Chapter 4
Algorithmic Correspondence for Relevance Logics I. The Algorithm PEARL

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Second Reader

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Dedicated to Alasdair Urquhart, on the occasion of his 75th birthday.

Abstract We apply and extend the theory and methods of algorithmic correspondence theory for modal logics, developed over the past 20 years, to the language $\mathcal{L}_R$ of relevance logics with respect to their standard Routley-Meyer relational semantics. We develop the non-deterministic algorithmic procedure PEARL for computing first-order equivalents of formulae of the language $\mathcal{L}_R$, in terms of that semantics. PEARL is an adaptation of the previously developed algorithmic procedures SQEMA (for normal modal logics) and ALBA (for distributive and non-distributive modal logics). We then identify a large syntactically defined class of *inductive formulae* in $\mathcal{L}_R$, analogous to previously defined such classes in the classical, distributive and non-distributive modal logic settings, and show that PEARL succeeds for every inductive formula and correctly computes a first-order definable condition which is equivalent to it with respect to frame validity. We also provide a detailed comparison with two earlier works, each extending the class of Sahlqvist formulae to relevance logics, and show that both are subsumed by simple subclasses of inductive formulae.

Keywords: Relevance logic, Routley-Meyer relational semantics, algorithmic correspondence, inductive formulae.

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4.1 Introduction

This paper brings together two important areas of active development in non-classical logics, viz. *relevance logics* and *algorithmic correspondence theory*. Since it is intended mainly for readers competent in relevance logics, but not necessarily so much in correspondence theory, we focus in this introduction on the latter topic by first providing a brief overview.

Overview of algorithmic correspondence theory

One of the classical results in modal logic since the invention of the possible worlds semantics was *Sahlqvist's theorem*[1] [26], which makes the following two important claims for all formulae from a certain syntactically defined class (subsequently called *Sahlqvist formulae*), including the modal principles appearing in axioms of the most important systems of normal modal logics:

(i) *FO correspondence*: all Sahlqvist formulae define conditions on Kripke frames that are also definable in the corresponding first-order language (FO), and

(ii) *Completeness via canonicity*: all normal modal logics axiomatized with Sahlqvist formulae are complete with respect to the class of Kripke frames that they define, because these logics are *canonical*, i.e. valid in their respective canonical frames.

That result set the stage for the emergence and development of the so called *correspondence theory in modal logic*, cf. [4]. Over the past 20 years that theory has been expanded significantly in at least three directions:

- The class of formulae covered by Sahlqvist's theorem was extended considerably to the class of so called *inductive formulae*[2] introduced first in [19] and [20], and further extended and refined in [9] and in [21], where all inductive formulae in the basic multi-modal languages and in some important extensions were proved both first-order definable and canonical, thus extending Sahlqvist's theorem.
- The method for eliminating propositional variables from modal formulae and computing their first-order equivalents was substantially extended in and made algorithmic in a series of papers developing *algorithmic correspondence theory* implemented by the algorithmic procedure SQEMA [10], which not only provably succeeds in computing the first-order equivalents of all inductive (and, in particular, all Sahlqvist) formulae, but also automatically proves their canonicity, just by virtue of succeeding on them. That enabled an algorithmic, and easily automatizable, approach to proving completeness of numerous old and new modal logics studied in the literature.

[1] Essentially the same result was also proved independently by van Benthem in his PhD thesis [3].

[2] The name refers to an inductive procedure for computing the *minimal valuations* of the occurring propositional variables that are to be computed in the right order and substituted in the formula to obtain the first-order frame condition defined by it.

The algorithm SQEMA was further extended to polyadic and hybrid modal languages in [11] and strengthened further in [12], [8], [13].

- Both the traditional and the algorithmic correspondence theory were subsequently developed further and extended significantly. First, SQEMA was generalized to the algorithm ALBA introduced in [14] to cover the inductive formulae for distributive modal logic, which respectively generalize the Sahlqvist formulae of [18]. This was extended to a wide range of logics including e.g. the intuitionistic modal mu-calculus [6] and non-normal modal logics [23], and ultimately to any logic algebraically captured by classes of normal (possibly non-distributive) lattice expansions [15]. As is evident from this list, this line of research has a strong algebraic flavour, and the reason for this is that, even while they pertain to relational semantics, the underlying mechanisms on which Sahlqvist-style results turn are ultimately order-theoretic. The line of research which develops this insight, and to which the cited papers belong, has been dubbed 'unified correspondence' [7].

Still, the scope and popularity of correspondence theory has remained mostly confined to modal logics in a broader sense, and to some extent to intuitionistic logics, whereas its use and impact in relevance logics have remained rather limited and largely unexplored, with just a couple of works, mainly [27] and [2], defining and exploring Sahlqvist-type formulae for relevance logics. Also, in the context of relevant algebras and their topological dual-spaces, in [29] Urquhart presents a correspondence result for algebraic inequalities built with fusion as the only operation. In [29], he also notes that "*Correspondence theory in the case of modal and intuitionistic logic has been extensively studied, but the analogous theory for the case of relevant logics is surprisingly neglected.*" This is indeed rather surprising, given that much work has been done in relevance logics (starting with the original paper [24] introducing the Routley-Meyer semantics, cf. also the classic book [25], as well as [17]) to identify the first-order conditions defined by numerous axioms of various systems of relevance logics, as well as proving their completeness with respect to several types of semantics, including Urquhart's semilattice semantics [28] and the Routley-Meyer relational semantics. Because of the more complex semantics, this kind of calculations can be significantly more involved than for modal logics with their standard Kripke semantics.

Notably, it turns out that the idea of inductive formulae is much more relevant (no pun intended) to relevance logics than to modal logics. This is because almost all important modal logic principles that are first-order definable and canonical fall in the smaller, but much better known, class of Sahlqvist formulae, whereas this is not the case for the important axioms of relevance logics. Almost all of these axioms turn out to be inductive, but only some of them are of Sahlqvist type, in terms of the natural analogue of Sahlqvist formulae for relevance logics. Briefly, this is because of the natural nesting of relevance implications, as well as fusions, in such axioms.

Thus, we argue that algorithmic correspondence theory is very naturally applicable and potentially quite useful for relevance logics. That was the main motivation of carrying out the present work.

Contributions of this paper

This work extends and adapts the theory and methods of algorithmic correspondence to relevance logics in the context of the Routley-Meyer relational semantics. We develop here a non-deterministic algorithmic procedure PEARL (acronym for Propositional variables Elimination Algorithm for Relevance Logic) for computing first-order equivalents in terms of frame validity of formulae of the language $\mathcal{L}_R$ for relevance logics. PEARL is an adaptation of the above mentioned procedures SQEMA [10] (for normal modal logics) and ALBA [14, 15] (for distributive and non-distributive modal logics). We define a large syntactically defined class of *inductive relevance formulae* in $\mathcal{L}_R$ and show that PEARL succeeds for all such formulae and correctly computes their first-order equivalents with respect to frame validity. We also provide a detailed comparison with the two closest earlier works on the topic, viz. [27] and [2], each extending the class of Sahlqvist formulae to relevance logics. We show that both are subsumed by a simple subclass of our inductive formulae.

We regards this work as initial exploration of algorithmic correspondence for relevance logics. There is much more to be done. Some extensions and continuations of the present work, like adding modal operators, are fairly routine. Others, such as proving canonicity of all formulae on which PEARL succeeds (in particular, all inductive formulae) are more involved, but seem feasible. These we intend to do in a follow-up part II of this work. Another non-trivial task is the development of algorithmic correspondence for Urquhart's semilattice semantics for relevance logics [28].

The structure of the paper

In the preliminary section 4.2 we summarise the basics of the Routley-Meyer relational semantics for relevance logics. In Section 4.3 we present the algorithmic procedure PEARL for computing first-order correspondents of formulae of relevance logic and prove its soundness with respect to the first-order equivalents that it computes for the frame conditions defined by the input formulae. Then, in Section 4.4 we define the class of inductive formulae for relevance logics, prove that PEARL succeeds on all inductive formulae, and compare with the classes of Sahlqvist formulae previously defined in the literature, viz. in [2] and [27], showing that they are all subsumed by subclasses of inductive formulae. We end with brief concluding remarks and directions for further work in Section 4.5. At the end of the paper we have added two appendices: Appendix 4.A with some proofs, and auxiliary Appendix 4.B with some axioms for relevance logics that we have copied there from [25] as a source of important examples of inductive formulae, for reference, and for the reader's convenience.

4.2 Preliminaries

We assume basic familiarity with the syntax and relational semantics of modal and relevance logics, general references for which are e.g. [5] (for modal logics) and [25], [17] (for relevance logics), from where we quote some of the definitions below and give a few additional definitions, not explicitly mentioned there.

4.2.1 Syntax and Routley-Meyer relational semantics for relevance logics

Hereafter we consider the language of propositional relevance logics $\mathcal{L}_R$ over a fixed set of propositional variables VAR containing the classical connectives $\wedge$, $\vee$, plus the relevant connectives **fusion** $\circ$, **(relevant) negation** $\sim$, **(relevant) implication** $\rightarrow$, and the special constant **(relevant) truth t**. The formulae of $\mathcal{L}_R$ are defined as expected:

$$A = p \mid \mathbf{t} \mid \sim A \mid (A \wedge A) \mid (A \vee A) \mid (A \circ A) \mid (A \rightarrow A)$$

where $p \in$ VAR.

A **relevance frame** is a tuple $\mathcal{F} = \langle W, O, R, ^* \rangle$, where:

- W is a non-empty set of states (possible worlds);
- $O \subseteq W$ is the subset of **normal** states;
- $R \subseteq W^3$ is a **relevant accessibility relation**;
- $^* : W \longrightarrow W$ is a function, called the **Routley star**, used to provide semantics for $\sim$.

The following binary relation $\leq$ is defined in every relevance frame:

$$u \leq v \text{ iff } \exists o(o \in O \wedge Rouv)$$

A **Routley-Meyer frame** (for short, **RM-frame**) is a relevance frame satisfying the following conditions for all $u, v, w, x, y, z \in W$:

1. $x \leq x$
2. If $x \leq y$ and $Ryuv$ then $Rxuv$.
3. If $x \leq y$ and $Ruyv$ then $Ruxv$.
4. If $x \leq y$ and $Ruvx$ then $Ruvy$.
5. If $x \leq y$ then $y^* \leq x^*$.
6. O is upward closed w.r.t. $\leq$, i.e. if $o \in O$ and $o \leq o'$ then $o' \in O$.

These properties ensure that $\leq$ is reflexive and transitive, hence a preorder, and that the semantics of the logical connectives has the monotonicity properties stated further. For the sake of comparing the definitions and results related to Sahlqvist formulae, here we have adopted the definition of Routley-Meyer frame as in [2].

Note that in the original paper [24] introducing the Routley-Meyer semantics, and in many subsequent sources, O is assumed to be an upwards closed set generated by a single element 0. Also, [24] and others assume that the Routley star * is an involution, i.e. $x^{**} = x$. We will not make either of these assumptions here. However, $\leq$ can be assumed to be a partial order (as originally assumed in [24]) w.l.o.g., since adding antisymmetry does not change the notion of frame validity.

A **Routley-Meyer model** (for short, **RM-model**) is a tuple $\mathcal{M} = \langle W, O, R, ^*, V \rangle$, where $\langle W, O, R, ^* \rangle$ is a Routley-Meyer frame and $V : \mathsf{VAR} \to \wp W$ is a mapping, called a **relevant valuation**, assigning to every atomic proposition $p \in \mathsf{VAR}$ a set $V(p)$ of states *upward closed* w.r.t. $\preceq$.

Truth of a formula A **in a RM-model** $\mathcal{M} = \langle W, O, R, ^*, V \rangle$ at a state $u \in W$, denoted $\mathcal{M}, u \Vdash A$, is defined as follows:

- $\mathcal{M}, u \Vdash p$ iff $u \in V(p)$;
- $\mathcal{M}, u \Vdash \mathbf{t}$ iff there is $o \in O$ such that $o \preceq u$; equivalently, iff $u \in O$;
- $\mathcal{M}, u \Vdash {\sim} A$ iff $\mathcal{M}, u^* \nVdash A$;
- $\mathcal{M}, u \Vdash A \land B$ iff $\mathcal{M}, u \Vdash A$ and $\mathcal{M}, u \Vdash B$;
- $\mathcal{M}, u \Vdash A \lor B$ iff $\mathcal{M}, u \Vdash A$ or $\mathcal{M}, u \Vdash B$;
- $\mathcal{M}, u \Vdash A \to B$ iff for every v, w such that $Ruvw$, if $\mathcal{M}, v \Vdash A$ then $\mathcal{M}, w \Vdash B$.
- $\mathcal{M}, u \Vdash A \circ B$ iff there exist v, w such that $Rvwu$, $\mathcal{M}, v \Vdash A$ and $\mathcal{M}, w \Vdash B$.

For every RM-model $\mathcal{M}$ and formula A we define the **extension of** A **in** $\mathcal{M}$ as

$$[[A]]_{\mathcal{M}} := \{u \in \mathcal{M} \mid \mathcal{M}, u \Vdash A\}$$

A formula A is declared:

- **true in an RM-model** $\mathcal{M}$, denoted by $\mathcal{M} \Vdash A$, if $O \subseteq [[A]]_{\mathcal{M}}$, i.e., $\mathcal{M}, o \Vdash A$ for every $o \in O$.
- **valid in an RM-frame** $\mathcal{F}$, denoted by $\mathcal{F} \Vdash A$, iff it is true in every RM-model over that frame.
- **RM-valid**, denoted by $\Vdash A$, iff it is true in every RM-model.

Remark 4.1. In [28], Urquhart proposed the well-known semilattice (or, operational) semantics for the relevant connectives. The states in these models can be thought of as pieces of information that can support assertions and can be combined. This combination of pieces of information imposes a natural (join) semilattice structure. In particular, a piece of information α supports an implication $\phi \to \psi$ (notation $\alpha \Vdash \phi \to \psi$) iff whenever we combine the α with any piece of information β which supports ϕ ($\beta \Vdash \phi$) the combination will support ψ ($\alpha \cdot \beta \Vdash \psi$).

An important property of this semantics is *Monotonicity*: for every RM-model $\mathcal{M}$ and formula A, the set $[[A]]_{\mathcal{M}}$ is $\preceq$*-upward closed*.

