\rangle\Box\varphi$ or $\neg\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$.*

For an ***ATL*** *formula φ, we write $\sim\varphi$ for the normal form of $\neg\varphi$.*

Clearly each **ATL** formula is logically equivalent to its normal form by the Boolean identities. We can also formulate the syntax of **ATL** in normal form directly.

Definition 10 (Normal Form ATL Syntax) *An* ***ATL*** *formula is one of the following:*

(1) p, $\neg p$ for propositions $p \in \Pi$.
(2) $\varphi_1 \vee \varphi_2$ or $\varphi_1 \wedge \varphi_2$ where φ_1, φ_2 are ***ATL*** *formulae.*
(3) $\langle\langle A\rangle\rangle\bigcirc\varphi$, $\neg\langle\langle A\rangle\rangle\bigcirc\varphi$, $\langle\langle A\rangle\rangle\Box\varphi$, $\neg\langle\langle A\rangle\rangle\Box\varphi$, $\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$, $\neg\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$ where $A \subseteq \Sigma$ and φ, φ_1, φ_2 are ***ATL*** *formulae.*

3.2 Semantic Interpretation for Normal Form Syntax

This section contains an alternative, but equivalent, interpretation of **ATL** over CGSs, corresponding to the normal form syntax.

3.2.1 Co-Moves and Co-Strategies

In order to describe the semantic interpretation for negated formulae in a more constructive way, and in terms of the normal form syntax we have introduced, we define concepts dual to that of an A-move and an A-strategy.

The dual concepts we will introduce have the effect of reversing the quantifier alternation inherent in the strategic concepts. A coalition A has a strategy to achieve a goal (at the next state) if there *exists* an A-move such that *for all* resulting outcomes the goal is accomplished. If a coalition A *does not* have such a strategy, it does not imply that the complementary coalition $\Sigma \setminus A$ has a strategy to prevent the goal from being accomplished. It merely implies that *for all* moves, there *exists* an outcome such that the resulting state does not accomplish the goal. A *co-move* represents this dual condition, thus giving a reversal of the quantifiers: if coalition A does not have a strategy to achieve a goal, then there *exists* a co-A-move (an outcome associated with each A-move) such that *for all* of these outcomes the goal is *not* accomplished. We use this dual notion to define a co-strategy for the extended temporal cases. The reversal of quantifiers then allows us to give a uniform presentation of the extended temporal constructions, and allow the proofs to take a constructive approach (proving that something exists), even in the case of negated formulae. We now formalise these ideas:

Definition 11 (Co-A-move) *A* co*-A-move σ^c for a set of players $A \in \Sigma$ at a state q associates with each A-move $\sigma \in D_A(q)$ an element $\sigma^c(\sigma) \in \mathrm{out}(\sigma)$. Thus σ^c is a mapping $\sigma^c : D_A(q) \to Q$ such that $\sigma^c(\sigma) \in \mathrm{out}(\sigma)$. We write $D^c_A(q)$ for the set of co-A-moves at state $q \in Q$.*

The combination of an A-move $\sigma \in D_A(q)$ and a co-A-move $\sigma^c \in D^c_A(q)$ defines a unique q-successor $q' = \sigma^c(\sigma)$ We say that a q-successor q' is *consistent* with a co-A-move $\sigma^c \in D^c_A(q)$ if for some A-move $\sigma \in D_A(q)$, $q' = \sigma^c(\sigma)$, and write $\mathrm{out}(\sigma^c)$ for the set of states consistent with σ^c. The players in A *cannot ensure* that the successor to state q is a state *not consistent* with the co-A-move $\sigma^c \in D^c_A(q)$. Whatever A-move is made by coalition A, the outcome might still be a state consistent with the co-A-move σ^c.

Based on the definition of an co-A-move, we define the corresponding dual to an A-strategy:

Definition 12 (Co-A-strategy) *A* co-A-strategy *F^c_A for a set of players $A \in \Sigma$ assigns to each sequence $\lambda \cdot q \in Q^+$ a co-A-move $F^c_A(\lambda \cdot q) \in D^c_A(q)$. A q-computation $\lambda = q_0, q_1, \ldots$, with $q_0 = q$, is* consistent *with F^c_A if for all positions $i \geq 0$ the state $q_{i+1} \in \mathrm{out}(F^c_A(\lambda[0, i]))$. We write $\lambda \in \mathrm{out}(q, F^c_A)$ if λ is a q-computation consistent with F^c_A.*

Whatever strategy is followed by coalition A, there are moves for coalition

$\Sigma\backslash A$ such that the resulting computation is consistent with the co-A-strategy F_A^c.

The combination of an A-strategy F_A and a co-A-strategy F_A^c defines a unique q-computation λ from any state q, with $\lambda[0] = q$ and for all $i \geq 0$, $\lambda[i+1] = F_A^c(\lambda[0,i])\,(F_A(\lambda[0,i]))$. This is the unique q-computation consistent with both F_A and F_A^c.

We will now make precise the sense in which the concept of a co-A-move is dual to that of an A-move. Then, by describing strategies and co-strategies in terms of fixed points of certain monotone operators, we will show that a similar relationship holds between strategies and co-strategies.

From the definitions for move and co-move, we have:

Proposition 13 (Move/Co-Move Duality) *Let $q \in Q$ and let $P \subseteq Q$ be a set of q-successors. The following are equivalent:*

- *There is some A-move $\sigma \in D_A(q)$ such that* $\mathrm{out}(\sigma) \subseteq P$.
- *There is no co-A-move $\sigma^c \in D_A^c(q)$ such that* $\mathrm{out}(\sigma^c) \subseteq Q \setminus P$.

PROOF. Suppose there is an A-move $\sigma \in D_A(q)$ such that $\mathrm{out}(\sigma) \subseteq P$. Then for any co-$A$-move $\sigma^c \in D_A^c(q)$, the resulting combination $q' = \sigma^c(\sigma)$ is consistent with σ, hence $q' \in P$. But then q' is a q-successor consistent with σ^c not in $Q \setminus P$.

For the converse we argue by contraposition: suppose there is no A-move $\sigma \in D_A(q)$ such that $\mathrm{out}(\sigma) \subseteq P$. Then for each A-move $\sigma \in D_A(q)$, there is some $q' \in \mathrm{out}(\sigma)$ such that $q' \in Q \setminus P$. This q' is the result of the combination of σ with some $\Sigma\backslash A$-move $\sigma'_\sigma \in D_{\Sigma\backslash A}(q)$. Let this define the co-A-move $\sigma^c \in D_A^c(q)$ by $\sigma^c(\sigma) = \sigma'_\sigma$. Then for every A-move σ, we have that $\sigma^c(\sigma) \in Q \setminus P$. Thus every q-successor consistent with σ^c is in $Q \setminus P$. $\square$

3.2.2 Fixed Point Theory

We will make use of the theory of fixed points of monotone operators, for which we briefly recall the necessary preliminaries:

Definition 14 (Monotone Operator, Least/Greatest Fixed Point) *Let W be a set. An operator $\Omega : \mathbf{2}^W \to \mathbf{2}^W$ is* monotone *if $Y_1 \subseteq Y_2$ implies $\Omega(Y_1) \subseteq \Omega(Y_2)$.*

A set $Y \subseteq W$ is:

- *A* pre-fixed point *of Ω, if $\Omega(Y) \subseteq Y$.*

- *A* post-fixed point *of* Ω, *if* $\Omega(Y) \supseteq Y$.
- *A* fixed point *of* Ω, *if* $\Omega(Y) = Y$.
- *A* least (pre-)fixed point *of* Ω, *if* Y *is a (pre-)fixed point and for every (pre-)fixed point* Z, $Y \subseteq Z$.
- *A* greatest (post-)fixed point *of* Ω, *if* Y *is a (post-)fixed point and for every (post-)fixed point* Z, $Y \supseteq Z$.

The key result regarding fixed points that we will use is the following (see e.g. [3]):

Theorem 15 (Knaster-Tarski) *Every monotone operator* $\Omega : \mathbf{2}^W \to \mathbf{2}^W$ *has a unique least pre-fixed point, which is also the least fixed point.*

Every monotone operator $\Omega : \mathbf{2}^W \to \mathbf{2}^W$ *has a unique greatest post-fixed point, which is also the greatest fixed point.*

3.2.3 Fixed Point Characterizations of Temporal Operators

Strategies and co-strategies have fixed point characterizations in terms of moves and co-moves. We start by defining some operators on the set of states.

Definition 16

- $[\langle\langle A\rangle\rangle\bigcirc] : \mathbf{2}^Q \to \mathbf{2}^Q$ *is defined by* $q \in [\langle\langle A\rangle\rangle\bigcirc](X)$ *iff there exists an A-move* $\sigma_A \in D_A(q)$ *such that* $\mathrm{out}(q, \sigma_A) \subseteq X$.
- $[\neg\langle\langle A\rangle\rangle\bigcirc] : \mathbf{2}^Q \to \mathbf{2}^Q$ *is defined by* $q \in [\neg\langle\langle A\rangle\rangle\bigcirc](X)$ *iff there exists a co-A-move* $\sigma^c_A \in D^c_A(q)$ *such that* $\mathrm{out}(q, \sigma^c_A) \subseteq Q \setminus X$.

The operator $[\langle\langle A\rangle\rangle\bigcirc]$ is similiar to the *Pre* operator defined for the symbolic model checking algorithm in [2]. The correspondence is that $[\langle\langle A\rangle\rangle\bigcirc](X)$ is exactly $Pre(A, X)$.

We denote by $\|\varphi\|$ the set $\{q \mid q \models \varphi\}$.

From the Move/Co-Move Duality proposition and the above operator definitions we have:

- $\|\langle\langle A\rangle\rangle\bigcirc\varphi\| = [\langle\langle A\rangle\rangle\bigcirc](\|\varphi\|)$
- $\|\neg\langle\langle A\rangle\rangle\bigcirc\varphi\| = [\neg\langle\langle A\rangle\rangle\bigcirc](\|\varphi\|) = Q \setminus \|\langle\langle A\rangle\rangle\bigcirc\varphi\|$

To simplify notation we write $[\alpha(\bullet)] : \mathbf{2}^Q \to \mathbf{2}^Q$ for the operator where $[\alpha(\bullet)](Y) = \alpha(Y)$.

If $[\alpha(\bullet)]$ is a monotone operator (i.e. $X \subseteq Y$ implies $\alpha(X) \subseteq \alpha(Y)$) we denote the least fixed point by $\mu X.\alpha(X)$ and the greatest fixed point by $\nu X.\alpha(X)$.

Lemma 17 *For all $q \in Q$ the following fixed point characterizations hold:*

(i) $q \in \|\langle\langle A\rangle\rangle \Box \varphi\|$ *iff* $q \in \nu X.\|\varphi\| \cap [\langle\langle A\rangle\rangle \bigcirc](X)$ *iff there exists an A-strategy F_A such that for each q-computation $\lambda \in \mathrm{out}(q, F_A)$ and all $i \geq 0$, $\lambda[i] \in \|\varphi\|$.*

(ii) $q \in \|\langle\langle A\rangle\rangle \varphi_1 \mathcal{U} \varphi_2\|$ *iff* $q \in \mu X.\|\varphi_2\| \cup (\|\varphi_1\| \cap [\langle\langle A\rangle\rangle \bigcirc](X))$ *iff there exists an A-strategy F_A such that for each q-computation $\lambda \in \mathrm{out}(q, F_A)$, there exists a position $i \geq 0$ such that $\lambda[i] \in \|\varphi_2\|$ and for all positions $0 \leq j < i$ we have $\lambda[j] \in \|\varphi_1\|$.*

PROOF. *(i)*: First we show that $[\|\varphi\| \cap [\langle\langle A\rangle\rangle \bigcirc](\bullet)]$ is monotone. Let $X_1 \subseteq X_2 \subseteq Q$. Then $s \in \|\varphi\| \cap [\langle\langle A\rangle\rangle \bigcirc](X_1)$ means that $x \in \|\varphi\|$ and there is some A-move $\sigma_A \in D_A(s)$ such that $\mathrm{out}(s, \sigma_A) \subseteq X_1$. But then $\mathrm{out}(s, \sigma_A) \subseteq X_2$, so that $s \in [\langle\langle A\rangle\rangle \bigcirc](X_2)$, so $s \in \|\varphi\| \cap [\langle\langle A\rangle\rangle \bigcirc](X_2)$.

So $[\|\varphi\| \cap [\langle\langle A\rangle\rangle \bigcirc](\bullet)]$ is monotone and has a greatest fixed point $\nu X.\|\varphi\| \cap [\langle\langle A\rangle\rangle \bigcirc](X)$.

Now we show that $\|\langle\langle A\rangle\rangle \Box \varphi\|$ is a post-fixed point of $[\|\varphi\| \cap [\langle\langle A\rangle\rangle \bigcirc](\bullet)]$. Let $q \in \|\langle\langle A\rangle\rangle \Box \varphi\|$. By the semantic interpretation of **ATL**, there is an A-strategy F_A such that for all q-computations $\lambda \in \mathrm{out}(q, F_A)$ and all $i \geq 0$, $\lambda[i] \in \|\varphi\|$. In particular, $\lambda[0] \in \|\varphi\|$. For any $q' \in \mathrm{out}(F_A(q))$, we define an A-strategy F'_A that represents the 'remainder' of the A-strategy F_A after q' is reached from q. Thus let F'_A be defined for every q'-computation κ by $F'_A(\kappa) = F_A(q \cdot \kappa)$. Then for all q'-computations $\lambda' \in \mathrm{out}(q', F'_A)$ we have $q \cdot \lambda' \in \mathrm{out}(q, F_A)$, hence for all positions $i \geq 0$, we have $\lambda'[i] = (q \cdot \lambda')[i+1] \in \|\varphi\|$. Thus $q' \in \|\langle\langle A\rangle\rangle \Box \varphi\|$, and thus $q \in [\langle\langle A\rangle\rangle \bigcirc](\|\langle\langle A\rangle\rangle \Box \varphi\|)$. So $q \in \|\varphi\| \cap [\langle\langle A\rangle\rangle \bigcirc](\|\langle\langle A\rangle\rangle \Box \varphi\|)$.

Next we show that for every post-fixed point Z of $[\|\varphi\| \cap [\langle\langle A\rangle\rangle \bigcirc](\bullet)]$, $Z \subseteq \|\langle\langle A\rangle\rangle \Box \varphi\|$. Let $z \in Z \subseteq \|\varphi\| \cap [\langle\langle A\rangle\rangle \bigcirc](Z)$ (since Z is a post-fixed point). We construct an A-strategy F_A by induction on the length of the finite prefix $\lambda \cdot q'$ of a z-computation $\lambda' = \lambda \cdot q' \cdot \ldots$ where $\lambda \cdot q'$ is a finite prefix consistent with F_A. Denote the length of $\lambda \cdot q'$ by $|\lambda \cdot q'|$. We maintain the invariant that $\mathrm{out}(F_A(\lambda \cdot q')) \subseteq \|\varphi\| \cap [\langle\langle A\rangle\rangle \bigcirc](Z)$.

The initial case: for $|\lambda \cdot q'| = 1$ we have that $\lambda \cdot q' = z$ since we are considering z-computations. Now $z \in Z \subseteq \|\varphi\| \cap [\langle\langle A\rangle\rangle \bigcirc](Z)$, so $z \in \|\varphi\|$ and there is some A-move $\sigma_A \in D_A(z)$ such that $\mathrm{out}(z, \sigma_A) \subseteq Z$. Let $F_A(z)$ be this choice, so $\mathrm{out}(z, F_A(z)) \subseteq Z \subseteq \|\varphi\| \cap [\langle\langle A\rangle\rangle \bigcirc](Z)$. Thus the invariant holds for the case $|\lambda \cdot q'| = 1$.

The induction step: for the case where $|\lambda \cdot q'| = k$, assume that F_A has been defined for prefixes of length $k-1$, thus that $F_A(\lambda)$ has been defined. By the invariant, $\mathrm{out}(F_A(\lambda)) \subseteq \|\varphi\| \cap [\langle\langle A\rangle\rangle \bigcirc](Z)$ and so $q' \in \mathrm{out}(F_A(\lambda))$ since $\lambda \cdot q'$ is a finite prefix consistent with F_A. So $q' \in \|\varphi\| \cap [\langle\langle A\rangle\rangle \bigcirc](Z)$, giving

$q' \in \|\varphi\|$ and also some A-move $\sigma_A \in D_A(q')$ such that $\mathrm{out}(q', \sigma_A) \in Z$, so that $\mathrm{out}(q', \sigma_A) \subseteq \|\varphi\| \cap [\langle\langle A\rangle\rangle\bigcirc](Z)$. Let $F_A(\lambda \cdot q') = \sigma_A$, and the invariant is maintained.

This defines F_A for all finite prefixes of z-computations consistent with F_A. For all other $\kappa \in Q^+$, $F_A(\kappa)$ can be chosen arbitrarily. For all z-computations λ consistent with F_A, thus for $\lambda \in \mathrm{out}(z, F_A)$, and all $i \geq 0$ we have that $\lambda[i] \in \|\varphi\|$. Thus $z \in \|\langle\langle A\rangle\rangle\Box\varphi\|$, and hence every post-fixed point of $[\|\varphi\| \cap [\langle\langle A\rangle\rangle\bigcirc](\bullet)]$ is included in $\|\langle\langle A\rangle\rangle\Box\varphi\|$.

Thus we have shown that $\|\langle\langle A\rangle\rangle\Box\varphi\| = \nu X.\|\varphi\| \cap [\langle\langle A\rangle\rangle\bigcirc](X)$.

(ii): An argument similar to the above shows that $\|\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2\|$ is the least pre-fixed point of $[\|\varphi_2\| \cup (\|\varphi_1\| \cap [\langle\langle A\rangle\rangle\bigcirc](\bullet)]$. $\Box$

Next, using co-moves, we relate the fixed point characterizations to co-strategies.

Lemma 18 *For all $q \in Q$ we have:*

(i) $q \in \|\neg\langle\langle A\rangle\rangle\Box\varphi\|$ *iff* $q \in \mu X.\|{\sim}\varphi\| \cup [\neg\langle\langle A\rangle\rangle\bigcirc](Q \setminus X)$ *iff there exists a co-A-strategy F_A^c such that for each q-computation $\lambda \in \mathrm{out}(q, F_A^c)$ there is some position $i \geq 0$, such that $\lambda[i] \in \|{\sim}\varphi\|$.*

(ii) $q \in \|\neg\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2\|$ *iff* $q \in \nu X.\|{\sim}\varphi_2\| \cap (\|{\sim}\varphi_1\| \cup [\neg\langle\langle A\rangle\rangle\bigcirc](Q \setminus X)$ *iff there exists a co-A-strategy F_A^c such that for each q-computation $\lambda \in \mathrm{out}(q, F_A^c)$, if there is a position $i \geq 0$ such that $\lambda[i] \in \|\varphi_2\|$ then there is some position $0 \leq j < i$ where we have $\lambda[j] \in \|{\sim}\varphi_1\|$.*

PROOF. *(i)*: Firstly note that $\|\neg\langle\langle A\rangle\rangle\Box\varphi\| = Q \setminus \|\langle\langle A\rangle\rangle\Box\varphi\| = Q \setminus \nu X.\|\varphi\| \cap [\langle\langle A\rangle\rangle\bigcirc](X) = \mu X.\|{\sim}\varphi\| \cup [\neg\langle\langle A\rangle\rangle\bigcirc](Q \setminus X)$.

Next we show that this least fixed point characterization corresponds to the existence of co-A-strategies.

Let $Z \subseteq Q$ be the set of states z where there exists a co-A-strategy F_A^c such that for each z-computation $\lambda \in \mathrm{out}(z, F_A^c)$ there is some position $i \geq 0$, such that $\lambda[i] \in \|{\sim}\varphi\|$. We will show that Z is a pre-fixed point of the operator $[\|{\sim}\varphi\| \cup [\neg\langle\langle A\rangle\rangle\bigcirc](Q \setminus \bullet)]$.

Let $z \in \|{\sim}\varphi\| \cup [\neg\langle\langle A\rangle\rangle\bigcirc](Q \setminus Z)$. If $z \in \|{\sim}\varphi\|$ then any co-A-strategy would have for every z-computation λ that $\lambda[0] = z \in \|{\sim}\varphi\|$. Otherwise $z \in [\neg\langle\langle A\rangle\rangle\bigcirc](Q \setminus Z)$. But then there is a co-A-move $\sigma_A^c \in D_A^c(z)$ such that $\mathrm{out}(z, \sigma_A^c) \subseteq Q\setminus(Q\setminus Z)$, so $\mathrm{out}(z, \sigma_A^c) \subseteq Z$. Thus for each $z' \in \mathrm{out}(z, \sigma_A^c)$, there a co-$A$-strategy $F_{A,z'}^c$ such that for each z'-computation $\lambda \in \mathrm{out}(z', F_{A,z'}^c)$ there is some position $i \geq 0$, such that $\lambda[i] \in \|{\sim}\varphi\|$. Now define a co-$A$-strategy $F_A^{c'}$

by $F_A^{c'}(z) = \sigma_A^c$ and for each $z' \in \sigma_A^c$, $F_A^{c'}(z \cdot z' \cdot \kappa) = F_{A,z'}^c(z' \cdot \kappa)$. Then for each computation $\lambda \in \mathrm{out}(z, F_A^{c'})$, $\lambda[0] = z \in \|{\sim}\varphi\|$, or for some $i \geq 1$, $\lambda[i] \in \|{\sim}\varphi\|$. Thus the existence of the co-A-strategy $F_A^{c'}$ shows that $z \in Z$.

Now we show that Z is included in every pre-fixed point of the operator $[\|{\sim}\varphi\| \cup [\neg\langle\langle A\rangle\rangle\bigcirc](Q \setminus \bullet)]$. Let Y be a pre-fixed point of $[\|{\sim}\varphi\| \cup [\neg\langle\langle A\rangle\rangle\bigcirc](Q \setminus \bullet)]$, thus $Y \supseteq \|{\sim}\varphi\| \cup [\neg\langle\langle A\rangle\rangle\bigcirc](Q \setminus Y)$. We argue by contraposition: suppose $y \notin Y$; we need to show that $y \notin Z$.

We have that $y \notin Y$ so $y \notin \|{\sim}\varphi\| \cup [\neg\langle\langle A\rangle\rangle\bigcirc](Q \setminus Y)$, thus $y \notin \|{\sim}\varphi\|$ and there is no co-A-move $\sigma_A^c \in D_A^c(y)$ such that $\mathrm{out}(y, \sigma_A^c) \subseteq Y$. Consider now any co-$A$-strategy F_A^c. For all y-computations $\lambda \in \mathrm{out}(y, F_A^c)$, $\lambda[0] = y \notin \|{\sim}\varphi\|$. Also, for the particular co-A-move $\sigma_A^c = F_A^c(y)$, we have for some $y_1 \in \mathrm{out}(y, \sigma_A^c)$, $y_1 \notin Y$. The same argument may then be applied to y_1, in order to show that $y_1 \notin \|{\sim}\varphi\|$ and to find a y_1-successor $y_2 \notin Y$. In this way we construct $\lambda' = y, y_1, y_2, \ldots$ which is a y-computation in $\mathrm{out}(y, F_A^c)$ such that for all $i \geq 0$ $\lambda[i] \notin \|{\sim}\varphi\|$.

Thus there is no co-A-strategy F_A^c such that for each y-computation $\lambda \in \mathrm{out}(y, F_A^c)$ there is some position $i \geq 0$, such that $\lambda[i] \in \|{\sim}\varphi\|$. Thus $y \notin Z$.

(ii): The argument, to show that the greatest fixed point corresponds to the existence of a co-A-strategy, is similar to the previous case. □

*3.2.4 Normal Form **ATL** Semantics*

The concepts of co-moves and co-strategies that we have introduced, together with the above lemmas, allow us to define an equivalent **ATL** semantic interpretation in terms of the normal form syntax, such that negated temporal formulae are described constructively by co-moves and co-strategies.

Definition 19 (Normal Form ATL Semantics) *The relation $\models$ is defined, for all states q of $\mathcal{S}$, inductively as follows:*

- *For $p \in \Pi$ we have $q \models p$ iff $p \in \pi(q)$.*
- *For $p \in \Pi$ we have $q \models \neg p$ iff $p \notin \pi(q)$.*
- *$q \models \varphi_1 \vee \varphi_2$ iff $q \models \varphi_1$ or $q \models \varphi_2$.*
- *$q \models \varphi_2 \wedge \varphi_2$ iff $q \models \varphi_1$ and $q \models \varphi_2$.*
- *$q \models \langle\langle A\rangle\rangle\bigcirc\varphi$ iff there exists an A-strategy F_A, such that for each computation $\lambda \in \mathrm{out}(q, F_A)$ we have $\lambda[1] \models \varphi$. Equivalently, iff there exists an A-move $\sigma \in D_A(q)$ such that for all $q' \in \mathrm{out}(\sigma)$ we have $q' \models \varphi$.*
- *$q \models \neg\langle\langle A\rangle\rangle\bigcirc\varphi$ iff there exists a co-A-strategy, F_A^c, such that for each computation $\lambda \in \mathrm{out}(q, F_A^c)$ we have $\lambda[1] \models {\sim}\varphi$. Equivalently, iff for each A-move $\sigma \in D_A(q)$ there is some $q' \in \mathrm{out}(\sigma)$ for which $q' \models {\sim}\varphi$. Equivalently, iff there exists a co-A-move $\sigma^c \in D_A^c(q)$ such that for all $q' \in \mathrm{out}(\sigma^c)$ we have*

$q' \models \sim\varphi$.

- $q \models \langle\langle A\rangle\rangle\Box\varphi$ *iff there exists an A-strategy* F_A *such that for each computation* $\lambda \in \mathrm{out}(q, F_A)$ *and all positions* $i \geq 0$, *we have* $\lambda[i] \models \varphi$.
- $q \models \neg\langle\langle A\rangle\rangle\Box\varphi$ *iff there exists a co-A-strategy* F^c_A *such that for each computation* $\lambda \in \mathrm{out}(q, F^c_A)$ *there is some position* $i \geq 0$, *such that* $\lambda[i] \models \sim\varphi$.
- $q \models \langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$ *iff there exists an A-strategy* F_A *such that for each computation* $\lambda \in \mathrm{out}(q, F_A)$ *there exists a position* $i \geq 0$ *such that* $\lambda[i] \models \varphi_2$ *and for all positions* $0 \leq j < i$ *we have* $\lambda[j] \models \varphi_1$.
- $q \models \neg\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$ *iff there exists a co-A-strategy* F^c_A *such that for each computation* $\lambda \in \mathrm{out}(q, F^c_A)$, *if there is a position* $i \geq 0$ *such that* $\lambda[i] \models \varphi_2$ *then there is some position* $0 \leq j < i$ *where we have* $\lambda[j] \models \sim\varphi_1$.

Remark 20 *The strategies and co-strategies used in the **ATL** semantics are perfect recall strategies, using complete information about the history of the computation to determine each move or co-move. It is noted in [2] that for **ATL** memory-free strategies suffice. The strategies and co-strategies that we define in this paper will not make use of the computation histories, and will thus be equivalent to memory-free strategies and co-strategies. However, for the sake of consistency we retain and use the formal definitions as perfect memory strategies.*

3.3 Invariants and Eventualities, Extended Closure

Inspired by the fixed point characterizations of the temporal operators, we define the concepts of eventuality and fulfilment. This allows us to define the extended closure of a formula.

Definition 21 (Eventuality, Fulfilment) *An* eventuality *is a formula of the form* $\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$ *or the form* $\neg\langle\langle A\rangle\rangle\Box\varphi$.

An eventuality $\psi = \langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$ *is* fulfilled at $q \in Q$ *if* $q \models \varphi_2$. *An eventuality* $\psi = \neg\langle\langle A\rangle\rangle\Box\varphi$ *is* fulfilled at $q \in Q$ *if* $q \models \sim\varphi$.

Definition 22 (Invariant) *Formulae of the form* $\langle\langle A\rangle\rangle\Box\varphi$ *and* $\neg\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$ *are* invariant *formulae that may be expressed by recursive local conditions. Specifically, we have that:*

$q \models \langle\langle A\rangle\rangle\Box\varphi$ *iff* $q \models \varphi \wedge \langle\langle A\rangle\rangle\bigcirc\langle\langle A\rangle\rangle\Box\varphi$, *and*
$q \models \neg\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$ *iff* $q \models \sim\varphi_2 \wedge (\sim\varphi_1 \vee \neg\langle\langle A\rangle\rangle\bigcirc\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2)$.

Accordingly, the eventualities $\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$ *and* $\neg\langle\langle A\rangle\rangle\Box\varphi$ *have local invariant conditions that are met at a state* q *where they hold:*

$q \models \neg\langle\langle A\rangle\rangle\Box\varphi$ *iff* $q \models \sim\varphi \vee \neg\langle\langle A\rangle\rangle\bigcirc\langle\langle A\rangle\rangle\Box\varphi$, *and*

$q \models \langle\langle A \rangle\rangle \varphi_1 \mathcal{U} \varphi_2$ *iff* $q \models \varphi_2 \vee (\varphi_1 \wedge \langle\langle A \rangle\rangle \bigcirc \langle\langle A \rangle\rangle \varphi_1 \mathcal{U} \varphi_2)$.

For the eventualities, these local conditions are not sufficient to characterize truth of the formulae, as the eventualities have to be fulfilled after a finite number of states.

When considering an **ATL** formula ψ, we will be interested in the subformulae of ψ, as well as the invariant unpackings as described above, and the negations of formulae under consideration.

We combine all of these to define a closure of a formula ψ, similar to the Fisher-Ladner closure for **PDL**:

Definition 23 (Closure of ψ) *The* closure $\mathrm{cl}(\psi)$ *of an* ***ATL****-formula* ψ *is the smallest set of formulae that satisfies the following closure conditions:*

- *Each subformula of* ψ*, including* ψ *itself, is included in* $\mathrm{cl}(\psi)$*.*
- *If* $\langle\langle A \rangle\rangle \Box \varphi \in \mathrm{cl}(\psi)$ *then* $\langle\langle A \rangle\rangle \bigcirc \langle\langle A \rangle\rangle \Box \varphi \in \mathrm{cl}(\psi)$*.*
- *If* $\neg \langle\langle A \rangle\rangle \Box \varphi \in \mathrm{cl}(\psi)$ *then* $\neg \langle\langle A \rangle\rangle \bigcirc \langle\langle A \rangle\rangle \Box \varphi \in \mathrm{cl}(\psi)$*.*
- *If* $\langle\langle A \rangle\rangle \varphi_1 \mathcal{U} \varphi_2 \in \mathrm{cl}(\psi)$ *then* $\langle\langle A \rangle\rangle \bigcirc \langle\langle A \rangle\rangle \varphi_1 \mathcal{U} \varphi_2 \in \mathrm{cl}(\psi)$*.*
- *If* $\neg \langle\langle A \rangle\rangle \varphi_1 \mathcal{U} \varphi_2 \in \mathrm{cl}(\psi)$ *then* $\neg \langle\langle A \rangle\rangle \bigcirc \langle\langle A \rangle\rangle \varphi_1 \mathcal{U} \varphi_2 \in \mathrm{cl}(\psi)$*.*
- *If* $\varphi \in \mathrm{cl}(\psi)$ *then* $\sim\varphi \in \mathrm{cl}(\psi)$*.*

If $|\psi|$ denotes the size of the formula ψ, then $|\mathrm{cl}(\psi)| = O(|\psi|)$.