A formula A of $\mathcal{L}_R$ not containing variables will be called a **constant formula**. Clearly, the truth of a constant formula in a RM-model does not depend on the valuation, i.e., the extension $[[A]]_{\mathcal{M}}$ is the same for every RM-model $\mathcal{M}$ based on a

RM-frame $\mathcal{F}$, so we will identify it with validity in $\mathcal{F}$ and denote it by $[[A]]_{\mathcal{F}}$. Then, $\mathcal{F}, u \Vdash A$ iff $O \subseteq [[A]]_{\mathcal{F}}$.

Formulae A and B from $\mathcal{L}_R$ are: **semantically equivalent**, hereafter denoted $A \equiv B$, iff they are true at the same states in every RM-model; **RM-model-equivalent**, if they are true at the same RM-models; **RM-frame-equivalent**, if they are valid in the same RM-frames. Hereafter, 'equivalent formulae of $\mathcal{L}_R$' will mean 'semantically equivalent formulae', unless otherwise specified.

Clearly, Routley-Meyer frames are first-order structures for the first-order language with unary predicate symbol O, unary function symbol $*$, ternary relation symbol R, and individual variables $x_1, x_2, x_3, \ldots$, informally denoted x, x', x'' etc. We will call this language FO_R. Moreover, the semantics of relevance logic can be transparently expressed in FO_R and every relevance formula is then equivalently translated into a formula in FO_R by the following **standard translation** $ST : \mathcal{L}_R \to \mathrm{FO}_R$, parametric in a first-order individual variable:

$$
\begin{aligned}
ST_x(p) &= P(x) \\
ST_x(\mathbf{t}) &= O(x) \\
ST_x({\sim} A) &= \exists x'(x' = x^* \wedge \neg ST_{x'}(A)) \\
ST_x(A \wedge B) &= ST_x(A) \wedge ST_x(B) \\
ST_x(A \vee B) &= ST_x(A) \vee ST_x(B) \\
ST_x(A \circ B) &= \exists x'x''(Rx'x''x \wedge ST_{x'}(A) \wedge ST_{x''}(B)) \\
ST_x(A \to B) &= \forall x'x''(Rxx'x'' \wedge ST_{x'}(A) \to ST_{x''}(B))
\end{aligned}
$$

where x' and x'' are fresh individual variables.

It is routine to check that for every Routley-Meyer model $\mathcal{M}$, state w in $\mathcal{M}$ and $\mathcal{L}_R$-formula A, it holds that $\mathcal{M}, w \Vdash A$ iff $\mathcal{M} \models ST_x(A)[x := w]$, where $[x := w]$ indicates that the free variable x in $ST_x(A)$ is interpreted as w.

Positive and negative occurrences of logical connectives and propositional variables in a formula A **of** $\mathcal{L}_R$ are defined inductively on the structure of formulae, by technically treating propositional variables both as formulae and as unary (identity) connectives, as follows:

- When $A = \mathbf{t}$, the constant $\mathbf{t}$ occurs positively in the formula A.
- When $A = p$, the variable p occurs positive in the formula A.
- When $A = {\sim} B$, all positive (resp. negative) occurrences of connectives (incl. variables) in the subformula B are negative (resp. positive) occurrences of these connectives in A, and the occurrence of $\sim$ as a main connective is positive in A.
- When $A = B \bullet C$, where $\bullet \in \{\wedge, \vee, \circ\}$, all positive (resp. negative) occurrences of connectives (incl. variables) in the subformulae B and C are also positive (resp. negative) occurrences of these connectives in A. Besides, the occurrence of $\bullet$ as a main connective is positive in A.
- When $A = B \to C$, all positive (resp. negative) occurrences of connectives (incl. variables) in the subformula C are also positive (resp. negative) occurrences of these connectives in A, whereas all positive (resp. negative) occurrences of

connectives (incl. variables) in the subformula B are negative (resp. positive) occurrences of these connectives in A. Besides, the occurrence of $\rightarrow$ as a main connective is positive in A.

We say that a **formula** $A \in \mathcal{L}_R$ **is positive (resp., negative) in a propositional variable** p iff all occurrences of p in A are positive (resp., negative).

A routine inductive argument over the structure of formulae shows that, if a formula A in $\mathcal{L}_R$ is positive in p, then its induced semantic operation $A_p^V(X)$ from $\mathcal{P}^{\uparrow}(W)$ into $\mathcal{P}^{\uparrow}(W)$, is monotone (i.e., order preserving), whereas it is antitone (i.e., order-reversing) if A is negative in p. Further, we will simply say that $A(p)$ is monotone (antitone) in p if $A_p^V(X)$ is monotone (antitone).

4.2.2 Complex algebras of RM-frames

The **complex algebra** of a Routley-Meyer frame $\mathcal{F} = \langle W, R, *, O\rangle$ is the structure $\mathcal{F}^+ = \langle \mathcal{P}^{\uparrow}(W), \cap, \cup, \rightarrow, \circ, \sim, O\rangle$ where $\mathcal{P}^{\uparrow}(W)$ is the set of all upwards closed subsets (hereafter called **up-sets**) of W, $\cap$ and $\cup$ are set-theoretic intersection and union, and for all $Y, Z \in \mathcal{P}^{\uparrow}(W)$ the following hold:

$Y \rightarrow Z = \{x \in W \mid \forall yz \in W, \text{ if } Rxyz \text{ and } y \in Y, \text{ then } z \in Z\},$

$Y \circ Z = \{x \in W \mid \exists y, z \in W, Ryzx \text{ and } y \in Y \text{ and } z \in Z\},$

$\sim Y = \{x \in W \mid x^* \notin Y\}.$

Note that for any RM-model $\mathcal{M} = \langle W, O, R, ^*, V\rangle$ based upon the RM-frame $\mathcal{F}$, and all $A, B \in \mathcal{L}_R$, the family $\mathcal{M}^+ = \{V(A) \mid A \in \mathcal{L}_R\}$ is a subalgebra of $\mathcal{F}^+$ and the following hold:

- $[[\mathbf{t}]]_{\mathcal{M}} = O.$
- $[[\sim A]]_{\mathcal{M}} = \sim[[A]]_{\mathcal{M}}.$
- $[[A \wedge B]]_{\mathcal{M}} = [[A]]_{\mathcal{M}} \cap [[B]]_{\mathcal{M}},$
- $[[A \vee B]]_{\mathcal{M}} = [[A]]_{\mathcal{M}} \cup [[B]]_{\mathcal{M}},$
- $[[A \rightarrow B]]_{\mathcal{M}} = [[A]]_{\mathcal{M}} \rightarrow [[B]]_{\mathcal{M}},$
- $[[A \circ B]]_{\mathcal{M}} = [[A]]_{\mathcal{M}} \circ [[B]]_{\mathcal{M}},$

The following proposition states properties of the complex operations $\rightarrow$, $\circ$ and $\sim$ that are easy to verify and will be used further.

Proposition 4.1. *For every complex algebra* $\mathcal{F}^+ = \langle \mathcal{P}^{\uparrow}(W), \cap, \cup, \rightarrow, \circ, \sim, O\rangle$ *of an RM-frame,* $X \in \mathcal{P}^{\uparrow}(W)$ *and family* $\{Y_i \mid i \in I\} \subseteq \mathcal{P}^{\uparrow}(W)$*, the following hold.*

1. $X \rightarrow \bigcap_{i\in I} Y_i = \bigcap_{i\in I}(X \rightarrow Y_i),$
2. $\bigcup_{i\in I} Y_i \rightarrow X = \bigcap_{i\in I}(Y_i \rightarrow X),$
3. $X \circ \bigcup_{i\in I} Y_i = \bigcup_{i\in I}(X \circ Y_i),$
4. $\bigcup_{i\in I} Y_i \circ X = \bigcup_{i\in I}(Y_i \circ X).$
5. $\sim\bigcup_{i\in I} Y_i = \bigcap_{i\in I}(\sim Y_i)$
6. $\sim\bigcap_{i\in I} Y_i = \bigcup_{i\in I}(\sim Y_i)$

In fact, every complex algebra $\mathcal{F}^+ = \langle \mathcal{P}^{\uparrow}(W), \cap, \cup, \rightarrow, \circ, \sim, O\rangle$, is a complete and perfect distributive right-residuated magma with a constant O and a unary DeMorgan operation $\sim$(see e.g.[22]). These algebras are called 'relevance algebras' by Urquhart [29], although he also includes lattice bounds and requires O to be a left identity of $\circ$.

Two families of elements of $\mathcal{P}^{\uparrow}(W)$ will be of particular interest to us, namely the set $J(\mathcal{F}^+) = \{\uparrow x \mid x \in W\}$ of all **principal up-sets** $\uparrow x = \{y \in W \mid y \geq x\}$, and the set $M(\mathcal{F}^+) = \{(\downarrow x)^c \mid x \in W\}$ of set-theoretic compliments of principal downwards closed subsets (hereafter called **co-downsets**). The families $J(\mathcal{F}^+)$ and $M(\mathcal{F}^+)$ consist, respectively, of exactly the join- and meet-irreducible elements of the lattice $\langle \mathcal{P}^{\uparrow}(W), \cap, \cup\rangle$ (see e.g. [16]). They have some easy to prove but important properties, summarised in the next proposition, which will be used further.

Proposition 4.2. *For every pre-ordered set* $(W, \prec)$ *and* $X \in \mathcal{P}^{\uparrow}(W)$*, the following hold.*

1. *For any up-set* $X \in \mathcal{P}^{\uparrow}(W)$*:*
   a. *X can be written as the union of elements of* $J(\mathcal{F}^+)$*,*
   viz. $X = \bigcup\{\uparrow x \mid x \in X\}$*. Thus,* $\mathcal{P}^{\uparrow}(W)$ *is* $\cup$***-generated by*** $J(\mathcal{F}^+)$.
   b. *X can be written as the intersection of elements of* $M(\mathcal{F}^+)$*,*
   viz. $X = \bigcap\{(\downarrow x)^c \mid x \notin X\}$*. Thus,* $\mathcal{P}^{\uparrow}(W)$ *is* $\cap$***-generated by*** $M(\mathcal{F}^+)$.
2. *For any* $x \in W$ *and family* $\{X_i \mid i \in I\} \subseteq \mathcal{P}^{\uparrow}(W)$*:*
   a. $\uparrow x \subseteq \bigcup_{i\in I} X_i$ *iff* $\uparrow x \subseteq X_{i_0}$ *for some* $i_0 \in I$*.*
   Thus, every element of $J(\mathcal{F}^+)$ *is* ***completely*** $\cup$***-prime****.*
   b. $(\downarrow x)^c \supseteq \bigcap_{i\in I} X_i$ *iff* $(\downarrow x)^c \supseteq X_{i_0}$ *for some* $i_0 \in I$*.*
   Thus, every element of $M(\mathcal{F}^+)$ *is* ***completely*** $\cap$***-prime****.*

4.3 PEARL: a calculus for computing first-order correspondents of formulae of relevance logic

In this section we present a calculus of rewrite rules, in the style of the algorithms SQEMA [10] and ALBA [14, 15], which is sound and complete for deriving first-order frame correspondents for a large class of formulae of $\mathcal{L}_R$, viz. the class of *inductive (relevance) formulae* defined in Section 4.4. As we will show later, the class of inductive formulae substantially extends the classes of Sahlqvist formulae in $\mathcal{L}_R$ defined in [27] and [2] and almost all axioms of important systems of relevance logic listed in Appendix 4.B (copied there from [25]) are inductive formulae, while many of them are not Sahlqvist.

4.3.1 The extended language $\mathcal{L}_R^+$

Here we extend the language $\mathcal{L}_R$ to $\mathcal{L}_R^+$ by adding connectives which are related, as residuals or adjoints, to some of the connectives in $\mathcal{L}_R$. In particular, we add the left adjoint $\sim^\flat$ and the right adjoint $\sim^\sharp$ of $\sim$, the Heyting implication $\Rightarrow$ (as right residual of $\wedge$), the co-Heyting implication $\prec$ as the left residual of $\vee$ and the operation $\hookrightarrow$ as the residual of $\circ$ in the second coordinate and of $\rightarrow$ in the first coordinate. $\mathcal{L}_R^+$ will be the working language of the algorithm PEARL. For its purpose we also include in $\mathcal{L}_R^+$ two (countably infinite) sets, $\mathsf{NOM} = \{\mathbf{j}_0, \mathbf{j}_1, \mathbf{j}_2, \ldots\}$ and $\mathsf{CNOM} = \{\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2, \ldots\}$, of special variables, respectively called **nominals** and **co-nominals**. Informally, we will denote nominals by $\mathbf{i}, \mathbf{j}, \mathbf{k}$, possibly with indices, while co-nominals will be denoted by $\mathbf{m}, \mathbf{n}$, possibly with indices. To distinguish visually from $\mathcal{L}_R$, the formulae of the extended language $\mathcal{L}_R^+$ will be denoted by lowercase Greek letters, typically $\alpha, \beta, \gamma, \phi, \psi, \xi$, etc. and are defined by the following grammar:

$$\phi = p \mid \mathbf{i} \mid \mathbf{m} \mid \top \mid \bot \mid \mathbf{t} \mid {\sim}\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \circ \phi)$$
$$\mid (\phi \rightarrow \phi) \mid {\sim^\flat}\phi \mid {\sim^\sharp}\phi \mid (\phi \prec \phi) \mid (\phi \Rightarrow \phi) \mid (\phi \hookrightarrow \phi)$$

where $p \in \mathsf{VAR}$, $\mathbf{i} \in \mathsf{NOM}$ and $\mathbf{m} \in \mathsf{CNOM}$. We denote $\mathsf{ATOMS} := \mathsf{VAR} \cup \mathsf{NOM} \cup \mathsf{CNOM}$. The elements of ATOMS will be called **atoms**.