Although the closure as presented above will be adequate for the automata-based procedures, we will use the extended closure defined below in the completeness proof for the axiomatization. In particular some of the axioms involve boolean operations on the coalitions that appear in formulae, and boolean combinations of formulae will be used in some constructions.

Definition 24 (Extended Closure of ψ) *The* extended closure *of an* ***ATL*** *formula* ψ*, written* $\mathrm{ecl}(\psi)$*, is constructed from the closure of* ψ *as follows:*

(1) Add all coalitions for 'nexttime' formulae:
$\mathrm{cl}^+(\psi) = \mathrm{cl}(\psi) \cup \{\langle\langle A' \rangle\rangle \bigcirc \varphi, \neg \langle\langle A' \rangle\rangle \bigcirc \varphi \mid \langle\langle A \rangle\rangle \bigcirc \varphi \in \mathrm{cl}(\psi)$ *and* $A' \subseteq \Sigma\}$*.*

(2) Close under positive Boolean combinations:
We let $\mathrm{ecl}(\psi)$ *be the (finite) set of positive Boolean combinations (i.e. constructed using* $\wedge$ *and* $\vee$*) of formulae from the set* $\mathrm{cl}^+(\psi)$*, taken up to tautological equivalence.*

The construction ensures that $\mathrm{ecl}(\psi)$ is finite.

In the construction for the completeness proof of the axiomatization, as well as the automata-theoretic decision procedure, we will use labelled tree models for **ATL**. This section begins with some standard definitions of words and trees. Next we describe how a labelled tree with an appropriate fixed branching degree may be viewed as a CGS, emphasizing how the move structure is encoded into the branching structure of the tree models.

3.4.1 Words and Trees

Definition 25 (Words, Trees) *We fix a nonempty alphabet Θ. Then Θ^* is the set of finite sequences $a_0, \ldots, a_n$ of symbols from Θ. A* finite word *is an element of Θ^*. By Θ^ω we denote the set of infinite sequences $a_0, a_1, \ldots$ of symbols from Θ. An* infinite word *is an element of Θ^ω.*

A tree *is a set $T \subseteq \mathbb{N}^*$ such that if $x \cdot c \in T$ where $x \in \mathbb{N}^*$ and $c \in \mathbb{N}$, then also $x \in T$, and for all $0 \leq c' < c$, $x \cdot c' \in T$. The elements of T are called* nodes*, and the empty word ϵ is the* root *of T. For every $x \in T$, the nodes $x \cdot c$ where $c \in \mathbb{N}$ are the* successors *of x. The number of successors of x is called the* degree *of x and is denoted by $d(x)$. The successor function* $\mathrm{succ} : T \to \mathbf{2}^T$ *maps each node $x \in T$ to the set* $\mathrm{succ}(t) \subseteq T$ *of successors of x. A node is a* leaf *if it has no successors, otherwise it is an* interior *node.*

A path *π in a tree T is a set $\pi \subseteq T$ such that $\epsilon \in \pi$ and for every $x \in \pi$, either x is a leaf or there exists a unique $c \in \mathbb{N}$ such that $x \cdot c \in \pi$.*

Given the alphabet Θ, a Θ-labelled tree *is a pair $\langle T, V \rangle$ where T is a tree and $V : T \to \Theta$ maps each node of T to a letter in Θ. Of special interest will be Θ-labelled trees in which $\Theta = \mathbf{2}^\Pi$ for some set Π of atomic propositions.*

A k-tree *is a tree in which all the interior nodes have degree k. A* simple tree *is a tree that consists only of a root node and a set of successors of the root node.*

3.4.2 Labelled Tree Models

In this section we describe how some labelled trees may be interpreted as models for **ATL**. In particular we will be interested in trees with a fixed degree k^n, labelled either by sets of propositions, or by arbitrary sets of formulae.

Fix a set of propositions Π and set of players $\Sigma = \{1, \ldots, n\}$. We view a labelled tree $\langle T, V \rangle$ with labels from $\mathbf{2}^\Pi$, of fixed branching degree k^n, for some

$k \in \mathbb{N}, k \geq 1$, as a CGS with uniform move degree k: $\mathcal{S}_{\langle T,V \rangle} = \langle n, T, \Pi, V, d, \delta \rangle$, with $d_a(q) = k$ for all $a \in \Sigma$ and $q \in Q$. It remains to define the transition function $\delta : Q \times \mathbb{N}^n \to Q$ for $\mathcal{S}_{\langle T,V \rangle}$.

Intuitively, each node $t \in T$ has k^n successors, which we can name by n-tuples of elements of $\{0, \ldots, k-1\}$. We then let $\delta(t, j_1, \ldots, j_n)$ be the successor of t named by the n-tuple $\langle j_1 - 1, \ldots, j_n - 1 \rangle$.

Formally we define the encoding of the n-tuples into the k^n successors of each node, the resulting transition function δ and some related notation that will be used later.

Let $K = \{0, \ldots, k-1\}$ and K^n be the set of n-tuples of elements of K.

We define a bijective encoding $\tau : K^n \to \{0, \ldots, k^n - 1\}$ of these n-tuples into the set $\{0, \ldots, k^n - 1\}$, allowing us to move freely between the two representations of the successors, as follows: $\tau(\langle a_1, a_2, \ldots, a_n \rangle) = a_1 k^{n-1} + a_2 k^{n-2} + \ldots + a_n k^0$. For a state $t \in T$ and move vector $\langle j_1, \ldots, j_n \rangle$ we define $\delta(t, j_1, \ldots, j_n) = t \cdot c$ where $\tau^{-1}(c) = \langle j_1 - 1, \ldots, j_n - 1 \rangle$.

So $\mathcal{S}_{\langle T,V \rangle}$ is well defined as a CGS, when $\langle T, V \rangle$ is a labelled tree with a fixed branching degree k^n.

Now for any **ATL** formula φ and node $t \in T$, we write $\langle T, V \rangle, t \models \varphi$ for $\mathcal{S}_{\langle T,V \rangle}, t \models \varphi$, and $\langle T, V \rangle \models \varphi$ for $\langle T, V \rangle, \epsilon \models \varphi$.

In a labelled tree with fixed branching degree k^n, for a coalition $A \subseteq \Sigma$, the set of A-moves $D_A(t)$ at a state $t \in T$ is always the same, since $d_a(t) = k$ for all a and t. We write Δ_A for this set of A-moves, and $\sigma_A \in \Delta_A$ as a specific A-move.

A tree $\langle T, V \rangle$ with fixed branching degree k^n labelled with sets of **ATL** formulae may be considered a CGS by restricting the propositional valuation at a node t to the set $V(t) \cap \Pi$.

Remark 26 *The CGS so constructed is a 'Moore synchronous' CGS in terms of [1] — the state space is composed of the product of local components, one for each player. The evolution of the system proceeds in 'lock-step' with every player determining its next local state.*

4 Sound and Complete Axiomatization of ATL

In this section we present an axiomatic system for **ATL** and prove its soundness and completeness.

The axiomatic system for **ATL** extends the Coalition Logic from [11] (the $\bot, \top, \Sigma$ and S axioms and the $\langle\langle A\rangle\rangle\bigcirc$-Monotonicity rule) with the fixed-point axioms for $\Box$ and $\mathcal{U}$, and the $\Box$-*Necessitation* rule.

Definition 27 *The* axiomatic system *for **ATL** consists of the following axioms and rules of inference, where $A, A_1, A_2 \subseteq \Sigma$:*

Axioms:

(TAUT) Enough propositional tautologies.
($\bot$) $\neg\langle\langle A\rangle\rangle\bigcirc\bot$
($\top$) $\langle\langle A\rangle\rangle\bigcirc\top$
(Σ) $\neg\langle\langle\varnothing\rangle\rangle\bigcirc\neg\varphi \to \langle\langle\Sigma\rangle\rangle\bigcirc\varphi$
(S) $\langle\langle A_1\rangle\rangle\bigcirc\varphi_1 \wedge \langle\langle A_2\rangle\rangle\bigcirc\varphi_2 \to \langle\langle A_1 \cup A_2\rangle\rangle\bigcirc(\varphi_1 \wedge \varphi_2)$ for disjoint A_1 and A_2
(FP$_\Box$) $\langle\langle A\rangle\rangle\Box\varphi \leftrightarrow \varphi \wedge \langle\langle A\rangle\rangle\bigcirc\langle\langle A\rangle\rangle\Box\varphi$
(GFP$_\Box$) $\langle\langle\varnothing\rangle\rangle\Box(\theta \to (\varphi \wedge \langle\langle A\rangle\rangle\bigcirc\theta)) \to \langle\langle\varnothing\rangle\rangle\Box(\theta \to \langle\langle A\rangle\rangle\Box\varphi)$
(FP$_\mathcal{U}$) $\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2 \leftrightarrow \varphi_2 \vee (\varphi_1 \wedge \langle\langle A\rangle\rangle\bigcirc\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2)$
(LFP$_\mathcal{U}$) $\langle\langle\varnothing\rangle\rangle\Box((\varphi_2 \vee (\varphi_1 \wedge \langle\langle A\rangle\rangle\bigcirc\theta)) \to \theta) \to \langle\langle\varnothing\rangle\rangle\Box(\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2 \to \theta)$

Rules of inference:

(Modus Ponens) $\dfrac{\varphi_1, \varphi_1 \to \varphi_2}{\varphi_2}$

($\langle\langle A\rangle\rangle\bigcirc$-Monotonicity) $\dfrac{\varphi_1 \to \varphi_2}{\langle\langle A\rangle\rangle\bigcirc\varphi_1 \to \langle\langle A\rangle\rangle\bigcirc\varphi_2}$

($\langle\langle\varnothing\rangle\rangle\Box$-Necessitation) $\dfrac{\varphi}{\langle\langle\varnothing\rangle\rangle\Box\varphi}$

Proposition 28 *The following are derivable in **ATL**:*

(1) Regularity $\langle\langle A\rangle\rangle\bigcirc\varphi \to \neg\langle\langle\Sigma\backslash A\rangle\rangle\bigcirc\neg\varphi$,
(2) Outcome monotonicity*:* $\langle\langle A\rangle\rangle\bigcirc(\varphi_1 \wedge \varphi_2) \to \langle\langle A\rangle\rangle\bigcirc\varphi_1$,
(3) Coalition-monotonicity: $\langle\langle A_1\rangle\rangle\bigcirc\varphi \to \langle\langle A_1 \cup A_2\rangle\rangle\bigcirc\varphi$ *if* $A_1 \cap A_2 = \varnothing$ [1], *and therefore: if $A_1 \subseteq A_2$ then **ATL**$\vdash \langle\langle A_1\rangle\rangle\bigcirc\varphi \to \langle\langle A_2\rangle\rangle\bigcirc\varphi$.*
(4) $\langle\langle A\rangle\rangle\Box$-Monotonicity:

$$\frac{\varphi_1 \to \varphi_2}{\langle\langle A\rangle\rangle\Box\varphi_1 \to \langle\langle A\rangle\rangle\Box\varphi_2}$$

[1] This condition is not necessary. Once the property is proved for this particular case, the general case follows quite easily.

(5) $\langle\langle A\rangle\rangle\mathcal{U}$-Monotonicity:

$$\frac{\varphi_1 \to \varphi_1', \varphi_2 \to \varphi_2'}{\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2 \to \langle\langle A\rangle\rangle\varphi_1'\mathcal{U}\varphi_2'}$$

(6) The $\langle\langle A\rangle\rangle\Box$-Necessitation rule:

$$\frac{\varphi}{\langle\langle A\rangle\rangle\Box\varphi}$$

(7) $\langle\langle A\rangle\rangle\Box$-Induction:

$$\frac{\theta \to (\varphi \wedge \langle\langle A\rangle\rangle\bigcirc\theta)}{\theta \to \langle\langle A\rangle\rangle\Box\varphi}$$

(8) $\langle\langle A\rangle\rangle\mathcal{U}$-Induction:

$$\frac{(\varphi_2 \vee (\varphi_1 \wedge \langle\langle A\rangle\rangle\bigcirc\theta)) \to \theta}{\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2 \to \theta}$$

(9) Σ*-maximality:* $\neg\langle\langle\varnothing\rangle\rangle\Box\neg\varphi \leftrightarrow \langle\langle\Sigma\rangle\rangle\Diamond\varphi$.
(10) $\varnothing$*-maximality:* $\neg\langle\langle\Sigma\rangle\rangle\Box\neg\varphi \leftrightarrow \langle\langle\varnothing\rangle\rangle\Diamond\varphi$.

PROOF.

(1) Immediate, from the axioms (S) and ($\bot$).
(2) Immediate, from the $\langle\langle A\rangle\rangle\bigcirc$-Monotonicity rule.
(3) Applying the axiom (S) to $\langle\langle A_1\rangle\rangle\bigcirc\varphi$ and $\langle\langle A_2\rangle\rangle\bigcirc\top$.
(4) First, given that **ATL**$\vdash \varphi_1 \to \varphi_2$, and using (FP$_\Box$) we obtain **ATL**$\vdash$ $\langle\langle A\rangle\rangle\Box\varphi_1 \to \varphi_2 \wedge \langle\langle A\rangle\rangle\bigcirc\langle\langle A\rangle\rangle\Box\varphi_1$, hence **ATL** $\vdash \langle\langle\varnothing\rangle\rangle\Box(\langle\langle A\rangle\rangle\Box\varphi_1 \to \varphi_2 \wedge \langle\langle A\rangle\rangle\bigcirc\langle\langle A\rangle\rangle\Box\varphi_1)$. Now, applying axiom (GFP$_\Box$) we get **ATL** $\vdash$ $\langle\langle\varnothing\rangle\rangle\Box(\langle\langle A\rangle\rangle\Box\varphi_1 \to \langle\langle A\rangle\rangle\Box\varphi_2)$, hence **ATL**$\vdash \langle\langle A\rangle\rangle\Box\varphi_1 \to \langle\langle A\rangle\rangle\Box\varphi_2$ because **ATL**$\vdash \langle\langle\varnothing\rangle\rangle\Box\theta \to \theta$.
(5) Likewise, from **ATL**$\vdash \varphi_1 \to \varphi_1'$, **ATL**$\vdash \varphi_2 \to \varphi_2'$, and using (FP$_\mathcal{U}$) we obtain **ATL**$\vdash \varphi_2 \vee (\varphi_1 \wedge \langle\langle A\rangle\rangle\bigcirc\langle\langle A\rangle\rangle\varphi_1'\mathcal{U}\varphi_2') \to \langle\langle A\rangle\rangle\varphi_1'\mathcal{U}\varphi_2'$ so **ATL** $\vdash$ $\langle\langle\varnothing\rangle\rangle\Box(\varphi_2 \vee (\varphi_1 \wedge \langle\langle A\rangle\rangle\bigcirc\langle\langle A\rangle\rangle\varphi_1'\mathcal{U}\varphi_2') \to \langle\langle A\rangle\rangle\varphi_1'\mathcal{U}\varphi_2')$, hence, by (LFP$_\mathcal{U}$) (where $\theta = \langle\langle A\rangle\rangle\varphi_1'\mathcal{U}\varphi_2'$) **ATL**$\vdash \langle\langle\varnothing\rangle\rangle\Box(\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2 \to \langle\langle A\rangle\rangle\varphi_1'\mathcal{U}\varphi_2')$, so **ATL**$\vdash \langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2 \to \langle\langle A\rangle\rangle\varphi_1'\mathcal{U}\varphi_2'$.
(6) From the $\langle\langle\varnothing\rangle\rangle\Box$-Necessitation and $\langle\langle A\rangle\rangle\Box$-Monotonicity rules.
(7) Straightforward from the (GFP$_\Box$) axiom and the $\langle\langle\varnothing\rangle\rangle\Box$-Necessitation rule.
(8) Likewise.
(9) First, **ATL**$\vdash \varphi \to \neg\langle\langle\varnothing\rangle\rangle\Box\neg\varphi$ by (FP$_\Box$) and contraposition. Also, **ATL**$\vdash \langle\langle\Sigma\rangle\rangle\bigcirc\neg\langle\langle\varnothing\rangle\rangle\Box\neg\varphi \to \neg\langle\langle\varnothing\rangle\rangle\bigcirc\langle\langle\varnothing\rangle\rangle\Box\neg\varphi$, by regularity, and **ATL** $\vdash \neg\langle\langle\varnothing\rangle\rangle\bigcirc\langle\langle\varnothing\rangle\rangle\Box\neg\varphi \to \neg\langle\langle\varnothing\rangle\rangle\Box\neg\varphi$ by (FP$_\Box$) and contraposition. Hence **ATL**$\vdash \varphi \vee (\top \wedge \langle\langle\Sigma\rangle\rangle\bigcirc\neg\langle\langle\varnothing\rangle\rangle\Box\neg\varphi) \to \neg\langle\langle\varnothing\rangle\rangle\Box\neg\varphi$. Applying the $\langle\langle\Sigma\rangle\rangle\mathcal{U}$-Induction we get: **ATL**$\vdash \langle\langle\Sigma\rangle\rangle\top\mathcal{U}\varphi \to \neg\langle\langle\varnothing\rangle\rangle\Box\neg\varphi$.