The additional connectives of $\mathcal{L}_R^+$ are interpreted in the same Routley-Meyer models as $\mathcal{L}_R$, except that the notion of valuation need to be adjusted so that instead of $V : \mathsf{VAR} \rightarrow \wp W$, we have $V : \mathsf{ATOMS} \rightarrow \wp W$ and V maps nominals to principal up-sets and co-nominals to compliments of principal down-sets, i.e., for all $\mathbf{i} \in \mathsf{NOM}$ and all $\mathbf{m} \in \mathsf{CNOM}$ we have $V(\mathbf{i}) = {\uparrow}w$ for some $w \in W$ and $V(\mathbf{m}) = ({\downarrow}v)^c$ for some $v \in W$. The semantics of the additional connectives of $\mathcal{L}_R^+$ are given as follows:

- $\mathcal{M}, w \Vdash \mathbf{i}$ iff $w \in V(\mathbf{i})$
- $\mathcal{M}, w \Vdash \mathbf{m}$ iff $w \in V(\mathbf{m})$
- $\mathcal{M}, w \Vdash \top$
- $\mathcal{M}, w \nVdash \bot$
- $\mathcal{M}, w \Vdash {\sim^\flat}\phi$ iff there is a v such that $v^* = w$ and $\mathcal{M}, v \nVdash \phi$
- $\mathcal{M}, w \Vdash {\sim^\sharp}\phi$ iff for all v such that $v^* = w$, it is the case that $\mathcal{M}, v \nVdash \phi$.
- $\mathcal{M}, w \Vdash \phi \prec \psi$ iff there exists v such that $v \leq w$, $\mathcal{M}, v \Vdash \phi$ and $\mathcal{M}, v \nVdash \psi$
- $\mathcal{M}, w \Vdash \phi \Rightarrow \psi$ iff for all $v \geq w$, if $\mathcal{M}, v \Vdash \phi$ then $\mathcal{M}, v \Vdash \psi$
- $\mathcal{M}, w \Vdash \phi \hookrightarrow \psi$ iff for all $v, u \in W$, if $Rvwu$ and $\mathcal{M}, v \Vdash \phi$ then $\mathcal{M}, u \Vdash \psi$

Under the assumption that * is an involution, i.e. that $w^{**} = w$ for all $w \in W$, the clauses for $\sim^\flat$ and $\sim^\sharp$ become

- $\mathcal{M}, w \Vdash {\sim^\flat}\phi$ iff $\mathcal{M}, w^* \nVdash \phi$ iff $\mathcal{M}, w \Vdash {\sim}\phi$ and
- $\mathcal{M}, w \Vdash {\sim^\sharp}\phi$ iff $\mathcal{M}, w^* \nVdash \phi$ iff $\mathcal{M}, w \Vdash {\sim}\phi$.

The standard translation ST can be extended to the language $\mathcal{L}_R^+$. For that purpose we will add sets of individual variables $\{y_0, y_1, y_2, \ldots\}$ and $\{z_0, z_1, z_2, \ldots\}$ to be used for the translations of nominals and co-nominals, respectively. We extend the translation with the following clauses:

$$
\begin{aligned}
ST_x(\mathbf{j}_i) &= x \geq y_i \\
ST_x(\mathbf{m}_i) &= \neg(x \leq z_i) \\
ST_x(\top) &= x = x \\
ST_x(\bot) &= \neg(x = x) \\
ST_x({\sim}^{\flat}\phi) &= \exists x'((x')^* = x \wedge \neg ST_{x'}(\phi)) \\
ST_x({\sim}^{\sharp}\phi) &= \forall x'((x')^* = x \rightarrow \neg ST_{x'}(\phi)) \\
ST_x(\phi \prec \psi) &= \exists x'(x' \leq x \wedge ST_{x'}(\phi) \wedge \neg ST_{x'}(\psi)) \\
ST_x(\phi \Rightarrow \psi) &= \forall x'(x' \geq x \wedge ST_{x'}(\phi) \rightarrow ST_{x'}(\psi)) \\
ST_x(\phi \hookrightarrow \psi) &= \forall x' \forall x''(Rx'xx'' \wedge ST_{x'}(\phi) \rightarrow ST_{x''}(\psi))
\end{aligned}
$$

where x', x'' are fresh individual variables, and $x \leq x'$ is shorthand for $\exists x''(O(x'') \wedge R(x''xx'))$.

The definition of **positive and negative occurrences of logical connectives, propositional variables, nominals and co-nominals** is extended to $\mathcal{L}_R^+$-formulae ϕ in the expected way. In particular:

- When $\phi = \mathbf{i}$ ($\phi = \mathbf{m}$ / $\top$ / $\bot$), the nominal $\mathbf{i}$ (co-nominal $\mathbf{m}$ / constant $\top$ / constant $\bot$) occurs positively in the formula ϕ.
- When $\phi = {\sim}^{\flat}\psi$ or $\phi = {\sim}^{\sharp}\psi$, all positive (resp. negative) occurrences of connectives (incl. variables, constants, nominals and co-nominals) in the subformula ψ are negative (resp. positive) occurrences of these connectives in ϕ. Besides, the occurrence of ${\sim}^{\flat}$ or ${\sim}^{\sharp}$ as the main connective is positive in ϕ.
- When $\phi = \psi \prec \chi$, all positive (resp. negative) occurrences of connectives (incl. variables, constants, nominals and co-nominals) in the subformula ψ are also positive (resp. negative) occurrences of these connectives in ϕ, whereas all positive (resp. negative) occurrences of connectives (incl. variables) in the subformula χ are negative (resp. positive) occurrences of these connectives in ϕ. Besides, the occurrence of $\prec$ as a main connective is positive in ϕ.
- When $\phi = \psi \Rightarrow \chi$, all positive (resp. negative) occurrences of connectives (incl. variables, constants, nominals and co-nominals) in the subformula χ are also positive (resp. negative) occurrences of these connectives in ϕ, whereas all positive (resp. negative) occurrences of connectives (incl. variables) in the subformula ψ are negative (resp. positive) occurrences of these connectives in ϕ. Besides, the occurrence of $\Rightarrow$ as a main connective is positive in ϕ.
- The clause for $\phi = \psi \hookrightarrow \chi$ is verbatim the same as for $\phi = \psi \Rightarrow \chi$, but replacing $\Rightarrow$ with $\hookrightarrow$.

Extending the complex algebraic operations for the additional connectives of $\mathcal{L}_R^+$ and identifying their salient properties is quite straightforward.

Some terminology: given an atom $a \in \mathsf{ATOMS}$, two RM-valuations, V and V', in a RM-frame $\mathcal{F}$ are called a**-variants** (notation $V \approx_a V'$), if $V(b) = V'(b)$ for all $b \in \mathsf{ATOMS} \setminus \{a\}$.

The following observations are routine to verify from the semantic definitions:

Proposition 4.3. *For every RM-model $\mathcal{M}$ and formulae $\phi, \psi, \chi \in \mathcal{L}_R^+$, the following equivalences hold:*

1. $[[\sim\phi]]_{\mathcal{M}} \subseteq [[\psi]]_{\mathcal{M}}$ *iff* $[[\sim^\flat\psi]]_{\mathcal{M}} \subseteq [[\phi]]_{\mathcal{M}}$
2. $[[\phi]]_{\mathcal{M}} \subseteq [[\sim\psi]]_{\mathcal{M}}$ *iff* $[[\psi]]_{\mathcal{M}} \subseteq [[\sim^\sharp\phi]]_{\mathcal{M}}$
3. $[[\phi]]_{\mathcal{M}} \subseteq [[\psi \vee \chi]]_{\mathcal{M}}$ *iff* $[[\phi \prec \psi]]_{\mathcal{M}} \subseteq [[\chi]]_{\mathcal{M}}$
4. $[[\phi \wedge \psi]]_{\mathcal{M}} \subseteq [[\chi]]_{\mathcal{M}}$ *iff* $[[\phi]]_{\mathcal{M}} \subseteq [[\psi \Rightarrow \chi]]_{\mathcal{M}}$
5. $[[\phi \circ \psi]]_{\mathcal{M}} \subseteq [[\chi]]_{\mathcal{M}}$ *iff* $[[\phi]]_{\mathcal{M}} \subseteq [[\psi \to \chi]]_{\mathcal{M}}$
6. $[[\phi \circ \psi]]_{\mathcal{M}} \subseteq [[\chi]]_{\mathcal{M}}$ *iff* $[[\psi]]_{\mathcal{M}} \subseteq [[\phi \hookrightarrow \chi]]_{\mathcal{M}}$
7. $[[\phi]]_{\mathcal{M}} \subseteq [[\psi \to \chi]]_{\mathcal{M}}$ *iff* $[[\psi]]_{\mathcal{M}} \subseteq [[\phi \hookrightarrow \chi]]_{\mathcal{M}}$

These equivalences say that the interpretations of the respective connectives in the complex algebra are each others' (co-)residuals or adjoints.

For the purpose of the algorithm PEARL we will combine formulae of $\mathcal{L}_R^+$ into set-theoretic versions of sequents of formulae, as follows: an **inclusion** is an expression of the form $\phi \subseteq \psi$ for $\phi, \psi \in \mathcal{L}_R^+$, while a **quasi-inclusion** is an expression $\phi_1 \subseteq \psi_1, \ldots, \phi_n \subseteq \psi_n \vdash \phi \subseteq \psi$ where $\phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi_n, \phi, \psi \in \mathcal{L}_R^+$. The semantics of these expressions is as expected: an inclusion $\phi \subseteq \psi$ is true in a Routley-Meyer model $\mathcal{M}$, denoted

$$\mathcal{M} \Vdash \phi \subseteq \psi,$$

iff $[[\phi]]_{\mathcal{M}} \subseteq [[\psi]]_{\mathcal{M}}$, while a quasi-inclusion $\phi_1 \subseteq \psi_1, \ldots, \phi_n \subseteq \psi_n \vdash \phi \subseteq \psi$ is true in $\mathcal{M}$, denoted

$$\mathcal{M} \Vdash \phi_1 \subseteq \psi_1, \ldots, \phi_n \subseteq \psi_n \vdash \phi \subseteq \psi$$

iff $[[\phi_i]]_{\mathcal{M}} \nsubseteq [[\psi_i]]_{\mathcal{M}}$ for some $1 \leq i \leq n$ or $[[\phi]]_{\mathcal{M}} \subseteq [[\psi]]_{\mathcal{M}}$. Now, the notions of validity of inclusions and quasi-inclusions in a RM-frame is defined in the expected way, as validity in all RM-models on that frame.

The formulae in $\mathcal{L}_R^+$ will be treated as special type of inclusions, viz. a formula ϕ will be identified with the inclusion $\mathbf{t} \subseteq \phi$. Clearly, this identification complies with the semantics of formulae and inclusions.

4.3.2 The rules of PEARL

Here we introduce the rewrite rules of our calculus.[3] Most of these rules will be invertible, indicated by a double line.

[3] The rules introduced in this section can be seen as specializations of the rules of the general-purpose algorithm ALBA [15] to the language and semantics of relevance logic. However, the fact that the complex algebras of Routley-Meyer frames are distributive lattice expansions allows us to present these rules in a simpler style closer to that of [14] and, to some extent, [10].

Every rule applies in the context of a set on inclusions, which are either free-standing (before the First approximation rule is applied), or are in the antecedent of a quasi-inclusion (after the application of the First approximation rule).

In order to claim soundness of each rule (to be shown in Section 4.3.4), we have to specify where and how it is applicable, with the following possible options:

1. to one or two individual inclusions, taken as its premises, but only in the antecedent of a quasi-inclusion.
2. to one or two individual inclusions, taken as its premises, in any context.
3. (only for the Ackermann-rules) to the set of all inclusions in the antecedent of a quasi-inclusion.

Unless otherwise indicated, case 2 above will be assumed by default.

Monotone variable elimination rules:

$$\frac{\alpha(p) \subseteq \beta(p)}{\alpha(\bot) \subseteq \beta(\bot)}\,(\bot) \qquad \frac{\gamma(p) \subseteq \chi(p)}{\gamma(\top) \subseteq \chi(\top)}\,(\top)$$

These two rules apply to inclusions and come with the following side conditions:

- for $(\bot)$: that α is negative in p and β is positive in p.
- for $(\top)$: that γ is positive in p and χ is negative in p.

First approximation rule:

$$\frac{\phi \subseteq \psi}{\mathbf{j} \subseteq \phi,\ \psi \subseteq \mathbf{m} \vdash \mathbf{j} \subseteq \mathbf{m}}\,(\mathrm{FA})$$

where $\mathbf{j}$ is a nominal and $\mathbf{m}$ is a co-nominal not occurring in ϕ or ψ[4]. These are implicitly universally quantified over in the quasi-inclusion. This rule applies to inclusions, possibly in the context of a list of other inclusions, which it turns into quasi-inclusions.

It *does not* apply to inclusions within the antecedents of quasi-inclusions.

Approximation rules:

$$\frac{\chi \to \phi \subseteq \mathbf{m}}{\mathbf{j} \to \phi \subseteq \mathbf{m},\ \ \mathbf{j} \subseteq \chi}\,(\to\text{Appr-Left}) \qquad \frac{\chi \to \phi \subseteq \mathbf{m}}{\chi \to \mathbf{n} \subseteq \mathbf{m},\ \ \phi \subseteq \mathbf{n}}\,(\to\text{Appr-Right})$$

[4] This requirement is only needed for the inverse rule, but we impose it on both, to preserve the equivalence.

$$\frac{\mathbf{i} \subseteq \chi \circ \phi}{\mathbf{i} \subseteq \mathbf{j} \circ \phi,\ \ \mathbf{j} \subseteq \chi}\ (\circ\text{Appr-Left}) \qquad \frac{\mathbf{i} \subseteq \chi \circ \phi}{\mathbf{i} \subseteq \chi \circ \mathbf{k},\ \ \mathbf{k} \subseteq \phi}\ (\circ\text{Appr-Right})$$

These four rules apply to inclusions in the antecedents of quasi-inclusions, and have the requirement that the nominals and co-nominals introduced by them need to be *fresh*, i.e., do not occur in the derivation thus far. Thus, they are introduced as witnesses of existentially quantified inclusions.

Residuation rules:

$$\frac{\phi \subseteq \chi \vee \psi}{\phi {-\!\!<}\ \chi \subseteq \psi}\ (\vee\text{Res}) \qquad \frac{\chi \wedge \psi \subseteq \phi}{\chi \subseteq \psi \Rightarrow \phi}\ (\wedge\text{Res}) \qquad \frac{\phi \subseteq \chi \rightarrow \psi}{\phi \circ \chi \subseteq \psi}\ (\rightarrow\text{Res})$$

$$\frac{\phi \circ \psi \subseteq \chi}{\psi \subseteq \phi \hookrightarrow \chi}\ (\circ\text{Res}) \qquad \frac{\phi \subseteq \psi \rightarrow \chi}{\psi \subseteq \phi \hookrightarrow \chi}\ (\rightarrow_1\text{Res})$$

Adjunction rules

$$\frac{\phi \vee \chi \subseteq \psi}{\phi \subseteq \psi \quad \chi \subseteq \psi}\ (\vee\text{Adj}) \qquad \frac{\psi \subseteq \phi \wedge \chi}{\psi \subseteq \phi \quad \psi \subseteq \chi}\ (\wedge\text{Adj})$$

$$\frac{\sim\phi \subseteq \psi}{\sim^{\flat} \psi \subseteq \phi}\ (\sim\text{Left-Adj}) \qquad \frac{\phi \subseteq\ \sim\psi}{\psi \subseteq\ \sim^{\sharp} \phi}\ (\sim\text{Right-Adj})$$

Not to clutter the procedure with extra rules, we allow commuting the arguments of $\wedge$ and $\vee$ whenever needed before applying the rules above. Recall that, in case * is assumed to be an involution, $\sim^{\flat}$ and $\sim^{\sharp}$ both coincide with $\sim$.

Ackermann-rules

The Ackermann-rules given apply to the set of **all** inclusions in the antecedent of a quasi-inclusion, but only involve the variable (call it p) that is being eliminated.

The following conditions apply to the rules below:

- p does not occur in α,
- $\beta_1(p), \ldots, \beta_m(p)$ are positive in p, and
- $\gamma_1(p), \ldots, \gamma_m(p)$ are negative in p.
- p does not occur in any other inclusion in the antecedent of the quasi-inclusion to which the rule is applied.

Right Ackermann-rule:

$$\frac{\alpha \subseteq p,\ \ \beta_1(p) \subseteq \gamma_1(p), \ldots, \beta_m(p) \subseteq \gamma_m(p)}{\beta_1(\alpha) \subseteq \gamma_1(\alpha), \ldots, \beta_m(\alpha) \subseteq \gamma_m(\alpha)}\ (RAR)$$

Left Ackermann-rule:

$$\frac{p \subseteq \alpha,\ \ \gamma_1(p) \subseteq \beta_1(p), \ldots, \gamma_m(p) \subseteq \beta_m(p)}{\gamma_1(\alpha) \subseteq \beta_1(\alpha), \ldots, \gamma_m(\alpha) \subseteq \beta_m(\alpha)}\ (LAR)$$

Note that the rules ($\perp$) and ($\top$) are, in fact, special cases of the Ackermann-rules (RAR) and (LAR), respectively.

Simplification rules.

In the rules below Γ is a possibly empty list of inclusions.

$$\frac{\Gamma,\ \mathbf{i} \subseteq \phi\ \vdash\ \mathbf{i} \subseteq \psi}{\Gamma\ \vdash\ \phi \subseteq \psi}\ \text{(Simpl-Left)} \qquad \frac{\Gamma,\ \psi \subseteq \mathbf{m}\ \vdash\ \phi \subseteq \mathbf{m}}{\Gamma\ \vdash\ \phi \subseteq \psi}\ \text{(Simpl-Right)}$$

In the rule (Simpl-Left) the nominal $\mathbf{i}$ must not occur in ϕ, or ψ, or any inclusion in Γ. Likewise, in the rule (Simpl-Right) the co-nominal $\mathbf{m}$ must not occur in ϕ, or ψ, or any inclusion in Γ. These rules are usually applied in the post-processing, to eliminate nominals and co-nominals introduced by the approximation rules.

4.3.3 Description of PEARL

PEARL is a non-deterministic algorithmic procedure, the purpose of which is to eliminate propositional variables from inclusions, while maintaining frame validity. It consists of 3 main phases, which we will describe further and will illustrate with a running example.

We will illustrate the phases of the algorithm PEARL with the formula

$$\psi = (p \to q) \wedge (q \to r) \to (p \to r).$$

I. Pre-processing.

The algorithm starts with an input formula $\psi \in \mathcal{L}_R$, represented as the **initial inclusion** $\mathbf{t} \subseteq \psi$. In our running example, the initial inclusion is

$$\mathbf{t} \subseteq (p \to q) \wedge (q \to r) \to (p \to r).$$

Remark: more generally, the algorithm can start with any input inclusion $\phi \subseteq \psi$ in $\mathcal{L}_R^+$, with no difference in what follows; in particular, it can be applied likewise to formulae in $\mathcal{L}_R^+$.

The pre-processing considers the inclusion $\phi \subseteq \psi$ and applies the following transformations:

1. **Distribution rules**. Apply the following equivalences to surface positive (negative) occurrences of $\vee$ and negative (positive) occurrences of $\wedge$ in the left-hand (right-hand) side of the inclusion.
   $(\phi \vee \psi) \to \theta \equiv (\phi \to \theta) \wedge (\psi \to \theta)$,
   $\phi \to (\psi \wedge \theta) \equiv (\phi \to \psi) \wedge (\psi \to \theta)$,
   $(\phi \vee \psi) \circ \theta \equiv (\phi \circ \theta) \vee (\psi \circ \theta)$,
   $\phi \circ (\psi \vee \theta) \equiv (\phi \circ \psi) \vee (\phi \circ \theta)$,
   $(\phi \vee \psi) \wedge \theta \equiv (\phi \wedge \theta) \vee (\psi \wedge \theta)$,
   $\theta \wedge (\phi \vee \psi) \equiv (\theta \wedge \phi) \vee (\theta \wedge \psi)$,
   $(\phi \wedge \psi) \vee \theta \equiv (\phi \vee \theta) \wedge (\psi \vee \theta)$,
   $\theta \vee (\phi \wedge \psi) \equiv (\theta \vee \phi) \wedge (\theta \vee \psi)$,
   $\sim(\phi \vee \psi) \equiv \sim\phi \wedge \sim\psi$,
   $\sim(\phi \wedge \psi) \equiv \sim\phi \vee \sim\psi$
2. Apply the ($\vee$Adj) and ($\wedge$Adj) rules to split inclusions into two, where possible.
3. Apply the **monotone variable elimination rules** ($\top$) and ($\bot$) wherever applicable.
   Thus, the pre-processing so far may split the original inclusion into a number of inclusions, on each of which the first two phases of the algorithm proceed separately.
4. Apply the **First approximation rule** (FA) to each inclusion. As a result, each inclusion is converted into a quasi-inclusion consisting of an implication with two inclusions in the antecedent, and one inclusion in the consequent. The inclusion in the consequent contains no propositional variables and thus all steps after this point are aimed only at eliminating propositional variables from the two inclusions in the antecedent.

The purpose of that pre-processing is to get the inclusions in the right shape so that the other rules can be applied.

The applicable pre-processing in our running example consists only of the last step, applying (FA), to produce

$$\mathbf{i} \subseteq \mathbf{t},\ (p \to q) \wedge (q \to r) \to (p \to r) \subseteq \mathbf{m} \quad \vdash \quad \mathbf{i} \subseteq \mathbf{m}$$

Remark: If O is generated by a singleton, then $\mathbf{t}$ itself is a nominal, semantically speaking, which can be used instead of $\mathbf{i}$ in this step.

II. Main (elimination) phase.

In this phase the resulting system of quasi-inclusions is transformed by alternating the following two sub-phases:

1. **Applying the residuation, adjunction and approximation rules**. These rules prepare the antecedents of quasi-inclusions for the application of the Ackermann-rules. The residuation and adjunction rules are straightforward applications of properties of the operations on complex algebras. The approximation rules are a little more intricate. They are based on the $\cup$-primeness of nominals and $\cap$-primeness of co-nominals and the proof of their soundness in Section 4.3.4 will use the properties listed in Propositions 4.1 and 4.2.

   Note that the alternation of the two sub-phases is only needed because the polarity (left or right) of applications of the Ackermann-rules to the variables to be eliminated determines how the residuation, adjunction and approximation rules should be applied. If the right polarity is known or guessed in advance, there is no need to alternate; otherwise, the second sub-phase may fail because a wrong polarity was chosen, and then backtracking may be needed to change the preparation for applying the Ackermann-rules with different polarity. When dealing with inductive inclusions defined in Section 4.4.3, a *strategy* dictating the polarity in which these rules are to be applied is determined by the way in which the formula is analysed syntactically when judged to be inductive, as it will be described there. Thus, there is no need for alternation of the sub-phases when applied to such formulae.
2. **Applying the Ackermann-rules** (RAR) and (LAR) to the quasi-inclusions to eliminate propositional variables. After each application, some of the other rules may become applicable again, before the next application of the Ackermann-rules is enabled.

Eventually, the algorithm either succeeds to eliminate all variables or it reaches a stage where there are still variables but no further applications of the Ackermann-rules can be enabled. Then the algorithm fails.

Here is the elimination phase for our running example.

1. The quasi-inclusion produced in the pre-processing:

$$\mathbf{i} \subseteq \mathbf{t},\ (p \to q) \wedge (q \to r) \to (p \to r) \subseteq \mathbf{m} \ \vdash\ \mathbf{i} \subseteq \mathbf{m}$$

2. Apply ($\to$Appr-Left), and then ($\to$Appr-Right) to produce:

$$\mathbf{i} \subseteq \mathbf{t},\ \mathbf{j} \subseteq (p \to q) \wedge (q \to r),\ p \to r \subseteq \mathbf{n},\ \mathbf{j} \to \mathbf{n} \subseteq \mathbf{m} \ \vdash\ \mathbf{i} \subseteq \mathbf{m}$$

3. Apply the adjunction rule ($\wedge$Adj) to the 2nd inclusion above, to obtain:

$$\mathbf{i} \subseteq \mathbf{t},\ \mathbf{j} \subseteq p \to q,\ \mathbf{j} \subseteq q \to r,\ p \to r \subseteq \mathbf{n},\ \mathbf{j} \to \mathbf{n} \subseteq \mathbf{m}\ \vdash\ \mathbf{i} \subseteq \mathbf{m}$$

4. Apply the approximation rule ($\to$Appr-Left) to $p \to r \subseteq \mathbf{n}$ to produce:

$$\mathbf{i} \subseteq \mathbf{t},\ \mathbf{j} \subseteq p \to q,\ \mathbf{j} \subseteq q \to r,\ \mathbf{k} \subseteq p,\ \mathbf{k} \to r \subseteq \mathbf{n},\ \mathbf{j} \to \mathbf{n} \subseteq \mathbf{m}\ \vdash\ \mathbf{i} \subseteq \mathbf{m}$$

5. Apply the Ackermann-rule with respect to p to the inclusions 2 and 4 that contain it, to obtain:

$$\mathbf{i} \subseteq \mathbf{t},\ \mathbf{j} \subseteq \mathbf{k} \to q,\ \mathbf{j} \subseteq q \to r,\ \mathbf{k} \to r \subseteq \mathbf{n},\ \mathbf{j} \to \mathbf{n} \subseteq \mathbf{m}\ \vdash\ \mathbf{i} \subseteq \mathbf{m}$$

6. Apply $\circ$-residuation to $\mathbf{j} \subseteq \mathbf{k} \to q$ to obtain:

$$\mathbf{i} \subseteq \mathbf{t},\ \mathbf{j} \circ \mathbf{k} \subseteq q,\ \mathbf{j} \subseteq q \to r,\ \mathbf{k} \to r \subseteq \mathbf{n},\ \mathbf{j} \to \mathbf{n} \subseteq \mathbf{m}\ \vdash\ \mathbf{i} \subseteq \mathbf{m},$$

7. Apply again the Ackermann-rule, now to eliminate q:

$$\mathbf{i} \subseteq \mathbf{t},\ \ \mathbf{j} \subseteq (\mathbf{j} \circ \mathbf{k}) \to r,\ \mathbf{k} \to r \subseteq \mathbf{n},\ \mathbf{j} \to \mathbf{n} \subseteq \mathbf{m}\ \vdash\ \mathbf{i} \subseteq \mathbf{m}.$$

8. Applying $\circ$-residuation to $\mathbf{j} \subseteq (\mathbf{j} \circ \mathbf{k}) \to r$ produces

$$\mathbf{i} \subseteq \mathbf{t},\ \ \mathbf{j} \circ (\mathbf{j} \circ \mathbf{k}) \subseteq r,\ \mathbf{k} \to r \subseteq \mathbf{n},\ \mathbf{j} \to \mathbf{n} \subseteq \mathbf{m}\ \vdash\ \mathbf{i} \subseteq \mathbf{m},$$

9. Now, r can be eliminated by on last application of the Ackermann-rule, to produce the pure quasi-inclusion

$$\mathbf{i} \subseteq \mathbf{t},\ \mathbf{k} \to \mathbf{j} \circ (\mathbf{j} \circ \mathbf{k}) \subseteq \mathbf{n},\ \mathbf{j} \to \mathbf{n} \subseteq \mathbf{m}\ \vdash\ \mathbf{i} \subseteq \mathbf{m}.$$

   Since all propositional variables have been successfully eliminated, this is the end of the elimination phase.

III. Post-processing.

This phase applies if/when the algorithm succeeds to eliminate all variables, thus ending with **pure quasi-inclusions**, containing only nominals and co-nominals, but no variables.

The purpose of the post-processing is to produce the first-order condition equivalent to the input formula. Each pure quasi-inclusion produced in the elimination phase is post-processed separately to produce a corresponding FO condition, and all these are then taken conjunctively to produce the corresponding FO condition of the input formula. So, we focus on the case of a single pure quasi-inclusion.

Computing a first-order equivalent of any pure quasi-inclusion can be done by straightforward application of the standard translation, but the result would usually be unnecessarily long and complicated. To avoid that, our post-processing of a pure quasi-inclusion starts with several optional simplification steps, involving the two simplification rules, as well as applications of residuation, adjunction and approximation rules and their inverses, wherever applicable, in order to reduce the number of nominals and co-nominals introduced in the elimination phase. Ideally, these simplification steps should end with a single pure inclusion.