Conversely, we have **ATL**$\vdash \neg\langle\!\langle\Sigma\rangle\!\rangle\top\mathcal{U}\varphi \to \neg\varphi \wedge \neg\langle\!\langle\Sigma\rangle\!\rangle\bigcirc\langle\!\langle\Sigma\rangle\!\rangle\top\mathcal{U}\varphi$ by $(\mathrm{FP}_{\mathcal{U}})$ and contraposition, and **ATL**$\vdash \neg\langle\!\langle\Sigma\rangle\!\rangle\bigcirc\langle\!\langle\Sigma\rangle\!\rangle\top\mathcal{U}\varphi \to \langle\!\langle\varnothing\rangle\!\rangle\bigcirc\neg\langle\!\langle\Sigma\rangle\!\rangle\top\mathcal{U}\varphi$ by Σ -maximality, hence **ATL**$\vdash \neg\langle\!\langle\Sigma\rangle\!\rangle\top\mathcal{U}\varphi \to \neg\varphi \wedge \langle\!\langle\varnothing\rangle\!\rangle\bigcirc\neg\langle\!\langle\Sigma\rangle\!\rangle\top\mathcal{U}\varphi$. Then, by $\Box$-induction we obtain **ATL**$\vdash \neg\langle\!\langle\Sigma\rangle\!\rangle\top\mathcal{U}\varphi \to \langle\!\langle\Sigma\rangle\!\rangle\Box\neg\varphi$, hence **ATL**$\vdash \neg\langle\!\langle\varnothing\rangle\!\rangle\Box\neg\varphi \to \langle\!\langle\Sigma\rangle\!\rangle\top\mathcal{U}\varphi$ by regularity and contraposition.

(10) Likewise. $\Box$

Proposition 29 (Soundness of ATL) *All axioms are valid in every CGS and all rules preserve validity in CGS.*

PROOF. Routine. $\Box$

Proposition 30 *Every **ATL** formula is provably equivalent in the axiomatic system for **ATL** to its normal form.*

PROOF. Routine. $\Box$

So, hereafter we may assume that **ATL** formulae are in normal form whenever necessary.

The rest of the section is devoted to proving completeness.

4.2 Completeness I - Overview

Here we give an overview of the completeness proof for the axiomatic system for **ATL**.

We fix an **ATL**-consistent formula ψ, over the set of players $\Sigma = \{1, \ldots, n\}$ and the set of atomic propositions Π. We will construct a CGS $\mathcal{S}$ such that for some state s in $\mathcal{S}$, we have $\mathcal{S}, s \models \psi$.

The CGS $\mathcal{S}$ will be an infinite tree, labeled with maximal consistent sets of formulae from $\mathrm{ecl}(\psi)$. The proof that our construction is correct will depend on a Truth Lemma which asserts that the labelling is correct in the sense that a formula appears in a label only when the formula is indeed true in the model. In order for the Truth Lemma to hold everywhere in the tree, we need to impose both local consistency conditions for successor nodes of the tree and longer term structure ensuring truth of the eventuality formulae.

Details of this construction are presented in three sections:

Local Constructions – Here we define the notion of *local consistency* of labelled trees and show that under certain conditions such locally consistent trees exist. These trees will form the local (next-time) transitions in our model.
Eventuality Realization – Next we use the existence of locally consistent transitions to prove that there exist finite trees within which given eventualy formulae are *realized* – a notion that corresponds to fulfillment of the eventualities in the final model.
Final Model Construction – Finally we use the finite trees that realize eventualities as components in the final infinite tree model. The final model is constructed from a regular arrangement and joining of these components. When then present the Truth Lemma, the resulting Completeness Theorem, and some related results.

4.3 Completeness II - Local Constructions

For the formula ψ we denote by $\Gamma \subseteq \mathbf{2}^{\mathrm{ecl}(\psi)}$ the set of maximal **ATL**-consistent subsets of $\mathrm{ecl}(\psi)$.

Lemma 31 (Disjoint Coalition Consistency)
*Let $\{\langle\langle A_1\rangle\rangle\bigcirc\varphi_1, \ldots, \langle\langle A_k\rangle\rangle\bigcirc\varphi_k, \neg\langle\langle A'\rangle\rangle\bigcirc\eta\}$ be an **ATL**-consistent set of formulae, with $A_1, \ldots, A_k$ and $\Sigma\backslash A'$ pairwise disjoint coalitions. Then $\{\varphi_1, \ldots, \varphi_k\}\cup \{\neg\eta\}$ is **ATL**-consistent.*

PROOF. First we note that in the case where $k = 0$, we may add $\langle\langle A'\rangle\rangle\bigcirc\top$ (an axiom) to the set of **ATL**-consistent formulae, ensuring that $k \geq 1$.

Since $\{\langle\langle A_1\rangle\rangle\bigcirc\varphi_1, \ldots, \langle\langle A_k\rangle\rangle\bigcirc\varphi_k\}$ is **ATL**-consistent, $\langle\langle A_1\rangle\rangle\bigcirc\varphi_1\wedge\ldots\wedge\langle\langle A_k\rangle\rangle\bigcirc\varphi_k$ is **ATL**-consistent. Repeated application of the axiom S gives **ATL** $\vdash \langle\langle A_1\rangle\rangle\bigcirc\varphi_1\wedge \ldots \wedge \langle\langle A_k\rangle\rangle\bigcirc\varphi_k \rightarrow \langle\langle A_1 \cup \ldots \cup A_k\rangle\rangle\bigcirc (\varphi_1 \wedge \ldots \wedge \varphi_k)$,
hence $\langle\langle A_1 \cup \ldots \cup A_k\rangle\rangle\bigcirc (\varphi_1 \wedge \ldots \wedge \varphi_k)$ is **ATL**-consistent.

Since $(A_1 \cup \ldots \cup A_k) \cap \Sigma\backslash A' = \emptyset$, $(A_1 \cup \ldots \cup A_k) \subseteq A'$. So by Coalition-monotonicity, **ATL** $\vdash \langle\langle A_1\cup\ldots\cup A_k\rangle\rangle\bigcirc (\varphi_1 \wedge \ldots \wedge \varphi_k) \rightarrow \langle\langle A'\rangle\rangle\bigcirc (\varphi_1 \wedge \ldots \wedge \varphi_k)$. It follows that $\langle\langle A'\rangle\rangle\bigcirc (\varphi_1 \wedge \ldots \wedge \varphi_k)$ is **ATL**-consistent.

Suppose now that $\{\varphi_1, \ldots, \varphi_k\} \cup \{\neg\eta\}$ is *not* **ATL**-consistent. Then **ATL** $\vdash (\varphi_1 \wedge \ldots \wedge \varphi_k) \rightarrow \eta$, so we would have **ATL** $\vdash \langle\langle A'\rangle\rangle\bigcirc (\varphi_1 \wedge \ldots \wedge \varphi_k) \rightarrow \langle\langle A'\rangle\rangle\bigcirc\eta$, by $\langle\langle A\rangle\rangle\bigcirc$-Monotonicity, giving **ATL** $\vdash \langle\langle A_1\rangle\rangle\bigcirc\varphi_1\wedge\ldots\wedge\langle\langle A_k\rangle\rangle\bigcirc\varphi_k \rightarrow \langle\langle A'\rangle\rangle\bigcirc\eta$. But this contradicts the hypothesis that $\{\langle\langle A_1\rangle\rangle\bigcirc\varphi_1, \ldots, \langle\langle A_k\rangle\rangle\bigcirc\varphi_k, \neg\langle\langle A'\rangle\rangle\bigcirc\eta\}$ is **ATL**-consistent. Thus $\{\varphi_1, \ldots, \varphi_k\} \cup \{\neg\eta\}$ *is* **ATL**-consistent. $\square$

We will work with trees with a fixed branching degree k^n, labelled with set of **ATL**-formulae. The interpretation of such a tree as a CGS is described in section 3.4.

Definition 32 *A labelled tree $\langle T, V\rangle$ labelled with* ***ATL****-consistent sets of formulae is* locally consistent *when for each interior node $t \in T$:*

(1) if $\langle\!\langle A\rangle\!\rangle \bigcirc \varphi \in V(t)$ then there exists $\sigma_A \in \Delta_A$ such that for all $c \in \mathrm{out}(\sigma_A)$, $\varphi \in V(t \cdot c)$, and
(2) if $\neg\langle\!\langle A\rangle\!\rangle \bigcirc \varphi \in V(t)$ then for all $\sigma_A \in \Delta_A$ there exists $c \in \mathrm{out}(\sigma_A)$ such that $\sim\!\varphi \in V(t \cdot c)$, equivalently there exists $\sigma_A^c \in \Delta_A^c$ such that for all $c \in \mathrm{out}(\sigma_A^c)$, $\sim\!\varphi \in V(t \cdot c)$.

Lemma 33 (Locally Consistent Tree Existence) *Let Φ be a finite* ***ATL****-consistent set of formulae (which can be assumed in normal form). Denote by $\Phi_\circ$ the set of formulae of the form $\langle\!\langle A\rangle\!\rangle \bigcirc \varphi$ or $\neg\langle\!\langle A\rangle\!\rangle \bigcirc \varphi$ in Φ. If $|\Phi_\circ| \le k$, then there is a locally consistent simple labelled tree $\langle T, V\rangle$, labelled with* ***ATL****-consistent sets of formulae, having branching degree k^n such that $V(\epsilon) = \Phi$.*

PROOF. We may assume that no formula of the form $\neg\langle\!\langle \Sigma\rangle\!\rangle \bigcirc \varphi$ appears in Φ, for any such formula is **ATL**-equivalent to $\langle\!\langle \emptyset\rangle\!\rangle \bigcirc \neg\varphi$ by axiom Σ and regularity.

We present one way to construct a transition such that the local consistency conditions hold. We interpret $c < k^n$ as an n-tuple from $\{0, \ldots, k-1\}^n$ in terms of the encoding from section 3.4.2, and write $c(i)$ for the i^{th} coordinate of c. For each successor c of the root of the tree we will define the set of formulae $V(c)$. We then show that each set of formulae $V(c)$ is consistent, and that indeed the local consistency conditions hold.

Let $\langle\!\langle A_0\rangle\!\rangle \bigcirc \varphi_0, \ldots, \langle\!\langle A_{m-1}\rangle\!\rangle \bigcirc \varphi_{m-1}$ be the positive formulae in Φ of the form $\langle\!\langle A\rangle\!\rangle \bigcirc \varphi$, and $\neg\langle\!\langle A_0\rangle\!\rangle \bigcirc \theta_0, \ldots, \neg\langle\!\langle A_{l-1}\rangle\!\rangle \bigcirc \theta_{l-1}$ be the negated formulae in Φ of the form $\neg\langle\!\langle A\rangle\!\rangle \bigcirc \theta$, (with $A_i \neq \Sigma$). Note that $m + l \le k$.

We define $V(c)$ by deciding for each formula $\langle\!\langle A\rangle\!\rangle \bigcirc \varphi \in \Phi$ ($\neg\langle\!\langle A\rangle\!\rangle \bigcirc \theta \in \Phi$) whether φ (respectively $\sim\!\theta$) should be in $V(c)$. Our decision will be such that lemma 31 is applicable – in particular at most one negated formula of the form $\sim\!\theta_j$ will be included in $V(c)$.

Positive Formulae: Consider a formula $\langle\!\langle A_p\rangle\!\rangle \bigcirc \varphi_p \in \Phi$. If $c(i) = p$ for all $i \in A_p$ then φ_p is to be in $V(c)$. Let Ψ denote the set of formulae that are selected like this.

Negated Formulae: From all negated formulae of the form $\neg\langle\!\langle A_q\rangle\!\rangle \bigcirc \theta_q \in \Phi$, at most one formula of the form $\sim\!\theta_q$ is selected for inclusion in $V(c)$. Let

$I = \{i \mid m \leq c(i) < m + l\}$ and $j = [\sum_{i \in I}(c(i) - m)] \bmod l$. Consider now the formula $\neg\langle\langle A_j \rangle\rangle \bigcirc \theta_j$: if $\Sigma \setminus A_j \subseteq I$, then we include $\sim\theta_j$ in $V(c)$, otherwise no negated formula is included in $V(c)$.

We will show that the set $V(c)$ so selected is **ATL**-consistent and ensures that $\langle T, V \rangle$ is locally consistent. For this we assume the case where positive formulae Ψ are selected and where $\sim\theta_j$ is included in $V(c)$; if no $\sim\theta_j$ is included in $V(c)$, we can use $\sim\perp$ in the argument that follows, since it does not affect the consistency of $V(c)$, as $\neg\langle\langle \emptyset \rangle\rangle \bigcirc \perp$ is an axiom.

Firstly note that all the coalitions A_p for $\varphi_p \in \Psi$, and the coalition $\Sigma \setminus A_j$ if $\sim\theta_j$ is included, are mutually disjoint. By lemma 31, it then follows that $\Psi \cup \{\sim\theta_j\}$ is consistent.

We now check condition 1 of the local consistency definition. The construction ensures that for each formula $\langle\langle A_p \rangle\rangle \bigcirc \varphi_p$ in Φ, there is some A_p-move $\sigma_{A_p} \in \Delta_{A_p}$ such that for all $c \in \text{out}(\sigma_{A_p})$, φ_p is in $V(c)$ – this is the A_p-move σ_{A_p} where $\sigma_i = p$ for all the players $i \in A_p$, and thus for every $c \in \text{out}(\sigma_{A_p})$, $c(i) = p$ for all the players $i \in A_p$.

Finally we check condition 2 of the locallay consistent definition. Consider a formula $\neg\langle\langle A_q \rangle\rangle \bigcirc \theta_q$ in Φ (recall that $A_q \neq \Sigma$). For the A_q-move $\sigma \in \Delta_{A_q}$ we show that there is a $c \in \text{out}(\sigma)$ such that $\sim\theta_q$ is in $V(c)$. Recall that we denote by σ_i the move of player i in the A_q-move σ, so that $c(i) = \sigma_i$ for $i \in A_q$. We are to define $c(i)$ for $i \in \Sigma \setminus A_q$. Let $I' = \{i \in A_q \mid m \leq \sigma_i < m + l\}$ and $j' = [\sum_{i \in I'}(\sigma_i - m)] \bmod l$. Now select one $i \in \Sigma \setminus A_q$, and let $c(i) = q - j' \bmod l$. For all the other $i' \in \Sigma \setminus A_q$, let $c(i') = m$. For $I = \{i \mid m \leq c(i) < m + l\}$, note that $\Sigma \setminus A_q \subseteq I$ and that $[\sum_{i \in I}(c(i) - m)] \bmod l = q$. Thus $\sim\theta_j$ is included in $V(c)$ $\square$

4.4 Completeness III - Eventuality Realization

For the extended temporal operators $\square$ and $\mathcal{U}$ we define the concept of *realization* in finite labelled trees. We then show that for each eventuality in a consistent set of formulae, there exists a finite labelled tree where the eventuality is realized. Such trees will be used as components in the final model we construct to prove completeness.

Intuitively an eventuality like $\langle\langle A \rangle\rangle \varphi_1 \mathcal{U} \varphi_2$ is realized when a state *labelled* with φ_2 is reached, and the intermediate states are *labelled* with φ_1. Realization is thus similar to truth, but relative to the labelling of states. Our final construction will ensure that labelling and truth correspond.

Definition 34 (Realization of eventualities and invariants) *The no-*

tions of strategy and co-strategy extend naturally to labelled trees that can be interpreted as CGSs.

- *An eventuality of the form* $\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$ *is* realized *from a node* t *of the labelled tree* $\langle T, V\rangle$ *over* Γ *when there exists an* A*-strategy* F_A *such that for all* $\lambda \in \text{out}(t, F_A)$*, there is some* i *such that* $\varphi_2 \in V(\lambda[i])$ *and for all* $0 \leq j < i$*,* $\varphi_1 \in V(\lambda[j])$*.*
- *An eventuality of the form* $\neg\langle\langle A\rangle\rangle\Box\varphi$ *is* realized *from a node* t *of the labelled tree* $\langle T, V\rangle$ *over* Γ *when there exists a co-*A*-strategy* F^c_A *such that for all* $\lambda \in \text{out}(t, F^c_A)$*, there is some* i *such that* $\sim\varphi \in V(\lambda[i])$*.*
- *An invariant of the form* $\langle\langle A\rangle\rangle\Box\varphi$ *is* realized *from a node* t *of the labelled tree* $\langle T, V\rangle$ *over* Γ *when there exists an* A*-strategy* F_A *such that for all* $\lambda \in \text{out}(t, F^c_A)$*, and all* $i \geq 0$ *we have* $\varphi \in V(\lambda[i])$*.*
- *An invariant of the form* $\neg\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$ *is* realized *from a node* t *of the labelled tree* $\langle T, V\rangle$ *over* Γ *when there exists a co-*A*-strategy* F^c_A *such that for all* $\lambda \in \text{out}(t, F^c_A)$*, if there is some* $i \geq 0$ *such that* $\varphi_2 \in V(\lambda[i])$ *then there is some* $0 \leq j < i$ *where* $\sim\varphi_1 \in V(\lambda[j])$*.*

Recall that Γ is finite, and that each element of Γ is a finite maximal **ATL**-consistent set of formulae.

We will show how to construct a finite locally consistent labelled tree realizing any given consistent eventuality. The construction will use the fact that subsets of Γ can be characterised by formulae in $\text{ecl}(\psi)$, hence the fixed point characterizations of eventuality formulae can be related to the corresponding axioms.

Lemma 35 (Characteristic Formula Lemma) *For any subset* $Y \subseteq \Gamma$ *there is a formula* $\chi_Y \in \text{ecl}(\psi)$ *(up to tautological equivalence), called a* characteristic formula of Y*, such that for every* $y \in \Gamma$*,* $\chi_Y \in y$ *iff* $y \in Y$ *.*

PROOF. For any $y \in Y$, define

$$\chi_{\{y\}} = \wedge y = \bigwedge\{\varphi \mid \varphi \in y\}.$$

Note that $\chi_{\{y\}} \in \text{ecl}(\psi)$ (up to tautological equivalence) since $\text{ecl}(\psi)$ is closed under finite conjunctions. $\chi_{\{y\}}$ so defined is indeed a characteristic formula for $\{y\}$: $\chi_{\{y\}} \in y$ since it is consistent with all formulae in y, and y is a maximal consistent set of formulae.

Conversly, consider an element $y' \in Y$, where $y \neq y'$. Then, since the elements of Y are maximal consistent sets, there is some formula $\theta \in \text{ecl}(\psi)$ such that $\theta \in y$ but $\neg\theta \in y'$. Thus θ is one of the conjuncts in $\chi_{\{y\}}$, and thus $\chi_{\{y\}} \wedge \neg\theta$ is inconsistent. Thus $\chi_{\{y\}} \notin y'$, since y' is a consistent set of formulae in $\text{ecl}(\psi)$.

Now for an arbitrary set $Y \subseteq \mathrm{ecl}(\psi)$, define

$$\chi_Y = \bigvee\{\chi_{\{y\}} \mid y \in Y\}$$

Again $\chi_Y \in \mathrm{ecl}(\psi)$.

For any $y \in Y$, since $\chi_{\{y\}}$ implies χ_Y, $\chi_Y \in y$. Conversely, for any $y' \notin Y$, $\chi_{\{y\}} \notin y'$ for all $y \in Y$ so the disjunction $\chi_Y \notin y'$. $\square$

The next-time formulae in $\mathrm{ecl}(\psi)$ are collected into the set $\Psi_\circ = \{\theta \in \mathrm{ecl}(\psi) \mid \theta = \langle\langle A\rangle\rangle \bigcirc \varphi \text{ or } \theta = \neg\langle\langle A\rangle\rangle \bigcirc \varphi\}$.

Lemma 36 (Eventuality Realization I) *For each eventuality of the form $\langle\langle A\rangle\rangle \varphi_1 \mathcal{U} \varphi_2$ and set $x \in \Gamma$ there is a finite labelled tree $\langle T, V\rangle$ over Γ such that:*

- *$\langle T, V\rangle$ is of fixed branching degree k^n, where $k = |\Psi_\circ| + 1$,*
- *$\langle T, V\rangle$ is locally consistent,*
- *$V(\epsilon) = x$, and*
- *if $\langle\langle A\rangle\rangle \varphi_1 \mathcal{U} \varphi_2 \in x$ then $\langle\langle A\rangle\rangle \varphi_1 \mathcal{U} \varphi_2$ is realized from ϵ.*

PROOF. Consider a specific eventuality $\langle\langle A\rangle\rangle \varphi_1 \mathcal{U} \varphi_2$. Let $Z \subseteq \Gamma$ be the set of maximal consistent sets of formulae where for each $x \in Z$ there is a finite, labelled tree that satisfies the conditions of the lemma. To prove the lemma we have to prove that $Z = \Gamma$. Now for any $x \in \Gamma$, if $\langle\langle A\rangle\rangle \varphi_1 \mathcal{U} \varphi_2 \notin x$ then we take the tree $\langle T, V\rangle$ to consist only of the root node ϵ, labelled by $V(\epsilon) = x$ and thus trivially $x \in Z$. Otherwise we have that $\langle\langle A\rangle\rangle \varphi_1 \mathcal{U} \varphi_2 \in x$ and we are to show that $x \in Z$.

Lemma 35 ensures that there is a characteristic formula χ_Z for the set Z, so it suffices to show that $\langle\langle A\rangle\rangle \varphi_1 \mathcal{U} \varphi_2 \rightarrow \chi_Z$ is an **ATL** theorem. From this it follows that if $x \in \Gamma$ is such that $\langle\langle A\rangle\rangle \varphi_1 \mathcal{U} \varphi_2 \in x$ then $\chi_Z \in x$, hence $x \in Z$.

To show that $\langle\langle A\rangle\rangle \varphi_1 \mathcal{U} \varphi_2 \rightarrow \chi_Z$ is an **ATL** theorem, it is sufficient by $\langle\langle A\rangle\rangle \mathcal{U}$-Induction to show that

$$\varphi_2 \vee (\varphi_1 \wedge \langle\langle A\rangle\rangle \bigcirc \chi_Z) \rightarrow \chi_Z \tag{1}$$

is an **ATL** theorem. Note, however, that $\langle\langle A\rangle\rangle \bigcirc \chi_Z$ need not be a formula in $\mathrm{ecl}(\psi)$, although $\chi_Z \in \mathrm{ecl}(\psi)$.

In order to show that the formula (1) is an **ATL** theorem, it is sufficient to show that (1) belongs to every maximal **ATL**-consistent set of formulae q (not only formulae from $\mathrm{ecl}(\psi)$). Note that the set $q \cap \mathrm{ecl}(\psi)$ is an element of Γ, so $\chi_Z \in q$ iff $q \cap \mathrm{ecl}(\psi) \in Z$.

We begin by taking care of two easy cases: If $\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2 \notin q$ then $q \cap \mathrm{ecl}(\psi) \in Z$, since we may consider the trivial tree, as described before. If $\varphi_2 \vee (\varphi_1 \wedge \langle\langle A\rangle\rangle\bigcirc\chi_Z) \notin q$ then (1) $\in q$ follows directly.

Otherwise we have that $\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2 \in q$ and $\varphi_2 \vee (\varphi_1 \wedge \langle\langle A\rangle\rangle\bigcirc\chi_Z) \in q$. We want to show $q \cap \mathrm{ecl}(\psi) \in Z$, that is to show that there is a finite labelled tree with root labelled by $q \cap \mathrm{ecl}(\psi)$ satisfying the conditions of the lemma. Since $\varphi_2 \vee (\varphi_1 \wedge \langle\langle A\rangle\rangle\bigcirc\chi_Z) \in q$ we have at least one of the following:

(a) $\varphi_2 \in q$, or
(b) $\varphi_1 \wedge \langle\langle A\rangle\rangle\bigcirc\chi_Z \in q$.

In case (a), we choose the required labelled tree to be the tree $\langle T, V\rangle$ that consists only of the root node ϵ, labelled by $V(\epsilon) = q \cap \mathrm{ecl}(\psi)$. The eventuality $\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$ is immediately realized from the root of $\langle T, V\rangle$, since $\varphi_2 \in V(\epsilon)$.

In case (b), we construct a labelled tree $\langle T, V\rangle$ with the root node labelled by $q \cap \mathrm{ecl}(\psi)$, and successors defined as follows:

Let Ψ' be the set containing all the formulae in $q \cap \mathrm{ecl}(\psi)$ of the form $\langle\langle A\rangle\rangle\bigcirc\varphi$ or $\neg\langle\langle A\rangle\rangle\bigcirc\varphi$, and also the formula $\langle\langle A\rangle\rangle\bigcirc\chi_Z$. Then $|\Psi'| \leq k$, where $k = |\Psi_\bigcirc| + 1$ as defined in the statement of the lemma. Thus, by lemma 33, there is a locally consistent simple labelled tree $\langle T', V'\rangle$ having a branching degree k^n such that $V'(\epsilon) = \Psi'$.

For any ϵ-successor $c < k^n$, take $V(c)$ to be any element of Γ that contains $V'(c) \cap \mathrm{ecl}(\psi)$. This preserves local consistency at ϵ, since each formula in $V(\epsilon)$ of the form $\langle\langle A\rangle\rangle\bigcirc\varphi$ or $\neg\langle\langle A\rangle\rangle\bigcirc\varphi$ has $\varphi \in \mathrm{ecl}(\psi)$.

Now, consider each ϵ-successor $c < k^n$, with $\chi_Z \in V(c)$. Since $\chi_Z \in V(c)$, we have that $V(c) \in Z$, thus there is a finite, locally consistent labelled tree $\langle T_c, V_c\rangle$ satisfying the conditions in the lemma. Replace the node c in T' with the labelled tree $\langle T_c, V_c\rangle$, i.e. identify c with the root of $\langle T_c, V_c\rangle$.

This construction creates a labelled tree $\langle T, V\rangle$ with the root node labelled by $q \cap \mathrm{ecl}(\psi)$ that is locally consistent and has fixed branching degree k^n.

Finally, we will show that the eventuality $\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$ is realized from the root of $\langle T, V\rangle$. An A-strategy that proves this is given by $F_A(\epsilon) = \sigma_A$, where σ_A is the A-move that follows from $\langle\langle A\rangle\rangle\bigcirc\chi_Z \in V'(\epsilon)$ and $\langle T', V'\rangle$ being locally consistent, noting that $\langle T, V\rangle$ has the same choice structure as $\langle T', V'\rangle$ from ϵ. Further, for every $c \in \mathrm{out}(\sigma_A)$, $\chi_Z \in V(c)$ so $V(c) \in Z$. Hence there is some A-strategy $F_{A,c}$ realizing $\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$ from c. We put $F_A(c \cdot \lambda) = F_{A,c}(\lambda)$. Since $F_{A,c}$ realizes $\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$ from c, it follows that F_A realizes $\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$ from the root ϵ. $\square$

Lemma 37 (Eventuality Realization II) *For each eventuality of the form* $\neg\langle\langle A\rangle\rangle\Box\varphi$ *and set* $x \in \Gamma$ *there is a finite, labelled tree* $\langle T, V\rangle$ *over* Γ *such that:*

- $\langle T, V\rangle$ *is of fixed branching degree* k^n*, where* $k = |\Psi_\bigcirc| + 1$,
- $\langle T, V\rangle$ *is locally consistent,*
- $V(\epsilon) = x$*, and*
- *if* $\neg\langle\langle A\rangle\rangle\Box\varphi \in x$ *then* $\neg\langle\langle A\rangle\rangle\Box\varphi$ *is realized from* ϵ.

PROOF. The proof is similar to that of the previous lemma.

Consider a specific eventuality $\neg\langle\langle A\rangle\rangle\Box\varphi$. Let $Z \subseteq \Gamma$ be the set of maximal consistent sets of formulae where for each $x \in Z$ there is a tree that satisfies the conditions of the lemma. To prove the lemma we have to prove that $Z = \Gamma$. Now for any $x \in \Gamma$, if $\neg\langle\langle A\rangle\rangle\Box\varphi \notin x$ then we take the tree $\langle T, V\rangle$ to consist only of the root node ϵ, labelled by $V(\epsilon) = x$ and thus trivially $x \in Z$. Otherwise we have that $\neg\langle\langle A\rangle\rangle\Box\varphi \in x$ and we are to show that $x \in Z$.

Lemma 35 ensures that there is a characteristic formula χ_Z for the set Z, so it suffices to show that $\neg\langle\langle A\rangle\rangle\Box\varphi \to \chi_Z$ is an **ATL** theorem. From this it follows that if $x \in \Gamma$ is such that $\neg\langle\langle A\rangle\rangle\Box\varphi \in x$ then $\chi_Z \in x$, hence $x \in Z$.

To show that $\neg\langle\langle A\rangle\rangle\Box\varphi \to \chi_Z$ is an **ATL** theorem, it is sufficient by the contrapositive in $\langle\langle A\rangle\rangle\Box$-Induction to show that

$$\neg\varphi \vee \neg\langle\langle A\rangle\rangle\bigcirc\neg\chi_Z \to \chi_Z \tag{2}$$

is an **ATL** theorem. Note however that $\neg\langle\langle A\rangle\rangle\bigcirc\neg\chi_Z$ need not be a formula in $\mathrm{ecl}(\psi)$, although $\chi_Z \in \mathrm{ecl}(\psi)$.

In order to show that the formula (2) is an **ATL** theorem, it is sufficient to show that (2) belongs to every maximal **ATL**-consistent set of formulae q (not only formulae from $\mathrm{ecl}(\psi)$). Note that the set $q \cap \mathrm{ecl}(\psi)$ is an element of Γ, so $\chi_Z \in q$ iff $q \cap \mathrm{ecl}(\psi) \in Z$.

We begin by taking care of two easy cases: If $\neg\langle\langle A\rangle\rangle\Box\varphi \notin q$ then $q \cap \mathrm{ecl}(\psi) \in Z$ since we may consider the trivial tree, as described before. If $\neg\varphi \vee \neg\langle\langle A\rangle\rangle\bigcirc\neg\chi_Z \notin q$ then (2) $\in q$ follows directly.