The simplification sub-phase is illustrated in the running example as follows:

1. Applying the simplification rule (Simpl-Left) to the pure quasi-inclusion produced in the elimination phase:

$$\mathbf{k} \to \mathbf{j} \circ (\mathbf{j} \circ \mathbf{k}) \subseteq \mathbf{n},\ \mathbf{j} \to \mathbf{n} \subseteq \mathbf{m} \ \vdash\ \mathbf{t} \subseteq \mathbf{m}.$$

   This step would be redundant if the FA rules is accordingly modified when applied to formulae.
2. Applying the simplification rule (Simpl-Right) to the result produces:

$$\mathbf{k} \to \mathbf{j} \circ (\mathbf{j} \circ \mathbf{k}) \subseteq \mathbf{n} \ \vdash\ \mathbf{t} \subseteq \mathbf{j} \to \mathbf{n}.$$

3. Then applying residuation (→Res) on the right produces:

$$\mathbf{k} \to \mathbf{j} \circ (\mathbf{j} \circ \mathbf{k}) \subseteq \mathbf{n} \ \vdash\ \mathbf{t} \circ \mathbf{j} \subseteq \mathbf{n}.$$

4. Again applying the simplification rule (Simpl-Right) produces:

$$\mathbf{t} \circ \mathbf{j} \subseteq \mathbf{k} \to \mathbf{j} \circ (\mathbf{j} \circ \mathbf{k})$$

5. After another residuation (to reduce the nesting depth on the right) we obtain:

$$(\mathbf{t} \circ \mathbf{j}) \circ \mathbf{k} \subseteq \mathbf{j} \circ (\mathbf{j} \circ \mathbf{k})$$

The next step is to compute the first-order equivalent. First, let us re-write the resulting pure inclusion to replace the metavariables with concrete nominals and co-nominals, always picking the first ones available in the respective lists of designated variables[5]. Here is the result of rewriting our example:

$$(\mathbf{t} \circ \mathbf{j}_1) \circ \mathbf{j}_2 \subseteq \mathbf{j}_1 \circ (\mathbf{j}_1 \circ \mathbf{j}_2)$$

Recall that all nominals and co-nominals, as well as the current state of evaluation, are implicitly universally quantified. So, now we re-instate the universal quantifiers over all of them in the first step of the standard translation which associates with each nominal $\mathbf{j}_i$ the designated variable x_i denoting the element generating the up-set $\uparrow x_i$ where $\mathbf{j}_i$ is true. Likewise, the standard translation associates with each co-nominal $\mathbf{m}_i$ the designated individual variable y_i denoting the generator of the co-downset where $\mathbf{m}$ is true, i.e. $[[\mathbf{m}]] = (\downarrow y_i)^c$. Lastly, recall that the constant $\mathbf{t}$ is translated into

[5] In practice, each rule should be applied to such concrete nominals and co-nominals, but we have used metavariables to avoid the extra technical bookkeeping.

the set O, so we also universally quantify over its elements. In the running example, that step produces:

$$\forall w \forall o \in O \forall x_i \forall x_2 \big(ST_w\left((\mathbf{t} \circ \mathbf{j}_1) \circ \mathbf{j}_2 \subseteq \mathbf{j}_1 \circ (\mathbf{j}_1 \circ \mathbf{j}_2) \right)\big)$$

The standard translation will now produce a simpler first-order equivalent, but it still does not take into account the monotonicity or anti-monotonicity of the valuations of nominals and co-nominals, as well as those of the relation R and the function *. Thus, further simplifications are possible, in fact desirable, at this stage. For these one can use a simple 'post-processing simplification guide", partly completed in Table 4.1. In that table:

- $[[\psi]]$ is the extension of the pure formula ψ in the given frame, expressed as a FO formula.
- for any sets $X, Y \subseteq W$ and $w \in W$, the expression $RXYw$ is a shorthand for $\exists x \in X \exists y \in Y\, Rxyw$, respectively simplified when X or Y is a singleton.

Truth of simple pure formulae	Corresponding FO conditions
$w \Vdash {\sim}\mathbf{i}$	$x_{\mathbf{i}} \not\leq w^*$
$w \Vdash {\sim}^{\flat}\mathbf{i}$	$\exists u(u^* = w \ \&\ x_{\mathbf{i}} \not\leq u)$
$w \Vdash {\sim}^{\sharp}\mathbf{i}$	$\forall u(u^* = w \Rightarrow x_{\mathbf{i}} \not\leq u)$
$w \Vdash \mathbf{i} \circ \mathbf{j}$	$Rx_{\mathbf{i}}x_{\mathbf{j}}w$
$w \Vdash \mathbf{t} \circ \mathbf{j}$	$x_{\mathbf{j}} \leq w$
$w \Vdash \mathbf{i} \circ \mathbf{t}$	$Rx_{\mathbf{i}}Ow$
$w \Vdash \mathbf{i} \circ \psi$	$Rx_{\mathbf{i}}[[\psi]]w$
$w \Vdash \phi \circ \mathbf{j}$	$R[[\phi]]x_{\mathbf{j}}w$
$w \Vdash \phi \circ \psi$	$R[[\phi]][[\psi]]w$
$w \Vdash \mathbf{i} \to \mathbf{j}$	$\forall z(Rwx_{\mathbf{i}}z \Rightarrow x_{\mathbf{j}} \leq z)$
$w \Vdash \mathbf{i} \to \psi$	$\forall z(Rwx_{\mathbf{i}}z \Rightarrow z \in [[\psi]])$
$w \Vdash \phi \to \mathbf{j}$	$\forall z(Rw[[\phi]]z \Rightarrow x_{\mathbf{j}} \leq z)$
$w \Vdash \phi \to \psi$	$\forall z(Rw[[\phi]]z \Rightarrow z \in [[\psi]])$

Table 4.1 Post-processing simplification table

This table can be used to simplify on-the-fly the computation of the first-order equivalent, and can be applied separately to the left- and right-hand sides of the inclusion. Our example is computed as follows, using a more intuitive shorthand language (note that the quantification $\forall o \in O$ is now redundant while we still work with the constant **t**):

$$\forall w \forall x_1 \forall x_2 \big(w \Vdash (\mathbf{t} \circ \mathbf{j}_1) \circ \mathbf{j}_2 \Rightarrow w \Vdash \mathbf{j}_1 \circ (\mathbf{j}_1 \circ \mathbf{j}_2) \big)$$

$$\forall w \forall x_1 \forall x_2 \big(R[[\mathbf{t} \circ \mathbf{j}_1]]x_2 w \Rightarrow Rx_1[[(\mathbf{j}_1 \circ \mathbf{j}_2)]]w \big)$$

$$\forall w \forall x_1 \forall x_2 \big(\exists u (u \in [[\mathbf{t} \circ \mathbf{j}_1]] \ \& \ Rux_2w) \Rightarrow \exists u (u \in [[(\mathbf{j}_1 \circ \mathbf{j}_2)]] \ \& \ Rx_1uw)\big)$$

$$\forall w \forall x_i \forall x_2 \big(\exists u (x_1 \leq u \ \& \ Rux_2w) \Rightarrow \exists u (Rx_1x_2u \ \& \ Rx_1uw)\big)$$

From the above, using the anti-monotonicity of R over the first argument and the definition of R^2, we obtain:

$$\forall w \forall x_i \forall x_2 \big(Rx_1x_2w \Rightarrow R^2x_1(x_1x_2)w\big)$$

which is the semantic condition for our input formula (known as axiom B2) known from [25].

Another example is worked out in detail in Section 4.4.3. More examples are sketched in less detail in Appendix 4.B.

4.3.4 Soundness of the rules and correctness of **PEARL**

Here we prove that the procedure PEARL is *correct*, in the sense of preserving validity in any given RM-frame both ways – from the initial inclusion to the final pure quasi-inclusion and vice versa. Here is the formal claim:

Theorem 4.1 (Correctness). *If* PEARL *transforms an initial inclusion* Γ_0 *into several quasi-inclusions* $\Delta_1, \ldots, \Delta_k$ *then for every RM-frame* $\mathcal{F}$*, the following holds:* $\mathcal{F} \vdash \Gamma_0$ *iff* $\mathcal{F} \vdash \Delta_i$ *for each* $i = 1, \ldots, k$*.*

We will prove the claim by showing that every rule is *sound* in the sense of preserving the validity of the current inclusion or quasi-inclusion to which it is applied in both directions. For most of the rules the argument is quite simple and an even stronger claim can be proved, viz. that the rule preserves validity both ways between premises and the conclusions *in every RM-model*. For some of the rules, however, viz the approximation rules and the Ackermann-rules, the argument must be done globally, for the entire quasi-inclusion. We proceed with the cases according to the various types of rules described in Section 4.3.2.

Pre-processing distribution rules. The soundness of these rules follow immediately from Proposition 4.1.

Monotone variable elimination rules. These rules are sound for preservation of frame validity, i.e.: under the assumptions on α, β, γ and δ, in any Routley-Meyer frame $\mathcal{T}$, it holds that $\mathcal{T} \Vdash \alpha(p) \subseteq \beta(p)$ iff $\mathcal{T} \Vdash \alpha(\bot) \subseteq \beta(\bot)$ and $\mathcal{T} \Vdash \gamma(p) \subseteq \chi(p)$ iff $\mathcal{T} \Vdash \gamma(\top) \subseteq \chi(\top)$. Indeed, the preservation from top to bottom in both rules is immediate, by substitution of p with $\bot$, resp. $\top$. The preservation from bottom to top for ($\bot$) is by the chain of inclusions $\alpha(p) \subseteq \alpha(\bot) \subseteq \beta(\bot) \subseteq \beta(p)$, using the antitonicity of α and the monotonicity of β. Likewise for the rule ($\top$).

Splitting rules for $\wedge$ and $\vee$. These rules are trivially sound.

First approximation rule. Soundness from top to bottom follows by the transitivity of set inclusion, i.e. the fact that for any $X, Y, Z, U \in \mathcal{P}^{\uparrow}(W)$, if $X \subseteq Y$, $Z \subseteq X$ and $Y \subseteq U$, then $Z \subseteq U$. Soundness from bottom to top follows from Proposition 4.2.

Approximation rules. For each of these, consider an arbitrary RM-frame $\mathcal{F} = \langle W, O, R, ^* \rangle$ and show preservation of validity of the entire quasi-inclusion in $\mathcal{F}$ from the premise to the conclusion and vice versa, recalling that all nominals and co-nominals in Γ are universally quantified.

(→Appr-Left) For any RM-model $\mathcal{M}$ over $\mathcal{F}$ the following chain of equivalences holds, by Propositions 4.1 and 4.2:
$\mathcal{M} \Vdash \chi \to \phi \subseteq \mathbf{m}$ iff
$[[\chi \to \phi]]_{\mathcal{M}} \subseteq [[\mathbf{m}]]_{\mathcal{M}}$ iff
$[[\chi]]_{\mathcal{M}} \to [[\phi]]_{\mathcal{M}} \subseteq [[\mathbf{m}]]_{\mathcal{M}}$ iff
$\left(\bigcup\{\uparrow x \mid x \in [[\chi]]_{\mathcal{M}}\}\right) \to [[\phi]]_{\mathcal{M}} \subseteq [[\mathbf{m}]]_{\mathcal{M}}$ iff
$\bigcap\left(\{\uparrow x \to [[\phi]]_{\mathcal{M}} \mid x \in [[\chi]]_{\mathcal{M}}\}\right) \subseteq [[\mathbf{m}]]_{\mathcal{M}}$ iff
$\uparrow x_0 \to [[\phi]]_{\mathcal{M}} \subseteq [[\mathbf{m}]]_{\mathcal{M}}$ for some $x_0 \in W$, such that $x_0 \in [[\chi]]_{\mathcal{M}}$ iff
$[[\mathbf{j}]]_{\mathcal{M}'} \subseteq [[\chi]]_{\mathcal{M}'}$ and $[[\mathbf{j} \to \phi]]_{\mathcal{M}'} \subseteq [[\mathbf{m}]]_{\mathcal{M}'}$,
where the model $\mathcal{M}'$ is an $\mathbf{j}$-variant of $\mathcal{M}$ such that $[[\mathbf{j}]]_{\mathcal{M}'} = \uparrow x_0$.
Now, the soundness claim in both directions follows immediately, because the implicit existential quantification over $\mathbf{j}$ in the antecedent of the quasi-inclusion in the last step above converts into universal quantification over $\mathbf{j}$ (i.e., over all $\mathbf{j}$-variants of the starting model $\mathcal{M}$) in the entire quasi-inclusion. Thus, the quasi-inclusion with the premise inclusion in its antecedent is valid in all RM-models over $\mathcal{F}$ iff the quasi-inclusion resulting from the application of the rule to that premise in the antecedent is valid in all RM-models over $\mathcal{F}$.

(→Appr-Right) The argument is similar, by showing the following chain of equivalences, again using Propositions 4.1 and 4.2:
$\mathcal{M} \Vdash \chi \to \phi \subseteq \mathbf{m}$ iff
$[[\chi \to \phi]]_{\mathcal{M}} \subseteq [[\mathbf{m}]]_{\mathcal{M}}$ iff
$[[\chi]]_{\mathcal{M}} \to [[\phi]]_{\mathcal{M}} \subseteq [[\mathbf{m}]]_{\mathcal{M}}$ iff
$[[\chi]]_{\mathcal{M}} \to \left(\bigcap\{(\downarrow x)^c \mid x \notin [[\phi]]_{\mathcal{M}}\}\right) \subseteq [[\mathbf{m}]]_{\mathcal{M}}$ iff
$\bigcap\left(\{[[\chi]]_{\mathcal{M}} \to (\downarrow x)^c \mid [[\phi]]_{\mathcal{M}} \subseteq (\downarrow x)^c\}\right) \subseteq [[\mathbf{m}]]_{\mathcal{M}}$ iff
$[[\chi]]_{\mathcal{M}} \to (\downarrow x_0)^c \subseteq [[\mathbf{m}]]_{\mathcal{M}}$ for some $x_0 \in W$, such that $[[\phi]]_{\mathcal{M}} \subseteq (\downarrow x_0)^c$ iff
$[[\phi]]_{\mathcal{M}'} \subseteq [[\mathbf{n}]]_{\mathcal{M}'}$ and $[[\chi \to \mathbf{n}]]_{\mathcal{M}'} \subseteq [[\mathbf{m}]]_{\mathcal{M}'}$,
where the model $\mathcal{M}'$ is an $\mathbf{n}$-variant of $\mathcal{M}$ such that $[[\mathbf{n}]]_{\mathcal{M}'} = (\downarrow x_0)^c$.
Now the argument for soundness in both directions is the same as above.

(∘Appr-Left) The argument is analogous to that for (→Appr-Left).

(∘Appr-Right) The argument is analogous to that for (→Appr-Right).

Residuation rules. The soundness in both directions of the residuation rules follows immediately from Proposition 4.3.

Adjunction rules. The soundness of (∨Adj) and (∧Adj) in both directions is straightforward. The soundness of (~Left-Adj) and (~Right-Adj) in both directions follows from Proposition 4.3.

Ackermann-rules. The soundness of the Ackermann-rules is based on a $\mathcal{L}_R$-version of the so called *Ackermann lemma*, proved by Ackermann in [1] in the context of second-order logic.

We now give two versions of Ackermann's lemma, essentially stating the soundness of the two respective Ackermann-rules.

Lemma 4.1 (Right Ackermann Lemma). *Let $\alpha \in \mathcal{L}_R^+$ with $p \notin \mathsf{PROP}(\alpha)$, let $\beta_1(p), \ldots, \beta_m(p) \in \mathcal{L}_R^+$ be positive in p, and let $\gamma_1(p), \ldots, \gamma_m(p) \in \mathcal{L}_R^+$ be negative in p. Take any valuation V on an RM-frame $\mathcal{F}$. Then:*

$$\mathcal{F}, V \Vdash \beta_j(\alpha/p) \subseteq \gamma_j(\alpha/p) \textit{ for all } 1 \leq j \leq m$$

iff there exists some $V' \approx_p V$ such that

$$\mathcal{F}, V' \Vdash \alpha \subseteq p \textit{ and } \mathcal{F}, V' \Vdash \beta_j(p) \subseteq \gamma_j(p), \textit{ for all } 1 \leq j \leq m.$$

Proof. For the implication from top to bottom, let $V'(p) = V(\alpha)$. Since α does not contain p, we have $V'(\alpha) = V(\alpha) = V'(p)$. Moreover, by assumption we get, for each $1 \leq j \leq m$:

$$V'(\beta_j(p)) = V(\beta_j(\alpha/p)) \subseteq V(\gamma_j(\alpha/p)) = V'(\gamma_j(p)).$$

For the implication from bottom to top, we make use of the fact that each β_j is monotone (being positive) in p, while each γ_j is antitone (being negative) in p.

We have $V(\alpha) = V'(\alpha) \subseteq V'(p)$, hence,

$$V(\beta_j(\alpha/p)) \subseteq V'(\beta_j(p)) \subseteq V'(\gamma_j(p)) \subseteq V(\gamma_j(\alpha/p)).$$

The proof of the following version of the lemma is completely analogous.

Lemma 4.2 (Left Ackermann Lemma). *Let $\alpha \in \mathcal{L}_R^+$ with $p \notin \mathsf{PROP}(\alpha)$, let $\beta_1(p), \ldots, \beta_m(p) \in \mathcal{L}_R^+$ be positive in p, and let $\gamma_1(p), \ldots, \gamma_m(p) \in \mathcal{L}_R^+$ be negative in p. Take any valuation V on an RM-frame $\mathcal{F}$. Then,*

$$\mathcal{F}, V \Vdash \gamma_j(\alpha/p) \subseteq \beta_j(\alpha/p) \textit{ for all } 1 \leq j \leq m$$

iff there exists some $V' \approx_p V$ such that

$$\mathcal{F}, V' \Vdash p \subseteq \alpha \textit{ and } \mathcal{F}, V' \Vdash \gamma_j(p) \subseteq \beta_j(p), \textit{ for all } 1 \leq j \leq m.$$

Now, the soundness claim in both directions follows immediately, just like in the cases of the approximation rules, because the implicit existential quantification over the valuation V' is in the antecedent of the quasi-inclusion, so it converts into universal quantification over all p-variants of the valuation V in the entire quasi-inclusion. Thus, the quasi-inclusion before the application of the (Left or Right) Ackermann-rule is valid in all RM-models over $\mathcal{F}$ iff the quasi-inclusion resulting from the application of the Ackermann-rule to the antecedent of that quasi-inclusion is valid in all RM-models over $\mathcal{F}$.

Simplification rules. The soundness of these rules follows from Proposition 4.2.

This completes the proof of correctness of PEARL.

Corollary 4.1 (Correspondence). *If* PEARL *succeeds in transforming an initial inclusion* $\mathbf{t} \subseteq \phi$ *into the system of pure quasi-inclusions* $\Delta_1, \ldots, \Delta_k$ *with respective FO equivalents* $FO(\Delta_1), \ldots, FO(\Delta_k)$*, then for every RM-frame* $\mathcal{F}$*, the following holds:*

$$\mathcal{F} \Vdash \phi \;\; \textit{iff} \;\; \mathcal{F} \Vdash FO(\Delta_1) \wedge \ldots \wedge FO(\Delta_k).$$

Almost all axioms used to define important systems of relevance logics studied in the literature are first-order definable and their first-order equivalents can be computed by PEARL. Actually, almost all of them fall in the syntactic class of *inductive formulae* defined in the next section. In particular, that is the case for all axioms copied from [25] and listed in Appendix 4.B. A few of them are worked out there, while the rest we leave to the reader to verify.

4.3.5 An example of failure of **PEARL**

The only exception of a first-order definable, but non-inductive axiom, mentioned in the literature that we are currently know is the following:

$$\zeta = ((p \to p) \to q) \to q$$

which is claimed in [25] to define the following frame condition:

$$\text{(C)} \quad \forall u \exists o \in O \; Ruou$$

The algorithm PEARL, as presented here, fails on this formula. However, we also claim that the FO condition above is not equivalent to it. Indeed, it is easy to check that the formula is valid in every RM-frame satisfying that condition. However, the following simple RM-frame is a counter-example for the other direction, stating the necessity of that condition. Consider $\mathcal{F} = \langle W, O, R, ^* \rangle$, where:

$$W = \{0, 1\}; \; O = \{0\}; \; R = \{(0,0,0), (0,1,1), (1,1,1)\}; \; 0^* = 1, 1^* = 0.$$

Note that the relation $\preceq$ in $\mathcal{F}$ is the identity, so checking that this is an RM-frame is easy. We now claim that:

1. ζ is valid in every RM-model $\mathcal{M}$ over $\mathcal{F}$.
   Indeed, to check $\mathcal{M}, 0 \Vdash ((p \to p) \to q) \to q$ it suffices to note that $p \to p$ is true everywhere for any valuation, because $Ruvw$ implies $v = w$.
2. However, the frame condition (C) fails in $\mathcal{F}$ because $R101$ does not hold.

We claim that the following frame condition for ζ is the correct one:

$$\text{(C')} \quad \forall u \exists v (\forall z (Rvuz \Rightarrow u \preceq z) \;\&\; Ruvu)$$

It is easy to see that (C) implies (C'). The converse, however, does not always hold, as seen from the frame above, which satisfies (C') but not (C).

A suitable extension of PEARL that does succeed on the formula ζ and computes the frame condition above is currently under construction.

4.4 Inductive formulae for relevance logics

In this section we first define the notions of Sahlqvist inclusions and formulae in the language $\mathcal{L}_R$ and then extend these to the more general class of inductive inclusions and formulae[6] (Subsections 4.4.1 and 4.4.3). We illustrate these definitions with a number of examples and then show that PEARL successfully computes first-order correspondents for all members of these classes (Subsection 4.4.4). We conclude the section by comparing our definitions to the two other proposals for Sahlqvist formulae in relevance logic in the literature, in [2] and [27] (Subsection 4.4.5).

For the purposes of proving that PEARL successfully computes first-order correspondents of all Sahlqvist and inductive formulae, it is convenient to define these classes of formulae in terms of their *signed generation trees* (Section 4.4.2) and the existence of suitable partitions of certain branches in the latter, following e.g. [15]. However, by exploiting the special features of the syntax of $\mathcal{L}_R$, it is possible to give much simpler definitions which are more convenient for practically identifying inductive and Sahlqvist $\mathcal{L}_R$ formulae, but less suitable for generalization or for use in proofs. We start off by giving these simpler definitions in Section 4.4.1. To readers who only want to know how to identify Sahlqvist and inductive $\mathcal{L}_R$ formulae, but are not interested in the technicalities of proving the success of PEARL on these classes, we recommend reading only Subsection 4.4.1 and skipping the rest of this section.

4.4.1 Sahlqvist and Inductive formulae: practical definitions

When referring to a positive (resp., negative) occurrence of a connectives, say $\wedge$, in a formula, we will often simply write "an occurrence of $+\wedge$" (resp., "an occurrence of $-\wedge$") instead of "a positive occurrence of $\wedge$" (resp., "a negative occurrence of $\wedge$"), and similarly for the other connectives and also for the propositional variables.

Definition 4.1. An $\mathcal{L}_R$-formula is **Sahlqvist** if, for each propositional variable p, at least one of the following holds:

- no occurrence of $+p$ is in the scope of any occurrence of $-\rightarrow$ or of $+\circ$, or

[6] These definitions are specializations of the general purpose definition given in [15] for modal logics algebraically captured by classes of normal lattice-expansions, which are based purely on the order-theoretic properties of the interpretations of the connectives. We refer readers who are interested in this level of generality and in the algebraic and order-theoretic analysis of the Sahlqvist an inductive classes to that paper.

- no occurrence of $-p$ is in the scope of any occurrence of $- \rightarrow$ or of $+\circ$.

Example 4.1. Consider the formula (K) $(p \rightarrow (q \rightarrow p))$, also discussed in Examples 4.4 and 4.5. Since there are no occurrences of either $- \rightarrow$ or $+\circ$, the formula is Sahlqvist.

Example 4.2. Consider the formula $(p \rightarrow \sim p) \rightarrow \sim p$, also discussed in Examples 4.4 and 4.6. The first two occurrences of p are positive and in the scope of a $- \rightarrow$, so the first clause of Definition 4.1 does not apply. However, the last occurrence of p is the only negative one, and this is not in the scope of any $- \rightarrow$ or $+\circ$, so the second clause of Definition 4.1 is satisfied and the formula is Sahlqvist.

Notice that, if we were to delete the first negation in the formula we would obtain the formula $(p \rightarrow p) \rightarrow \sim p$ where we have both a positive and a negative occurrence of p in the scope of $- \rightarrow$, so the obtained formula would not be Sahlqvist.

Definition 4.2. A **polarity-type** over a set of propositional variables S is a map $\epsilon : S \rightarrow \{+, -\}$. For every polarity-type ϵ, we denote its **opposite polarity-type** by ϵ^{∂}, that is, $\epsilon^{\partial}(p) = +$ iff $\epsilon(p) = -$, for every $p \in S$. We will sometime talk of a **polarity-type over a formula** ϕ when we mean a polarity type over the set $var(\phi)$ of propositional variables occurring in ϕ.

A positive (negative) occurrence of variable p in a formula ϕ **agrees with a polarity type** ϵ **over** ϕ if $\epsilon(p) = +$ ($\epsilon(p) = -$), otherwise it **disagrees with** ϵ.

Definition 4.3. An occurrence of a propositional variable in a formula is in **good scope** if it is not in the scope of any occurrence of $-\circ$ or $+ \rightarrow$ which is the scope of an occurrence of $- \rightarrow$ or $+\circ$.

Definition 4.4. Given a strict partial ordering $<_{\Omega}$ over $var(\phi)$ and a polarity type ϵ over ϕ, we say that ϕ is (Ω, ϵ)**-inductive** if

1. every variable occurrence that agrees with ϵ is in good scope, and
2. if an occurrence of a variable p which agrees with ϵ is in the scope of an occurrence of $- \rightarrow$ $(+\circ)$, then all variables q occurring in the other argument of this $- \rightarrow$ $(+\circ)$ disagree with ϵ and $q <_{\Omega} p$.
3. if an occurrence of a variable p which agrees with ϵ is in the scope of an occurrence of $-\vee$ $(+\wedge)$ which is the scope of an occurrence of $- \rightarrow$ or $+\circ$, then all variables q occurring in the other argument of this $-\vee$ $(+\wedge)$ disagree with ϵ and $q <_{\Omega} p$.

The formula ϕ is said to be **inductive** if it is (Ω, ϵ)**-inductive** for some partial ordering $<_{\Omega}$ and polarity type ϵ.

Example 4.3. Consider the axiom (WB) $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$, also considered further in Example 4.7. For ease of reference we indicate the polarity of all variable and connective occurrences:

$$(\overset{+}{p} \overset{-}{\rightarrow} \overset{-}{q}) \overset{-}{\wedge} (\overset{+}{q} \overset{-}{\rightarrow} \overset{-}{r}) \overset{+}{\rightarrow} (\overset{-}{p} \overset{+}{\rightarrow} \overset{+}{r}))$$

Note that this formula is not Sahlqvist, as there is both a $-q$ and a $+q$ occurring in the scope of $-\rightarrow$. However, it is (Ω, ϵ)-inductive where $p <_\Omega q$, $r <_\Omega q$, and $\epsilon(p) = -$, $\epsilon(q) = -$, $\epsilon(r) = +$. Notice that all variable occurrences (and therefore all variable occurrences that agree with ϵ) are in good scope. As there are no occurrences of $+\circ$, $-\vee$ or $+\wedge$ we only need to check the two occurrences of $-\rightarrow$ in order to verify that Definition 4.4 is satisfied. Indeed, in $(\overset{+}{p} \overset{-}{\rightarrow} \overset{-}{q})$, the occurrence $-q$ agrees with ϵ, while $+p$, which is the other argument of $-\rightarrow$, disagrees with ϵ and moreover $p <_\Omega p$; in $(\overset{+}{q} \overset{-}{\rightarrow} \overset{-}{r})$ neither $+q$ nor $-r$ agree with ϵ.

Keeping the same choice of dependency order $<_\Omega$, other choices of ϵ under which the formula would be (Ω, ϵ)-inductive are possible, too. For example, $\epsilon(p) = -$, $\epsilon(q) = +$, $\epsilon(r) = +$, or $\epsilon(p) = \epsilon(q) = \epsilon(r) = +$ would both work. However, any choice with $\epsilon(p) = +$ and $\epsilon(q) = -$ would not work, as this would violate the condition 3 of Definition 4.4 in the subformula $(\overset{+}{p} \overset{-}{\rightarrow} \overset{-}{q})$.

4.4.2 Signed generation trees for $\mathcal{L}_R$-formulae

In this subsection we restrict all definitions to $\mathcal{L}_R$-formulae and inclusions. They can all be easily extended to $\mathcal{L}_R^+$, as well as with modal operators, but here we restrict our language of interest to $\mathcal{L}_R$.

We now define the notion of a signed-generation tree of a formula. As mentioned in the introduction to this section, for the purpose of the forthcoming proofs, the definition of Sahlqvist and inductive inclusions (and formulae) is most conveniently given in terms of these trees. This style of definition first appears in the definition of the Sahlqvist formulae for distributive modal logic [18].