Otherwise we have that $\neg\langle\langle A\rangle\rangle\Box\varphi \in q$ and $\neg\varphi \vee \neg\langle\langle A\rangle\rangle\bigcirc\neg\chi_Z \in q$. We want to show $q \cap \mathrm{ecl}(\psi) \in Z$, that is to show that there is a finite labelled tree with root labelled by $q \cap \mathrm{ecl}(\psi)$ satisfying the conditions of the lemma. Since $\neg\varphi \vee \neg\langle\langle A\rangle\rangle\bigcirc\neg\chi_Z \in q$ we have at least one of the following:

(a) $\neg\varphi \in q$, or
(b) $\neg\langle\langle A\rangle\rangle\bigcirc\neg\chi_Z \in q$.

In case (a), we choose the required labelled tree be the tree $\langle T, V \rangle$ that consists only of the root node ϵ, labelled by $V(\epsilon) = q \cap \mathrm{ecl}(\psi)$. The eventuality $\neg\langle\langle A \rangle\rangle \Box \varphi$ is immediately realized from the root of $\langle T, V \rangle$, since $\neg\varphi \in V(\epsilon)$.

In case (b), we construct a labelled tree $\langle T, V \rangle$ with the root node labelled by $q \cap \mathrm{ecl}(\psi)$, and successors defined as follows:

Let Ψ' be the set containing all the formulae in $q \cap \mathrm{ecl}(\psi)$ of the form $\langle\langle A \rangle\rangle \bigcirc \varphi$ or $\neg\langle\langle A \rangle\rangle \bigcirc \varphi$, and also the formula $\neg\langle\langle A \rangle\rangle \bigcirc \chi_Z$. Then $|\Psi'| \leq k$, where $k = |\Psi_\circ| + 1$ as defined in the statement of the lemma. Thus, by lemma 33, there is a locally consistent simple labelled tree $\langle T', V' \rangle$ having a branching degree k^n such that $V'(\epsilon) = \Psi'$.

For any ϵ-successor $c < k^n$, take $V(c)$ to be any element of Γ that contains $V'(c) \cap \mathrm{ecl}(\psi)$. This preserves local consistency at ϵ, since each formula in $V(\epsilon)$ of the form $\langle\langle A \rangle\rangle \bigcirc \varphi$ or $\neg\langle\langle A \rangle\rangle \bigcirc \varphi$ has $\varphi \in \mathrm{ecl}(\psi)$.

Now, consider each ϵ-successor $c < k^n$, with $\chi_Z \in V(c)$. Since $\chi_Z \in V(c)$, we have that $V(c) \in Z$, thus there is a finite, locally consistent labelled tree $\langle T_c, V_c \rangle$ satisfying the conditions in the lemma. Replace the node c in T' with the labelled tree $\langle T_c, V_c \rangle$, i.e. identify c with the root of $\langle T_c, V_c \rangle$.

This construction creates a labelled tree $\langle T, V \rangle$ with the root node labelled by $q \cap \mathrm{ecl}(\psi)$ that is locally consistent and has fixed branching degree k^n.

Finally, we will show that the eventuality $\neg\langle\langle A \rangle\rangle \Box \varphi$ is realized from the root of $\langle T, V \rangle$. A co-A-strategy that proves this is given by $F_A^c(\epsilon) = \sigma_A^c$, where σ_A^c is the co-A-move that follows from $\neg\langle\langle A \rangle\rangle \bigcirc \neg\chi_Z \in V(\epsilon)$ and $\langle T, V \rangle$ being locally consistent, noting that $\langle T, V \rangle$ has the same choice structure as $\langle T', V' \rangle$ from ϵ. Further, for every $c \in \mathrm{out}(\sigma_A^c)$, $\chi_Z \in V(c)$. Hence there is some co-A-strategy $F_{A,c}^c$ realizing $\neg\langle\langle A \rangle\rangle \Box \varphi$ from c. We put $F_A^c(c \cdot \lambda) = F_{A,c}^c(\lambda)$. Since $F_{A,c}^c$ realizes $\neg\langle\langle A \rangle\rangle \Box \varphi$ from c, it follows that F_A^c realizes $\neg\langle\langle A \rangle\rangle \Box \varphi$ from the root ϵ. $\Box$

4.5 Completeness IV - Final Model Construction

Using the above lemmas, we now define a family of labelled trees, called the final model tree components, one for each pair (set in Γ, eventuality in $\mathrm{ecl}(\psi)$). These component trees will then be combined to form a final model for the formula ψ. The final model will be an infinite tree labelled with maximal consistent sets of formulae, and we prove a Truth Lemma that relates this labelling with truth in the associated model. We also note how a finite model may be similarly constructed, giving a Finite Model Theorem for **ATL**.

Definition 38 (Final Model Tree Components) *For $x \in \Gamma$ and an even-*

tuality $\varphi \in \mathrm{ecl}(\psi)$*, define the tree* $\langle T_{x,\varphi}, V_{x,\varphi}\rangle$ *as follows:*

- *If* $\varphi \notin x$*, let* $\langle T_{x,\varphi}, V_{x,\varphi}\rangle$ *be any locally consistent simple tree labelled by* x *at the root, according to the Locally Consistent Tree Existence Lemma.*
- *If* $\varphi \in x$*, let* $\langle T_{x,\varphi}, V_{x,\varphi}\rangle$ *be the tree that realizes* φ *from its root node, according to the Eventuality Realization I and II Lemmas.*

Thus, we have the following:

Lemma 39 *The trees selected as final model tree components have the following properties:*

- *They have fixed branching degree* k^n*, where* $k = |\Psi_\circ| + 1$*.*
- *They are locally consistent.*
- *They are finite.*

Using these components we will construct an infinite labelled tree $\langle T, V\rangle$, with root node labelled by a set containing ψ.

Definition 40 (Final Labelled Tree) *The* final labelled tree $\langle T_\psi, V_\psi\rangle$ *for the satisfiable formula* ψ *is defined as follows.*

Let the eventualities in $\mathrm{ecl}(\psi)$ *be listed as* $\varphi(0), \varphi(1), \ldots, \varphi(m)$*. The tree is constructed inductively.*

We first select an initial tree component: Select any $x \in \Gamma$ *such that* $\psi \in x$*. Such a set exists since* ψ *is* ***ATL****-consistent and in* $\mathrm{ecl}(\psi)$*. Let* $\langle T_{x,\varphi(0)}, V_{x,\varphi(0)}\rangle$ *be the initial tree.*

Next we describe how the tree construction is extended. Given the tree constructed so far, with the last eventuality used being $\varphi(i)$*, for every leaf node of the tree, if it is labelled with* y*, we identify it with the root node of* $\langle T_{y,\varphi(i+1)}, T_{y,\varphi(i+1)}\rangle$ *if* $i < m$*, or* $\langle T_{y,\varphi(0)}, T_{y,\varphi(0)}\rangle$ *if* $i = m$*. Thus, we adjoin to the leaves of the tree the corresponding tree components for the 'next' eventuality, restarting with eventuality* $\varphi(0)$ *after* $\varphi(m)$*.*

In preparation for the Truth Lemma we prove two lemmas showing that all eventualities in labels are realized in the final labelled tree.

Lemma 41 *If an eventuality* φ *is in the label of a node* t *of the final labelled tree* $\langle T_\psi, V_\psi\rangle$ *then* φ *is realized from* t*.*

PROOF. Given a node $t \in T_\psi$, let $\varphi(i)$ be an eventuality in $V_\psi(t)$. We consider the two cases for the form of φ.

- $\varphi(i)$ is an eventuality of the form $\langle\!\langle A\rangle\!\rangle \varphi_1 \mathcal{U} \varphi_2$.

The node t appears in $\langle T_\psi, V_\psi\rangle$ as the interior node of some component $\langle T_{x,\varphi(j)}, V_{x,\varphi(j)}\rangle$. If t is the root node of the component and $\varphi(i)$ is the eventuality associated with the component, then t is realized in the component and hence also in the final labelled tree.

Otherwise, we construct an A-strategy F_A as follows: Since $\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2 \in V(t)$ and $V(t)$ is a maximal **ATL**-consistent set, $\varphi_2\vee(\varphi_1\wedge\langle\langle A\rangle\rangle\bigcirc\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2) \in V(t)$. If $\varphi_2 \in V(t)$, we are done and the eventuality is realized immediately. If $\varphi_2 \notin V(t)$, then $\varphi_1 \wedge \langle\langle A\rangle\rangle\bigcirc\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2 \in V(t)$, so by local consistency there is an A-move $\sigma_A \in \Delta_A$ such that for all $c \in \mathrm{out}(\sigma_A)$, $\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2 \in V(t\cdot c)$. We let $F_A(t) = \sigma_A$.

We can continue with this argument for the definition of F_A, until a node t' is reached that is the root node of a component that has associated eventuality $\varphi(i)$. At this point the A-strategy that realizes $\varphi(i)$ within the component is followed by F_A. We are sure to reach such a node, since each component used in the construction of the final labelled tree is finite, and the eventualities are cycled through.

- $\varphi(i)$ is an eventuality of the form $\neg\langle\langle A\rangle\rangle\Box\varphi'$.

  The argument in this case is similar, but a co-A-strategy is constructed.

  If the node t is the root of a component with associated eventuality $\varphi(i)$, we follow the co-A-strategy that ensures realization within this component. Otherwise we proceed along the tree, constructing the co-A-strategy according to the co-A-moves that ensure local consistency of $\neg\varphi' \vee \neg\langle\langle A\rangle\rangle\bigcirc\langle\langle A\rangle\rangle\Box\varphi'$ until we reach the root node of a component that has associated eventuality $\varphi(i)$. Within this component we then follow the co-A-strategy that ensures realization of $\varphi(i)$. $\Box$

Lemma 42 *If a formula of the form $\langle\langle A\rangle\rangle\Box\varphi'$ or $\neg\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$ is in the label of a node t of the final labelled tree $\langle T_\psi, V_\psi\rangle$ then it is realized from t.*

PROOF.

- $\varphi = \langle\langle A\rangle\rangle\Box\varphi'$: We construct an A-strategy F_A inductively.

  Since $\langle\langle A\rangle\rangle\Box\varphi' \in V(t)$, and $V(t)$ is a maximal **ATL**-consistent set, $\varphi' \wedge \langle\langle A\rangle\rangle\bigcirc\langle\langle A\rangle\rangle\Box\varphi' \in V(t)$. So we have that $\varphi' \in V(t)$ and by local consistency there exists some A-move $\sigma_A \in \Delta_A$ such that for all $c \in \mathrm{out}(\sigma_A)$, $\langle\langle A\rangle\rangle\Box\varphi' \in V(t\cdot c)$. Let $F_A(t) = \sigma_A$.

  With F_A defined for a t-computation $\lambda\cdot c$, note that $\langle\langle A\rangle\rangle\Box\varphi' \in V(c)$. As before it follows that $\varphi' \in V(c)$ and there exists some A-move $\sigma'_A \in \Delta_A$ such that for all $c' \in \mathrm{out}(\sigma'_A)$, $\langle\langle A\rangle\rangle\Box\varphi' \in V(\lambda\cdot c\cdot c')$. Let $F_A(\lambda\cdot c) = \sigma'_A$.

  Consider now any t-computation $\lambda \in \mathrm{out}(t, F_A)$. The construction ensures that for all $i \geq 0$ we have $\varphi' \in V(\lambda[i])$, so that F_A is the A-strategy we require.
- $\varphi = \neg\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$: We construct a co-$A$-strategy F^c_A inductively as in the previous case. For all $t' \in T_\psi$, $\neg\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2 \in V(t)$ implies that $\sim\varphi_2 \in$

$V(t)$ and $\sim\varphi_1 \vee \neg\langle\langle A\rangle\rangle\bigcirc\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2 \in V(t)$. The co-$A$-strategy is thus constructed from the co-A-moves implied by $\neg\langle\langle A\rangle\rangle\bigcirc\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2 \in V(t)$, and the inductive construction might end (for a given t-computation) if a node t' is reached where $\sim\varphi_1 \in V(t')$. $\square$

The final labelled tree induces a CGS $\mathcal{S}_{\langle T_\psi, V_\psi\rangle} = \langle n, T_\psi, \Pi, \pi, d, \delta\rangle$, where $\pi(t) = V_\psi(t) \cap \Pi$, $d_a(q) = k$ and δ is defined as in section 3.4. We now prove that the labelling of the final labelled tree corresponds to truth in the induced CGS.

Lemma 43 (Truth Lemma) *For every node $t \in T_\psi$ and every formula $\varphi \in$ ecl(ψ), if $\varphi \in V_\psi(t)$ then $\mathcal{S}_{\langle T_\psi, V_\psi\rangle}, t \models \varphi$.*

PROOF. The proof is by induction on the structure of the formula φ:

- $\varphi = p$, where $p \in \Pi$: The definition of π ensures that $p \in V_\psi(t)$ implies $p \in \pi(t)$.
- $\varphi = \neg p$, where $p \in \Pi$: $V_\psi(t)$ is a maximal **ATL**-consistent set, so $p \notin V_\psi(t)$ and the definition of π ensures that $p \notin \pi(t)$.
- $\varphi = \varphi_1 \vee \varphi_2$: $V_\psi(t)$ is a maximal **ATL**-consistent set, so $\varphi_1 \in V_\psi(t)$ or $\varphi_2 \in V_\psi(t)$.
- $\varphi = \varphi_1 \wedge \varphi_2$: $V_\psi(t)$ is a maximal **ATL**-consistent set, so $\varphi_1 \in V_\psi(t)$ and $\varphi_2 \in V_\psi(t)$.
- $\varphi = \langle\langle A\rangle\rangle\bigcirc\varphi'$: Local consistency of the final labelled tree ensures that there exists some A-move $\sigma_A \in \Delta_A$ such that for all $c \in \mathrm{out}(\sigma_A)$, $\varphi' \in V_\psi(t \cdot c)$.
- $\varphi = \neg\langle\langle A\rangle\rangle\bigcirc\varphi'$: Local consistency of the final labelled tree ensures that there exists some co-A-move $\sigma^c_A \in \Delta^c_A$ such that for all $c \in \mathrm{out}(\sigma^c_A)$, $\sim\varphi' \in V_\psi(t\cdot c)$.
- $\varphi = \langle\langle A\rangle\rangle\Box\varphi'$, $\varphi = \neg\langle\langle A\rangle\rangle\Box\varphi'$, $\varphi = \langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$, or $\varphi = \neg\langle\langle A\rangle\rangle\varphi_1\mathcal{U}\varphi_2$: By the previous lemmas these formulae are realized from t in $\langle T_\psi, V_\psi\rangle$. By the inductive hypothesis, this realization implies truth at t. $\square$

In particular this ensures that ψ is satisfied at the root node of the model, giving:

Theorem 44 (Completeness Theorem) *The axiomatic system for **ATL** is complete: every **ATL**-consistent formula ψ is satisfiable in a CGS.*

The tree construction shows that satisfiable formulae are satisfiable in tree models with a fixed branching degree. This property of **ATL** will be used in the automata-based analysis, and is formalised as follows:

Corollary 45 (Bounded-Branching Model Theorem) *We consider **ATL** over the set of players Σ with $|\Sigma| = n$. If a formula ψ is satisfiable, then there* *is a labelled tree $\langle T, V\rangle$ over Π of fixed branching degree k^n, and with k dependent only on the length of the formula ψ, such that $\langle T, V\rangle \models \psi$.*

Instead of constructing an infinite tree model, we can also construct a finite cyclic model from the family of labelled components. The family of tree component are arranged as a grid, with each column associated with a set $x \in \Gamma$, and each row associated with an eventuality in $\varphi(0), \cdots, \varphi(m)$. The leaf nodes of each component is identified with the root node of another component as follows: if t is the leaf node of a component in the row for $\varphi(i)$ (with $0 \leq i < m$) and t is labelled with the set $y \in \Gamma$, then t is identified with the root node of the component in the row for $\varphi(i+1 \bmod m)$ and in the column for y. The construction is thus similar to that of the tree model, but components are reused as needed to produce a cyclic model.

As each component in the cyclic model described above is finite, we have built a finite model that satisfies the given **ATL** formula. The truth lemma that proves the correspondance between the labelling of nodes and semantic truth is as for the tree model. We thus have:

Theorem 46 (Finite Model Theorem) *Every satisfiable formula ψ is satisfiable in a finite CGS.*

5 Satisfiability of ATL by Alternating Tree Automata

In this section we describe an Alternating Büchi Tree Automaton that will accept the bounded-branching labelled tree models of an **ATL** formula ψ.

We begin with a description of alternating tree automata. Next we describe the alternating tree automaton for an **ATL** formula and prove correctness of the construction. The resulting decision procedure for **ATL** is in exponential time.

5.1 Alternating Tree Automata

This section provides a brief overview of the automata-theoretic concepts that will be used later. The definitions in this section are mainly taken from [13] and [7], where the reader may find an expanded discussion as well as references to the definitions and proofs of propositions noted here.

Alternating transitions are constructed from positive Boolean formulae:

Definition 47 (Positive Boolean Formula) *For a given set X, let $\mathcal{B}^+(X)$*

be the set of positive Boolean formulae over X (i.e. Boolean formulae built from elements in X using $\wedge$ and $\vee$), where we also allow the formulae true *and* false *and, as usual, $\wedge$ has precedence over $\vee$. For a set $Y \subseteq X$ and a formula $\theta \in \mathcal{B}^+(X)$, we say that Y satisfies θ iff assigning* true *to elements in Y and assigning* false *to elements in $X \setminus Y$ makes θ true.*

Note that if Y_1 satisfies θ_1 and Y_2 satisfies θ_2 then $Y_1 \cup Y_2$ satisfies $\theta_1 \wedge \theta_2$.

Definition 48 (Alternating Büchi Tree Automaton) *A (finite) alternating Büchi tree automaton (ATA) is a tuple $\mathcal{A} = \langle \Theta, k, S, s^0, \rho, F \rangle$, with:*

- *Θ is a finite alphabet,*
- *k is a finite branching degree,*
- *S is a finite set of states,*
- *$s^0 \in S$ is an initial state,*
- *$\rho : S \times \Theta \to \mathcal{B}^+(\{0, \ldots, k-1\} \times S)$ is a partial transition function, and*
- *$F \subseteq S$ is a set defining the acceptance condition.*

A run of an ATA $\mathcal{A}$ over a Θ-labelled leafless k-tree $\langle T, V \rangle$ is a tree $\langle T_r, r \rangle$ in which every node is labelled by an element of $\mathbb{N}^* \times S$. Each node of T_r is associated with a node of T. A node in T_r labelled by (x, s) describes a copy of the automaton that reads the node x of T in the state s. Note that many nodes of T_r can correspond to the same node of T.

The labels of a node and its children have to satisfy the transition function. Formally a run $\langle T_r, r \rangle$ is a Θ_r-labelled tree where $\Theta_r = \mathbb{N}^* \times S$ and $\langle T_r, r \rangle$ satisfies the following:

(1) $r(\epsilon) = (\epsilon, s^0)$.

(2) Let $y \in T_r$ with $r(y) = (x, s)$ and $\rho(s, V(x)) = \theta$. Then there is a (possibly empty) set $Q = \{(c_0, s_0), (c_1, s_1), \ldots, (c_p, s_p)\} \subseteq \{0, \ldots, k-1\} \times S$ such that the following hold:
   - *Q satisfies θ, and*
   - *for all $0 \leq i \leq p$, we have $y \cdot i \in T_r$ and $r(y \cdot i) = (x \cdot c_i, s_i)$.*

A run is accepting if all its infinite paths λ satisfy $\inf(\lambda) \cap F \neq \varnothing$. An automaton accepts a tree if and only if there exists a run that accepts it. We denote by $\mathcal{T}_\omega(\mathcal{A})$ the set of all Θ-labelled k-trees that $\mathcal{A}$ accepts.

Alternating Büchi tree automata may be translated to nondeterministic Büchi tree automata with an exponential increase in the number of states – a recent proof is presented in [8]. From the quadratic decidability of the nonemptiness problem for nondeterministic Büchi tree automata we thus have:

Proposition 49 (Nonemptiness of ATA) *The nonemptiness problem for Alternating Büchi Tree Automata is decidable in exponential time.*

The construction of the ATA follows directly from the labelled tree semantics.

Theorem 50 (ATL Formula Tree Automaton) *Given an **ATL** formula ψ over propositions Π and players Σ, with $|\Sigma| = n$, there is an Alternating Büchi Tree Automaton $\mathcal{A}_\psi = \langle \mathbf{2}^\Pi, k^n, \mathrm{cl}(\psi), \psi, \rho, F \rangle$ such that $\mathcal{T}_\omega(\mathcal{A}_\psi)$ is exactly the set of labelled trees of fixed branching degree k^n that satisfy ψ.*

PROOF. We complete the description of the automaton by defining the set of accepting states F and the transition function ρ.

Let F contain all invariants (formulae of the form $\langle\langle A \rangle\rangle \Box \varphi$ or $\neg \langle\langle A \rangle\rangle \varphi_1 \mathcal{U} \varphi_2$) that appear in $\mathrm{cl}(\psi)$. These formulae may be regenerated by the automaton and by inclusion in the set F we allow them to be regenerated infinitely many times along the branch of a run of $\mathcal{A}_\psi$.

As for the transition function ρ, this is defined for all $\pi \in \mathbf{2}^\Pi$ as follows: (Recall that we write Δ_A for the set of of A-moves in a labelled tree with fixed branching degree k^n. For an A-move $\sigma \in \Delta_A$ we let $c \in \mathrm{out}(\sigma)$ denote those successor nodes consistent with σ.)

- $\rho(p, \pi) = \mathsf{true}$ if $p \in \pi$
- $\rho(p, \pi) = \mathsf{false}$ if $p \notin \pi$
- $\rho(\neg p, \pi) = \mathsf{false}$ if $p \in \pi$
- $\rho(\neg p, \pi) = \mathsf{true}$ if $p \notin \pi$
- $\rho(\varphi_1 \wedge \varphi_2, \pi) = \rho(\varphi_1, \pi) \wedge \rho(\varphi_2, \pi)$
- $\rho(\varphi_1 \vee \varphi_2, \pi) = \rho(\varphi_1, \pi) \vee \rho(\varphi_2, \pi)$
- $\rho(\langle\langle A \rangle\rangle \bigcirc \varphi, \pi) = \bigvee_{\sigma \in \Delta_A} (\bigwedge_{c \in \mathrm{out}(\sigma)} (c, \varphi))$
- $\rho(\neg \langle\langle A \rangle\rangle \bigcirc \varphi, \pi) = \bigwedge_{\sigma \in \Delta_A} (\bigvee_{c \in \mathrm{out}(\sigma)} (c, \sim\varphi))$
- $\rho(\langle\langle A \rangle\rangle \Box \varphi, \pi) = \rho(\varphi, \pi) \wedge \rho(\langle\langle A \rangle\rangle \bigcirc \langle\langle A \rangle\rangle \Box \varphi, \pi)$
- $\rho(\neg \langle\langle A \rangle\rangle \Box \varphi, \pi) = \rho(\sim\varphi, \pi) \vee \rho(\neg \langle\langle A \rangle\rangle \bigcirc \langle\langle A \rangle\rangle \Box \varphi, \pi)$
- $\rho(\langle\langle A \rangle\rangle \varphi_1 \mathcal{U} \varphi_2, \pi) = \rho(\varphi_2, \pi) \vee (\rho(\varphi_1, \pi) \wedge \rho(\langle\langle A \rangle\rangle \bigcirc \langle\langle A \rangle\rangle \varphi_1 \mathcal{U} \varphi_2, \pi))$
- $\rho(\neg \langle\langle A \rangle\rangle \varphi_1 \mathcal{U} \varphi_2, \pi) = \rho(\sim\varphi_2, \pi) \wedge (\rho(\sim\varphi_1, \pi) \vee \rho(\neg \langle\langle A \rangle\rangle \bigcirc \langle\langle A \rangle\rangle \varphi_1 \mathcal{U} \varphi_2, \pi))$

The construction is correct since $\mathcal{T}_\omega(\mathcal{A}_\psi)$ contains exactly all the k^n-branching labelled trees that satisfy ψ. We prove soundness and completeness of the construction in the following two sections.

We first prove that $\mathcal{A}_\psi$ is sound, that is, given an accepting run $\langle T_r, r\rangle$ of $\mathcal{A}_\psi$ over a k^n-branching labelled tree $\langle T, V\rangle$, we prove that $\langle T, V\rangle \models \psi$.

To facilitate the proof we define the concept of a *sub-run*. For some node $y \in T_r$, and formula $\varphi \in \mathrm{cl}(\psi)$ such that $r(y) = (x, \varphi)$, the sub-run $\langle T_r^{y,\varphi}, r^{y,\varphi}\rangle$ is the tree with nodes $z \in T_r^{y,\varphi}$ iff $y{\cdot}z \in T_r$ and $r^{y,\varphi}(\epsilon) = (x, \varphi)$, $r^{y,\varphi}(z) = r(y{\cdot}z)$ for $z \neq \epsilon$.

Thus, $\langle T_r^{y,\varphi}, r^{y,\varphi}\rangle$ is the sub-tree of $\langle T_r, r\rangle$ taking node $y \in T_r$ as root node and modified so that the new root node is labelled by φ. Note that $\langle T_r^{y,\varphi}, r^{y,\varphi}\rangle = \langle T_r, r\rangle$ when $y = \epsilon$ and $\varphi = \psi$.

Also, define $\mathcal{A}_\psi^\varphi$ to be the automaton $\mathcal{A}_\psi$ modified to have an initial state φ. Then $\mathcal{A}_\psi^\psi = \mathcal{A}_\psi$.

The sub-runs are defined to have the following property: a sub-run $\langle T_r^{y,\varphi}, r^{y,\varphi}\rangle$ is an accepting run of the automaton $\mathcal{A}_\psi^\varphi$. This follows from the condition that $r(y) = (x, \varphi)$, ensuring that the sub-run satisfies the conditions for a run of $\mathcal{A}_\psi^\varphi$ at the root node.

We now show that for all nodes $y \in T_r$ with $r(y) = (x, \varphi)$ that if $\langle T_r^{y,\varphi}, r^{y,\varphi}\rangle$ is an accepting run of $\mathcal{A}_\psi^\varphi$, then $\langle T, V\rangle, x \models \varphi$. Thus, in particular, since $\langle T_r^{\epsilon,\psi}, r^{\epsilon,\psi}\rangle$ is an accepting run of $\mathcal{A}_\psi^\psi$, we have $\langle T, V\rangle, \epsilon \models \psi$.

Let $y \in T_r$ with $r(y) = (x, \varphi)$. The proof is by induction on the structure of the formula φ.

- Suppose φ is an atomic proposition $p \in \Pi$. If $\langle T_r^{y,p}, r^{y,p}\rangle$ is an accepting run of $\mathcal{A}_\psi^p$, then $\rho(p, V(x)) = \mathsf{true}$, so $p \in V(x)$ and $\langle T, V\rangle, x \models p$.
- Similarly if φ is of the form $\neg p \in \Pi$. If $\langle T_r^{y,\neg p}, r^{y,\neg p}\rangle$ is an accepting run of $\mathcal{A}_\psi^{\neg p}$, then $\rho(\neg p, V(x)) = \mathsf{true}$, so $p \notin V(x)$ and $\langle T, V\rangle, x \models \neg p$.
- If φ is a conjunction $\varphi = \varphi_1 \wedge \varphi_2$, then $\langle T_r^{y,\varphi}, r^{y,\varphi}\rangle$ accepting means that $\rho(\varphi_1 \wedge \varphi_2, V(x)) = \rho(\varphi_1, V(x)) \wedge \rho(\varphi_2, V(x)) = \theta$ is satisfied by some set $Q \subseteq \{0, \ldots, k^n{-}1\} {\times} \mathrm{cl}(\psi)$. But then $\langle T_r^{y,\varphi_1}, r^{y,\varphi_1}\rangle$ is an accepting run of $\mathcal{A}_\psi^{\varphi_1}$, using Q as the required set to satisfy $\rho(\varphi_1, V(x))$. Then by the induction hypothesis, $\langle T, V\rangle, x \models \varphi_1$. Likewise $\langle T, V\rangle, x \models \varphi_2$, hence $\langle T, V\rangle, x \models \varphi_1 \wedge \varphi_2$, hence $\langle T, V\rangle, x \models \varphi$.
- If φ is a disjunction $\varphi = \varphi_1 \vee \varphi_2$, then $\langle T_r^{y,\varphi}, r^{y,\varphi}\rangle$ accepting means that $\rho(\varphi_1 \vee \varphi_2, V(x)) = \rho(\varphi_1, V(x)) \vee \rho(\varphi_2, V(x)) = \theta$ is satisfied by some set $Q \subseteq \{0, \ldots, k^n - 1\} \times \mathrm{cl}(\psi)$. But then at least one of $\rho(\varphi_1, V(x))$ or $\rho(\varphi_2, V(x))$ is satisfied by Q, so at least one of $\langle T_r^{y,\varphi_1}, r^{y,\varphi_1}\rangle$ or $\langle T_r^{y,\varphi_2}, r^{y,\varphi_2}\rangle$ is an accepting run of $\mathcal{A}_\psi^{\varphi_1}$ or $\mathcal{A}_\psi^{\varphi_2}$ respectively. By the induction hypothesis then, $\langle T, V\rangle, x \models \varphi_1$ or $\langle T, V\rangle, x \models \varphi_2$, giving $\langle T, V\rangle, x \models \varphi_1 \vee \varphi_2$, hence

$\langle T, V\rangle, x \models \varphi$.

- If φ is of the form $\varphi = \langle\langle A\rangle\rangle \bigcirc \varphi'$, and $\langle T_r^{y,\varphi}, r^{y,\varphi}\rangle$ accepting, we have $\rho(\langle\langle A\rangle\rangle \bigcirc \varphi', V(x)) = \bigvee_{\sigma\in\Delta_A}(\bigwedge_{c\in \mathrm{out}(\sigma)}(c,\varphi'))$ satisfied by Q. So for some $\sigma \in \Delta_A$, $\{(c,\varphi') \mid c \in \sigma\} \subseteq Q$. Also, $r(c) = (c,\varphi')$ for each $c \in \sigma$, and thus $\langle T_r^{y\cdot c,\varphi'}, r^{y\cdot c,\varphi'}\rangle$ is an accepting run of $\mathcal{A}_\psi^{\varphi'}$ for each $c \in \sigma$, since sub-runs are accepting. By the induction hypothesis then, $\langle T, V\rangle, x\cdot c \models \varphi'$ for all $c \in \sigma$. The A-move $\sigma \in \Delta_A$ now corresponds to an A-move $\sigma \in D_A(x)$ such that for all $x \cdot c \in \mathrm{out}(\sigma)$ we have $\langle T, V\rangle, x\cdot c \models \varphi'$. Then $\langle T, V\rangle, x \models \langle\langle A\rangle\rangle \bigcirc \varphi'$, hence $\langle T, V\rangle, x \models \varphi$.
- If φ is of the form $\varphi = \neg\langle\langle A\rangle\rangle \bigcirc \varphi'$, and $\langle T_r^{y,\varphi}, r^{y,\varphi}\rangle$ accepting, we have $\rho(\neg\langle\langle A\rangle\rangle \bigcirc \varphi', V(x)) = \bigwedge_{\sigma\in\Delta_A}(\bigvee_{c\in \mathrm{out}(\sigma)}(c,\varphi'))$ satisfied by Q. So for each $\sigma \in \Delta_A$, there is some $c \in \mathrm{out}(\sigma)$ such that $(c,\varphi') \in Q$. Also, $r(c) = (c, \neg\varphi')$ and thus $\langle T_r^{y\cdot c,\neg\varphi'}, r^{y\cdot c,\neg\varphi'}\rangle$ is an accepting run of $\mathcal{A}_\psi^{\neg\varphi'}$.