Definition 4.5 (Signed Generation Tree). The **positive** (resp. **negative**) *generation tree* of any $\mathcal{L}_R$-formula ϕ is defined by labelling the root node of the generation tree of ϕ with the sign $+$ (resp. $-$), and then propagating the labelling on each remaining node as follows:

- For any node labelled with $\vee$, $\wedge$ or $\circ$, assign the same sign to its children.
- If a node is labelled with $\sim$, assign the opposite sign to its child.
- If a node is labelled with $\rightarrow$, assign the opposite sign to its left child and the same sign to its right child.

Nodes in signed generation trees are **positive** (resp. **negative**) if they are signed $+$ (resp. $-$).

Example 4.4. Fig. 4.1 shows the negative generation tree for the formula $p \rightarrow (q \rightarrow p)$, know as the axiom (K), while Figure 4.2 shows the negative generation tree of the formula $(p \rightarrow\sim p) \rightarrow\sim p$. The negative generation tree of the (WB) axiom $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is displayed in Figure 4.3.

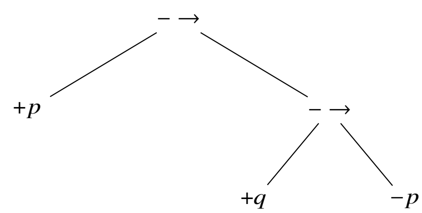


Fig. 4.1 The signed generation tree of $-[p \to (q \to p)]$. If $\epsilon(p) = -$ and $\epsilon(q) = +$, then the branches ending in $-p$ and $+q$ are ϵ-critical. All branches in the tree are excellent.

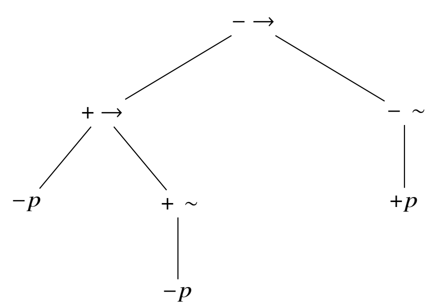


Fig. 4.2 The signed generation tree of $-[(p \to \sim p) \to \sim p]$. If $\epsilon(p) = +$, then the branch ending in $+p$ is ϵ-critical. The rightmost branch is excellent, while the leftmost and middle branches are good but not excellent.

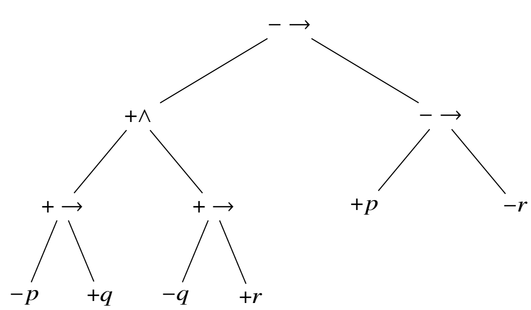


Fig. 4.3 The signed generation tree of $-[(p \to q) \wedge (q \to r) \to (p \to r)]$. If $\epsilon(p) = \epsilon(q) = \epsilon(r) = 1$, then the branches ending in $+p$, $+q$ and $+r$ are all ϵ-critical. The first four branches from the left are good, but not excellent, whereas the last two branches are excellent.

Signed generation trees will be mostly used in the context of inclusions $\phi \subseteq \psi$. In this context we will typically consider the positive generation tree $+\phi$ for the left-hand side and the negative one $-\psi$ for the right-hand side[7].

For any formula $\phi(p_1, \ldots, p_n)$, any polarity-type ϵ over $\{p_1, \ldots p_n\}$, and any $1 \leq i \leq n$, an **ϵ-critical node** in a signed generation tree of ϕ is a leaf node labelled with $\epsilon(p_i)p_i$. An **ϵ-critical branch** in the tree is a branch with an ϵ-critical leaf node. As we will see later, the variable occurrences corresponding to ϵ-critical nodes will be of special importance in the algorithm.

Hereafter we will use sg to denote $+$ or $-$. For every formula $\phi = \phi(p_1, \ldots p_n)$ and every polarity-type ϵ, we say that the signed formula $\mathsf{sg}\,\phi$ **agrees with** ϵ, and write $\epsilon(\mathsf{sg}\,\phi)$, if every leaf in the signed generation tree of $\mathsf{sg}\,\phi$ is ϵ-critical, i.e., signed in agreement with ϵ. Given an occurrence of a subformula ψ of a formula ϕ, we will write $\mathsf{sg}'\psi \prec \mathsf{sg}\,\phi$ to indicate that the sign of that occurrence of ψ in the signed generation tree $\mathsf{sg}\,\phi$ is sg'. We will also write $\epsilon(\psi) \prec \mathsf{sg}\,\phi$ (resp. $\epsilon^\partial(\psi) \prec \mathsf{sg}\,\phi$) to indicate that the signed subtree of the given occurrence of the subformula ψ of the signed tree $\mathsf{sg}\,\phi$, agrees with ϵ (resp. with ϵ^∂).

4.4.3 Sahlqvist and inductive formulae in $\mathcal{L}_R$

For what follows, we will need to distinguish three syntactic types of signed formulae, occurring as nodes in signed generation trees, depending on their sign and main connective, as indicated in Table 4.2. This terminology indicates how formulas with these main connectives occurring on the left and right of inclusions are dealt with by PEARL[8]. More precisely, a $+$-sign (resp., $-$-sign) indicates that the connective occurs on the right (left) side of an inclusion, at some point *after* first-approximation. Approximation nodes are dealt with by approximation rules (except $+\vee$ and $-\wedge$ that are dealt with by splitting with ($\vee$Adj) and ($\wedge$Adj) *before* first approximation on the left and right sides of an inclusion, respectively). Similarly, adjunction and residuation-nodes (collectively called 'inner nodes') are dealt with by adjunction and approximation rules, respectively. Note that some nodes are listed as both approximation and inner nodes. This is because approximation rules require either the left-hand side of an inclusion to be a nominal or its right-hand side to be a co-nominal, whereas residuation and adjunction rules have no such requirements. In particular, after a residuation rule has been applied, this syntactic requirement may be lost and only residuation and adjunction may be available to surface a particular variable occurrence in order to prepare for the Ackerman rule. This makes the correct

[7] The convention of considering the positive generation tree of the left-hand side and the negative generation tree of the right-hand side of an inclusion goes back to [18]. Although this convention might seem counter-intuitive at first glance, it is by now well established in this line of research, and we therefore maintain it to facilitate easier comparisons.

[8] Most papers in the unified correspondence literature, e.g.[14] and [15], classify nodes similarly, but with terminology referring not to the rules, but rather to the order-theoretic and algebraic properties of the interpretations of connectives upon which the soundness of these rules is based.

nesting order of connectives essential for the success of PEARL. For example, $+\circ$, corresponding to $\circ$ as the main connective on the right-hand side of an inclusion, can *only* be approximated, and so having it nested under $+\rightarrow$, which can only be residuated, may make it impossible to surface variable occurrences in their scope. The definitions of the Sahlqvist and inductive formulas that follow below therefore aim to stipulate sufficient conditions on the nesting order which will guarantee the success of the algorithm. In the case of the inductive formulae, the requirements on nesting imposed in the definition of the Sahlqvist formulae are relaxed, at the cost of imposing conditions on the co-occurrences (together in a subformula) of variables in the arguments of residuation nodes.

Remark 4.2. Here is an alternative, logic-based terminology and intuition for the classification of signed connectives and nodes, coming from modal logic which considers two main types of modal operators:

- **diamond-operators**, for which the semantic truth condition (for the positively signed), respectively falsity condition (for the negatively signed), are given by existential quantification over accessible worlds. In the case of $\mathcal{L}_R$ formulae, these are the positively signed fusion operator $+\circ$, as well as the negatively signed implication $-\rightarrow$, because the falsity of the implication is given by an existentially quantified semantic condition.
- **box-operators**, for which the semantic respective truth conditions are given by universal quantification over accessible worlds. In the case of $\mathcal{L}_R$ formulae, these are the positively signed implication $+\rightarrow$, as well as the negatively signed fusion operator $-\circ$, because the falsity of the fusion is given by a universally quantified semantic condition.

The signed connectives as described above can be called **proper diamonds**, respectively **proper boxes**. Besides, the propositional connectives $\wedge$, $\vee$, as well as the negation $\sim$ can be treated as either diamonds or boxes. Added to the above, these define what one can call **(generalised) diamonds and boxes**, which respectively correspond to the Approximation nodes and the Inner nodes in the table below. Still, the signed connectives of *disjunctive type*, viz. $+\vee$ and $-\wedge$, are more naturally treated as boxes, as they typically distribute over conjunctions, whereas the signed connectives of *conjunctive type*, viz. $-\vee$ and $+\wedge$, are more naturally treated as diamonds, as they typically distribute over disjunctions. This is only intuition, to help the modal-logic minded reader with remembering the types, but we emphasise that it is too coarse to serve as a viable alternative in the precise definition of the inductive formulae given below.

Definition 4.6. A branch in a signed generation tree $\mathsf{sg}\,\phi$, is called a **good branch** if it can be represented as a concatenation of two paths P_1 and P_2, any of which may possibly be of length 0, such that P_1 is a path starting from the leaf of the branch and consisting (apart from variable nodes) only of inner-nodes, and P_2 consists (apart from variable nodes) only of approximation-nodes. A good branch is **excellent** if it is a concatenation of paths P_1 and P_2 as above (each possibly empty), where only adjunction-nodes (and no residuation-nodes) can occur on P_1.

Approximation nodes	Inner nodes	
	Adjunction nodes	Residuation nodes
$+ \vee \wedge \circ \sim$	$+ \wedge \sim$	$+ \vee \to$
$- \wedge \vee \to \sim$	$- \vee \sim$	$- \wedge \circ$

Table 4.2 Types of nodes formulae in $\mathcal{L}_R$.

Definition 4.7. Given an order type ϵ and a formula $\phi = \phi(p_1, \ldots p_n)$ of $\mathcal{L}_R$, the signed generation tree $\mathsf{sg}\,\phi$ of ϕ is **ϵ-Sahlqvist** if every ϵ-critical branch is excellent. An inclusion $\phi \subseteq \psi$ is **ϵ-Sahlqvist** if both signed trees $+\phi$ and $-\psi$ are ϵ-Sahlqvist. An inclusion $\phi \subseteq \psi$ is **Sahlqvist** if it is ϵ-Sahlqvist for some ϵ. A formula ψ is **Sahlqvist** if the inclusion $\mathbf{t} \subseteq \psi$ is Sahlqvist.

Example 4.5. Consider the formula (K) mentioned in Examples 4.1 and 4.4 (Fig. 4.1), rewritten as the inclusion $\mathbf{t} \subseteq p \to (q \to p)$. The positive generation tree of $\mathbf{t}$ consists of the single node $+\mathbf{t}$ and so the only branch in this tree is trivially excellent. (This observation applies to any inclusion with $\mathbf{t}$ on the left, so we will not repeat it further.) The negative generation tree of $p \to (q \to p)$ is given in Figure 4.1, and each of the three branches in this tree is excellent, as they consist entirely of approximation-nodes. It follows that the inclusion $\mathbf{t} \subseteq p \to (q \to p)$ is ϵ-Sahlqvist for *any* polarity-type ϵ, hence the inclusion $\mathbf{t} \subseteq p \to (q \to p)$ and, consequently, the formula $p \to (q \to p)$, are both Sahlqvist.

Example 4.6. Consider the formula $(p \to \sim p) \to \sim p$, also discussed in Examples 4.4 and 4.2. It corresponds to the inclusion $\mathbf{t} \subseteq (p \to \sim p) \to \sim p$, so we only need focus on the negative generation tree of $(p \to \sim p) \to \sim p$, which is pictured in Figure 4.2. The right-most branch consists entirely of approximation-nodes, and is hence excellent, while the left-most and middle branches are good, but not excellent, as both contain the proper residuation node $(+ \to)$. Consequently, this formula is ϵ-Sahlqvist for any polarity-type ϵ with $\epsilon(p) = +$, but not for any polarity-type ϵ with $\epsilon(p) = -$.

Many other well known axioms of relevance logic are, in fact, Sahlqvist, as stated in Propositions 4.6, 4.7 and 4.8 in Appendix 4.B.

We now introduce the more general classes of *inductive formulae and inclusions for $\mathcal{L}_R$*:

Definition 4.8 (Inductive formulae and inclusions). For any polarity-type ϵ and any strict partial order Ω on $p_1, \ldots p_n$, the signed generation tree $\mathsf{sg}\,\phi$ of a formula $\phi = \phi(p_1, \ldots p_n)$ is **(Ω, ϵ)-inductive** if

1. every ϵ-critical branch is good (cf. Definition 4.6);
2. for every residuation-node occurring in a critical branch ending in leaf $\epsilon(p_i)p_i$ and labelled with a subformula of the form $\gamma \odot \beta$ or $\beta \odot \gamma$ where $\odot$ is a signed residuation-connective, the following hold:

   a. the leaf $\epsilon(p_i)p_i$ of the critical branch occurs in the subtree corresponding to the subformula β,

b. $\epsilon^\partial(\gamma) \prec \mathsf{sg}\,\phi$, i.e., the signed subtree corresponding to subformula γ has no critical leaves, and
c. $p_k <_\Omega p_i$ for every p_k occurring in γ.

Thus, clauses (b) and (c) above say that every leaf in the signed subtree corresponding to the subformula γ is non-critical (i.e. of the form $\epsilon^\partial(p_j)p_j$) with $p_j <_\Omega p_i$. (Note that we write $p_j <_\Omega p_i$ for $\Omega(p_i, p_i)$.)

We will refer to Ω as the **dependency order** on the variables.

An **inclusion** $\phi \subseteq \psi$ **is** (Ω, ϵ)**-inductive** if the signed generation trees $+\phi$ and $-\psi$ are both (Ω, ϵ)-inductive[9].

An **inclusion** $\phi \subseteq \psi$ **is inductive** if it is (Ω, ϵ)-inductive for some Ω and ϵ.

A **formula** ψ **is inductive** if the inclusion $\mathbf{t} \subseteq \psi$ is inductive.