  By the inductive hypothesis then, $\langle T, V\rangle, x\cdot c \models \neg\varphi'$. The choice of an element $c \in \mathrm{out}(\sigma)$ for each A-move $\sigma \in \Delta_A$ now corresponds to an co-A-move $\sigma^c \in D_A^c(x)$ such that for all $x\cdot c \in \mathrm{out}(\sigma^c)$ we have $\langle T, V\rangle, x\cdot c \models \neg\varphi'$. Then $\langle T, V\rangle, x \models \neg\langle\langle A\rangle\rangle \bigcirc \varphi'$, hence $\langle T, V\rangle, x \models \varphi$.
- If φ is of the form $\varphi = \langle\langle A\rangle\rangle \varphi_1 \mathcal{U} \varphi_2$, we construct an A-strategy $F_A : T \to \Delta_A$ as follows:

  Consider the run $\langle T_r^{y,\varphi}, r^{y,\varphi}\rangle$. Every path in $T_r^{y,\varphi}$ is such that at the root node, $\rho(\varphi_2, V(x))$ is satisfied, or $\rho(\varphi_1 \wedge \langle\langle A\rangle\rangle \bigcirc \varphi, V(x))$ is satisfied. If the latter is the case, there is some $\sigma_x \in \Delta_A$ such that $\langle\langle A\rangle\rangle \bigcirc \varphi$ is satisfied, and the same property holds at the next node of the path. However along each path eventually $\rho(\varphi_2, V(x))$ is satisfied, for otherwise the path would remain in the state $\langle\langle A\rangle\rangle \bigcirc \varphi$ forever. Such a path λ will only have one state in $\inf(\lambda)$, with $\inf(\lambda) \cap F = \varnothing$. (Acceptable infinite paths enter states $\neg\langle\langle A'\rangle\rangle \varphi_1' \mathcal{U} \varphi_2'$ or $\langle\langle A\rangle\rangle \Box \varphi'$ infinitely often.) We define $F_A(x) = \sigma_x$ at each of the nodes $x \in T_r^{y,\varphi}$ where $\rho(\langle\langle A\rangle\rangle \bigcirc \varphi, V(x))$ is satisfied.
- Formulae of the form $\neg\langle\langle A\rangle\rangle \Box \varphi'$ provide the other eventualities. For these we construct a co-A-strategy similar to the A-strategy construction above.
- The cases $\neg\langle\langle A\rangle\rangle \varphi_1 \mathcal{U} \varphi_2$ and $\langle\langle A\rangle\rangle \Box \varphi'$ are just invariants, and the construction follows directly from the inductive construction.

5.4 Correctness II - Completeness

We now prove that $\mathcal{A}_\psi$ is complete, i.e. given a k^n-branching labelled tree $\langle T, V\rangle$, such that $\langle T, V\rangle \models \psi$, we prove that $\mathcal{A}_\psi$ accepts $\langle T, V\rangle$, with accepting run $\langle T_r, r\rangle$.

We define $\langle T_r, r\rangle$ as follows:

The run starts from the initial state, so $\epsilon \in T_r$ and $r(\epsilon) = (\epsilon, \psi)$. We now proceed to add nodes to the run, ensuring that for all nodes $y \in T_r$ that are added, if $r(y) = (x, \varphi)$ we have that $\langle T, V\rangle, x \models \varphi$. This property holds for node ϵ by the assumption that $\langle T, V\rangle, \epsilon \models \psi$.

Consider now a node $y \in T_r$ with $r(y) = (x, \varphi)$ and $\langle T, V\rangle, x \models \varphi$. We have that $\rho(\varphi, V(x)) = \theta$ with $\theta \in \mathcal{B}^+(\{0, \ldots, k^n - 1\} \times \mathrm{cl}(\psi))$. We need to show that there exists a set $Q = \{(c_1, \varphi_1), (c_2, \varphi_2), \ldots, (c_p, \varphi_p)\} \subseteq \{0, \ldots, k^n - 1\} \times \mathrm{cl}(\psi)$ such that Q satisfies θ and for all $1 \leq i \leq p$ we have that $y \cdot i \in T_r$ and $r(y \cdot i) = (x \cdot c_i, \varphi_i)$, and $\langle T, V\rangle, x \cdot c_i \models \varphi_i$. During the construction of $\langle T_r, r\rangle$ we also associate with each node in T_r, a strategy or co-strategy for some of the eventualities in $\mathrm{cl}(\psi)$. Finally we show that the selection procedure ensures that all infinite paths λ satisfy $\inf(\lambda) \cap F \neq \varnothing$.

The proof that some set Q can be selected at a node $y \in T_r$ with $r(y) = (x, \varphi)$ is by induction on the structure of φ. The inductive hypothesis is that if $\langle T, V\rangle, x \models \varphi$ then there is some set Q that satisfies $\rho(\varphi, V(x))$ and that $Q = \{(c_1, \varphi_1), (c_2, \varphi_2), \ldots, (c_p, \varphi_p)\}$ is such that $\langle T, V\rangle, x \cdot c_i \models \varphi_i$. The successors of y in T_r are then the nodes $y \cdot i$, with $r(y \cdot i) = (x \cdot c_i, \varphi_i)$, thus constructing the accepting run $\langle T_r, r\rangle$.

- Suppose φ is an atomic proposition $p \in \Pi$. Then $\langle T, V\rangle, x \models p$ implies that $p \in V(x)$ so $\rho(p, V(x)) = \mathsf{true}$. No elements are needed in Q in this case.
- Likewise if φ is of the form $\neg p$ for $p \in \Pi$.
- If φ is a conjunction of the form $\varphi_1 \wedge \varphi_2$ then $\rho(\varphi, V(x)) = \rho(\varphi_1, V(x)) \wedge \rho(\varphi_2, V(x))$. $\langle T, V\rangle, x \models \varphi$ implies $\langle T, V\rangle, x \models \varphi_1$ and $\langle T, V\rangle, x \models \varphi_2$. Thus, there are sets Q_1 and Q_2 that satisfy $\rho(\varphi_1, V(x))$ and $\rho(\varphi_2, V(x))$ respectively. But then $Q_1 \cup Q_2$ satisfies $\rho(\varphi, V(x))$ (since these are $\mathcal{B}^+$-expressions). For all $(c_i, \varphi_i) \in Q_1$ we also have $\langle T, V\rangle, x \cdot c_i \models \varphi_i$ and, likewise, for $(c_i, \varphi_i) \in Q_2$ we have $\langle T, V\rangle, x \cdot c_i \models \varphi_i$. Thus, for the union $Q_1 \cup Q_2$ this property will hold.
- If φ is a disjunction of the form $\varphi_1 \vee \varphi_2$ then $\rho(\varphi, V(x)) = \rho(\varphi_1, V(x)) \vee \rho(\varphi_2, V(x))$. Then $\langle T, V\rangle, x \models \varphi$ implies $\langle T, V\rangle, x \models \varphi_1$ or $\langle T, V\rangle, x \models \varphi_2$. Without loss of generality, assume the former disjunct holds. Then $\langle T, V\rangle, x \models \varphi_1$ implies that there is a sets Q that satisfies $\rho(\varphi_1, V(x))$. But then this same set Q also satisfies $\rho(\varphi, V(x))$. As for the conjunctive case, Q is such that for all $(c_i, \varphi_i) \in Q$ we have $\langle T, V\rangle, x \cdot c_i \models \varphi_i$.
- If φ is of the form $\langle\!\langle A\rangle\!\rangle \bigcirc \varphi'$ then $\langle T, V\rangle, x \models \langle\!\langle A\rangle\!\rangle \bigcirc \varphi'$ implies that for some $\sigma \in \Delta_A$ all $c \in \mathrm{out}(\sigma)$ is such that $\langle T, V\rangle, x \cdot c \models \varphi'$. Then $Q = \{(c, \varphi') \mid c \in \mathrm{out}(\sigma)\}$ is the required set that satisfies $\rho(\langle\!\langle A\rangle\!\rangle \bigcirc \varphi', V(x)) = \bigvee_{\sigma \in \Delta_A} (\bigwedge_{c \in \mathrm{out}(\sigma)} (c, \varphi'))$.
- Similarly if φ is of the form $\neg\langle\!\langle A\rangle\!\rangle \bigcirc \varphi'$ then $\langle T, V\rangle, x \models \neg\langle\!\langle A\rangle\!\rangle \bigcirc \varphi'$ implies the existence of a co-A-move that leads to an appropriate set Q.
- If φ is of the form $\varphi = \langle\!\langle A\rangle\!\rangle \varphi_1 \mathcal{U} \varphi_2$ then $\langle T, V\rangle, x \models \varphi$ implies that $\langle T, V\rangle, x \models \varphi_2$ or ($\langle T, V\rangle, x \models \varphi_1$ and $\langle T, V\rangle, x \models \langle\!\langle A\rangle\!\rangle \bigcirc \varphi$). In the case where $\langle T, V\rangle, x \models \varphi_2$, by the inductive hypothesis there is a set Q that satisfies $\rho(\varphi_2, V(x))$. But then this same set Q also satisfies $\rho(\varphi, V(x))$ and for all $(c_i, \varphi_i') \in Q$,

$\langle T, V\rangle, x \cdot c_i \models \varphi'_i$.

Alternatively, we have that $\langle T, V\rangle, x \models \varphi_1$ and $\langle T, V\rangle, x \models \langle\langle A\rangle\rangle \bigcirc \varphi$. This gives an associated set Q_1 that satisfies $\rho(\varphi_1, V(x))$, and for $(c_i, \varphi'_i) \in Q_1$, $\langle T, V\rangle x \cdot c_i \models \varphi'_i$. For the second part ($\langle T, V\rangle, x \models \langle\langle A\rangle\rangle \bigcirc \varphi$), we ensure that the run is continued according to the strategy assigned to node y for the eventuality φ. If an A-strategy F_A has been assigned to y for the eventuality φ, let $\sigma = F_A(y)$. Otherwise, if no A-strategy has been assigned to y for the eventuality φ, then note that $\langle T, V\rangle, x \models \varphi$ implies there is some A-strategy F_A that witnesses the satisfaction of φ from x. Assign this strategy to y for the eventuality φ, and let $\sigma = F_A(y)$. Thus, we now have that for all $c \in \sigma$, $\langle T, V\rangle, x \cdot c \models \varphi$. Let $Q_2 = \{(c, \varphi) \mid c \in \sigma\}$. The set we require for the conjunction $\varphi_2 \wedge \langle\langle A\rangle\rangle \bigcirc \varphi$ is then given by $Q = Q_1 \cup Q_2$. We update the strategy information by assigning the same A-strategy F_A for the eventuality φ to the nodes $y \cdot c$ where $(c, \varphi) \in Q_2$.

Note that by following a fixed A-strategy, we ensure that the eventuality is fulfilled, and thus, that the state φ is not regenerated infinitely often along a specific path of the run.

- If φ is of the form $\varphi = \neg\langle\langle A\rangle\rangle \Box \varphi'$ it is handled in a similar way, except that a co-A-move dictated by a co-A-strategy is followed to select the elements in Q. Maintenance of the co-strategy information for successor nodes is similar.
- If φ is of the form $\neg\langle\langle A\rangle\rangle \varphi_1 \mathcal{U} \varphi_2$ or $\langle\langle A\rangle\rangle \Box \varphi'$ then Q is constructed according to the invariant expansions. No strategies or co-strategies need to be maintained, as the formula φ may be regenerated forever.

Finally consider any infinite path λ in the run $\langle T_r, r\rangle$ so constructed. The only states that are regenerated infinitely often are of the form $\neg\langle\langle A\rangle\rangle \varphi_1 \mathcal{U} \varphi_2$ and $\langle\langle A\rangle\rangle \Box \varphi'$. But these are exactly the states that are in the accepting set F. $\Box$

5.5 Satisfiability of ***ATL***

The alternating tree automaton construction, combined with the bounded-branching tree model property give the required satisfiability result:

Theorem 51 (Satisfiability via alternating tree automata) *An **ATL** formula is satisfiable iff the associated alternating tree automaton has a nonempty tree language.*

Finally we consider the computational complexity of the resulting automata-based decision procedure.

Theorem 52 (Complexity of Satisfiability for ATL) *The satisfiability problem for **ATL** (over a fixed finite set of players) is complete for exponential time.*

PROOF. By the bounded-branching tree model property, we have that if an **ATL** formula ψ is satisfiable in an CGS, then it is satisfiable in a bounded branching labelled tree, with branching degree a polynomial in the length of the formula (as we assume a fixed finite set of players). Furthermore, by the **ATL** formula tree automaton theorem there is an Alternating Büchi Tree Automaton that will accept exactly the bounded branching labelled trees that satisfy ψ, with the number of states linear in the length of the formula (since $|\mathrm{cl}(\psi)| = O(|\psi|)$). Finally, the nonemptiness problem for the constructed alternating tree automaton is decidable in exponential time. Thus, the satisfiability problem for ψ is in exponential time. This complexity is also the lower bound: **ATL** contains **CTL** as a fragment, and the problem of testing satisfiability for **CTL** is EXPTIME-complete [4]. $\square$

Remark 53 *We should make clear that the automata-based proof of our complexity result applies only to the decision problem for the language defined over a fixed finite set of players Σ. Over an unbounded set of players, the branching degree may increase exponentially with the length of the formula, as the set of players referred to grows. In this case the time complexity of the presented automata-based decision procedure would no longer be (singly) exponential in the length of the formula.*

Considering the language over an unbounded set of players may sometimes be appropriate, but gives rise to a number of subtly different satisfiability problems. In subsequent work by Walther et al. [14] these issues are considered in more detail. The exponential-time decision procedure for ***ATL*** *presented there is based on a type elimination procedure using constructions similar to those used to prove completeness of the axiomatization in the present work.*

6 Concluding remarks

We have presented a sound and complete axiomatic system, and described an automata-theoretic decision procedure for the Alternating-time Temporal Logic.

The axiomatic system builds on the axiomatization of the Coalition Logic from [11], and adds the fixed point axioms for the temporal operators, carried over from the logic **CTL**. Likewise, the proof of completeness combines the local game-like aspects with fixed point characterizations of the temporal operators.

By describing the relationship between Alternating-time Temporal Logic and alternating tree automata, we formalise the connection between these two forms of alternation and demonstrate the ability of alternating tree automata to elegantly express game-like properties, also in multi-player situations with explicit coalitional aspects.

We note that the language of **ATL**, and all results obtained here, can be strengthened by replacing $\langle\langle A\rangle\rangle\Box\varphi$ with the binary operation $\mathcal{R}$ (*Release*), dual to *Until*, in terms of which $\langle\langle A\rangle\rangle\Box\varphi$ is definable as $\langle\langle A\rangle\rangle\bot\mathcal{R}\varphi$.

Future work in this field may include extension of the results to the extended logics **ATL*** and the alternating μ-Calculus, both defined in [2]. In this regard a concern is that completeness proofs of the corresponding axiomatizations for **CTL*** and the modal μ-Calculus are already highly complicated. However elegant automata-theoretic decision procedures exist for both. The automata-theoretic approach defined here for **ATL** may also be used to provide a unified approach to model checking and satisfiability checking for the logic.

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