The intuition linking this definition to PEARL is that the polarity-type ϵ tells us for which version of the Ackermann-rule to prepare: if $\epsilon(p) = +$ (respectively, $\epsilon(p) = -$) we prepare for the right (respectively, left) Ackermann-rule by 'solving for' or 'displaying' positive (negative) occurrences of p. The acyclicity of the dependency order Ω and conditions (b) and (c) guarantee that this will be possible for all variables without including an occurrence of a variable to be solved for (according to ϵ) in the formula α of the Ackermann-rule and thereby possibly substituting it (through the application of the Ackermann-rule) into a scope from which it cannot be extracted in order to be displayed.

Example 4.7. The negative generation tree of the axiom (WB) $(p \to q) \wedge (q \to r) \to (p \to r)$ was displayed in Figure 4.3. (Recall that this formula was already considered in Example 4.3.) The two branches on the right of this tree consist only of approximation-nodes, and so they are excellent. The four branches on the left (ending in leaves $-p$, $+q$, $-q$ and $+r$, respectively) are good, all consisting of the signed variable, followed by the residuation-node $(+\to)$ and then above them the approximation-nodes $(+\wedge)$ and $(-\to)$. However, because of the presence of the residuation-node, these four branches are not excellent.

This formula is *not* ϵ-Sahlqvist for any polarity-type ϵ, since neither all $+q$-nodes nor all $-q$-nodes occur as the leaves of excellent branches, so it is impossible to choose an ϵ according to which all critical branches would be excellent.

However, it is (Ω, ϵ)-inductive where $p <_\Omega q$, $r <_\Omega q$, and $\epsilon(p) = +$, $\epsilon(q) = +$, $\epsilon(r) = -$. Under this choice of $<_\Omega$ and ϵ the critical branches ending in $+p$ and $-r$ are excellent, while the critical branch ending in $+q$ is good. Moreover, at the only residuation-node $(+\to)$ occurring on the latter branch, p plays the role of the subformula γ in Definition 4.8, while β is q. In the single-node subtree $-p$ corresponding to p, the only leaf (namely $-p$) is non-critical, and $p <_\Omega q$, so the requirements of the definition are satisfied.

Keeping the same choice of dependency order $<_\Omega$, other choices of ϵ under which the formula would be (Ω, ϵ)-inductive are possible, too. For example, $\epsilon(p) = +$,

[9] For formulae, this is equivalent the notion of (Ω, ϵ)-inductiveness given in Definition 4.4, modulo considering the opposite polarity type ϵ^∂.

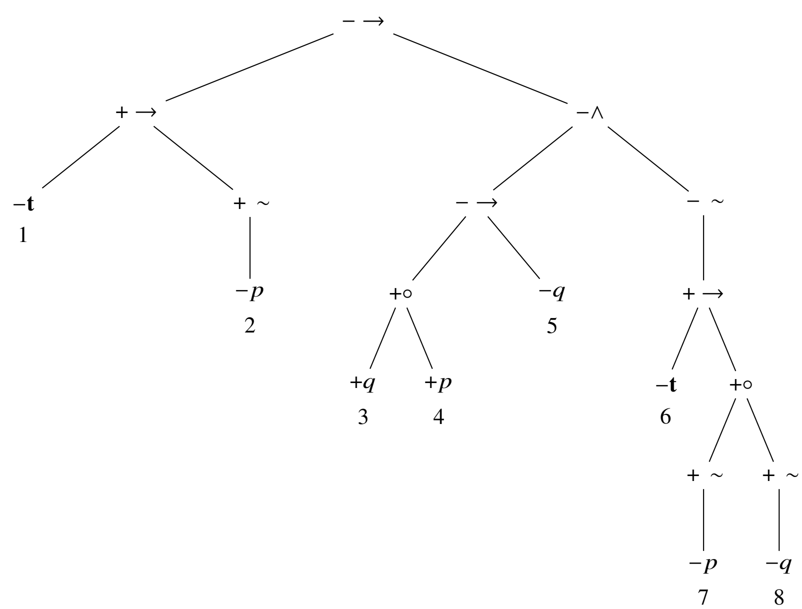
$\epsilon(q) = -$, $\epsilon(r) = -$, or $\epsilon(p) = \epsilon(q) = \epsilon(r) = -$ would both work. However, any choice with $\epsilon(p) = -$ and $\epsilon(q) = +$ would not work, as this would violate the conditions imposed on residuation-nodes occurring on critical branches.

Several well-known axioms of relevance logic are not Sahlqvist, but are inductive. See some examples in Appendix 4.B.

Example 4.8. This is an example of a $\mathcal{L}_R$-formula which requires some preprocessing which splits the execution of the algorithm into two branches.

$$\psi = (\mathbf{t} \to {\sim} p) \to (((q \circ p) \to q) \wedge {\sim}(\mathbf{t} \to ({\sim} p \circ {\sim} q))).$$

The signed generation tree $-\psi$ is given on Figure 4.8. The branches are numbered



from 1 to 8 for ease of reference. Note that ϕ is not inductive because, for example, branches 3 and 8 are not good and have leaves $+q$ and $-q$ respectively, making it impossible to find a polarity type for which all critical branches will be good. However, we will show that the preprocessing of PEARL splits this into two formulae, one of which is monotone, while the other is inductive. Indeed, PEARL takes

$$\mathbf{t} \subseteq (\mathbf{t} \to {\sim} p) \to (((q \circ p) \to q) \wedge {\sim}(\mathbf{t} \to ({\sim} p \circ {\sim} q)))$$

as input and then applies the equivalence $\phi \to (\psi \wedge \theta) \equiv (\phi \to \psi) \wedge (\psi \to \theta)$ after which the $\wedge$-split rule becomes applicable, yielding the inclusions

$$\mathbf{t} \subseteq (\mathbf{t} \rightarrow \sim p) \rightarrow ((q \circ p) \rightarrow q) \tag{4.1}$$

and

$$\mathbf{t} \subseteq (\mathbf{t} \rightarrow \sim p) \rightarrow \sim(\mathbf{t} \rightarrow (\sim p \circ \sim q)). \tag{4.2}$$

Now, the right-hand side of (4.2) is positive in both p and q while the left-hand side is vacuously negative in both these variables, so the monotone variable elimination rule ($\bot$) may be applied to transform (4.2) into

$$\mathbf{t} \subseteq (\mathbf{t} \rightarrow \sim\bot) \rightarrow \sim(\mathbf{t} \rightarrow (\sim\bot \circ \sim\bot)), \tag{4.3}$$

thereby eliminating all occurring propositional variables.

We note that (4.1) is (Ω, ϵ)-inductive for $\Omega = \varnothing$ and $\epsilon(p) = \epsilon(q) = -$. The algorithm now proceeds on (4.1) and applies the first-approximation rule to obtain

$$(\mathbf{t} \rightarrow \sim p) \rightarrow ((q \circ p) \rightarrow q) \subseteq \mathbf{m}_0 \vdash \mathbf{t} \subseteq \mathbf{m}_0.$$

Applications of the $\rightarrow$-approximation rule transforms this into

$$\mathbf{j}_1 \subseteq \mathbf{t} \rightarrow \sim p,\ \ (q \circ p) \rightarrow \mathbf{m}_2 \subseteq \mathbf{m}_1,\ \ q \subseteq \mathbf{m}_2,\ \ \mathbf{j}_1 \rightarrow \mathbf{m}_1 \subseteq \mathbf{m}_0 \vdash \mathbf{t} \subseteq \mathbf{m}_0.$$

Applying the left Ackermann-rule to eliminate q yields

$$\mathbf{j}_1 \subseteq \mathbf{t} \rightarrow \sim p,\ \ (\mathbf{m}_2 \circ p) \rightarrow \mathbf{m}_2 \subseteq \mathbf{m}_1,\ \ \mathbf{j}_1 \rightarrow \mathbf{m}_1 \subseteq \mathbf{m}_0 \vdash \mathbf{t} \subseteq \mathbf{m}_0.$$

We next solve for p by applying the $\rightarrow$-residuation and the $\sim$Right-Adj rule to produce

$$p \subseteq \sim^{\sharp}(\mathbf{j}_1 \circ \mathbf{t}),\ \ (\mathbf{m}_2 \circ p) \rightarrow \mathbf{m}_2 \subseteq \mathbf{m}_1,\ \ \mathbf{j}_1 \rightarrow \mathbf{m}_1 \subseteq \mathbf{m}_0 \vdash \mathbf{t} \subseteq \mathbf{m}_0,$$

to which the left Ackermann-rule can be applied with respect to p to produce the pure quasi-inclusion

$$(\mathbf{m}_2 \circ \sim^{\sharp}(\mathbf{j}_1 \circ \mathbf{t})) \rightarrow \mathbf{m}_2 \subseteq \mathbf{m}_1,\ \ \mathbf{j}_1 \rightarrow \mathbf{m}_1 \subseteq \mathbf{m}_0 \vdash \mathbf{t} \subseteq \mathbf{m}_0.$$

Applying a simplification rules turns this into

$$(\mathbf{m}_2 \circ \sim^{\sharp}(\mathbf{j}_1 \circ \mathbf{t})) \rightarrow \mathbf{m}_2 \subseteq \mathbf{m}_1 \vdash \mathbf{t} \subseteq \mathbf{j}_1 \rightarrow \mathbf{m}_1,$$

to the consequent of which we may apply the $\rightarrow$-residuation rule,

$$(\mathbf{m}_2 \circ \sim^{\sharp}(\mathbf{j}_1 \circ \mathbf{t})) \rightarrow \mathbf{m}_2 \subseteq \mathbf{m}_1 \vdash \mathbf{t} \circ \mathbf{j}_1 \subseteq \mathbf{m}_1,$$

from which another simplification rule application produces

$$\mathbf{t} \circ \mathbf{j}_1 \subseteq (\mathbf{m}_2 \circ \sim^{\sharp}(\mathbf{j}_1 \circ \mathbf{t})) \rightarrow \mathbf{m}_2. \tag{4.4}$$

Inclusions (4.3) and (4.4) can now be translated into first-order conditions, the conjunction of which is the first-order condition defined by ψ.

4.4.4 *Completeness of* PEARL *for all inductive formulae*

In this section we prove that PEARL successfully computes first-order frame correspondents for all inductive inclusions and formulae. The proof essentially follows the outline of the proof of the analogous result given in [15] for the more general ALBA algorithm, but specialized to the setting of relevance logic. The proofs of all the lemmas can be found in Appendix 4.A.

The following definition captures the shape of an inductive inclusion resulting from preprocessing, i.e. one in which the splitting rules have been applied to eliminate all disjunctions on the left and conjunctions on the right that could have been surfaced through distribution.

Definition 4.9. An (Ω, ϵ)-inductive inclusion is **definite** if its critical branches contain no occurrences of $+\vee$ or $-\wedge$ as approximation nodes.

Lemma 4.3. *Let* $\{\phi_i \subseteq \psi_i \mid 1 \leq i \leq n\}$ *be the set of inclusions obtained by preprocessing an* (Ω, ϵ)*-inductive inclusion* $\phi \subseteq \psi$*. Then each* $\phi_i \subseteq \psi_i$ *is a definite* (Ω, ϵ)*-inductive inclusion.*

The following definition intends to capture the state of a quasi-inclusion after approximation rules have been applied exhaustively:

Definition 4.10. Call a quasi-inclusion $\Gamma \vdash \mathbf{i} \subseteq \mathbf{m}$ (Ω, ϵ)-*stripped* if for each $\xi \subseteq \chi \in \Gamma$ the following conditions hold:

1. one of $-\xi$ and $+\chi$ is pure, and the other is (Ω, ϵ)-inductive;
2. apart from the leaves, every ϵ-critical branch in $-\xi$ and $+\chi$ consists entirely of inner-nodes.

Lemma 4.4. *For any definite* (Ω, ϵ)*-inductive inclusion* $\phi \subseteq \psi$ *the system the quasi-inclusion* $\mathbf{i} \subseteq \phi, \psi \subseteq \mathbf{m} \vdash \mathbf{i} \subseteq \mathbf{m}$ *arising from first approximation can be transformed into an* (Ω, ϵ)*-stripped quasi-inclusion by the application of approximation rules.*

Definition 4.11. An (Ω, ϵ)-stripped quasi-inclusion $\Gamma \vdash \mathbf{i} \subseteq \mathbf{m}$ is *Ackermann-ready* with respect to a propositional variable p_i with $\epsilon_i = +$ (respectively, $\epsilon_i = -$) if every inclusion $\xi \subseteq \chi \in \Gamma$ is of one of the following forms:

1. $\xi \subseteq p$ where ξ is pure (i.e., not containing propositional variables); respectively, $p \subseteq \chi$, where χ is pure, or
2. $\xi \subseteq \chi$ where neither $-\xi$ nor $+\chi$ contains any $+p_i$ (respectively, $-p_i$) leaves.

Note that the right or left Ackermann-rule (depending on whether $\epsilon_i = +$ or $\epsilon_i = -$) is applicable to a system which is Ackermann-ready with respect to p_i. In fact, this would still have been the case had we weakened the requirement that ξ and χ must be pure, to simply require that they do not contain p_i.

Lemma 4.5. *If* $\Gamma \vdash \mathbf{i} \subseteq \mathbf{m}$ *is* (Ω, ϵ)*-stripped and* p_i *is* Ω*-minimal among propositional variables occurring in* $\Gamma \vdash \mathbf{i} \subseteq \mathbf{m}$*, then* $\Gamma \vdash \mathbf{i} \subseteq \mathbf{m}$ *can be transformed, through the application of residuation- and adjunction-rules, into a system which is* Ackermann-ready *with respect to* p_i*.*

Lemma 4.6. *Applying the appropriate Ackermann-rule with respect to* p_i *to an* (Ω, ϵ)*-stripped quasi-inclusion which is Ackermann-ready with respect to* p_i*, again yields an* (Ω, ϵ)*-stripped quasi-inclusion.*

Theorem 4.2. PEARL *succeeds on all inductive inclusions.*

Proof. Let $\phi \subseteq \psi$ be an (Ω, ϵ)-inductive inclusion. By Lemma 4.3, applying preprocessing yields a finite set of definite (Ω, ϵ)-inductive inclusions, each of which gives rise to a quasi-inclusion $\mathbf{i} \subseteq \phi', \psi' \subseteq \mathbf{m} \vdash \mathbf{i} \subseteq \mathbf{m}$. By Lemma 4.4, applications of the approximation rules convert this into an (Ω, ϵ)-stripped quasi-inclusion $\Gamma \vdash \mathbf{i} \subseteq \mathbf{m}$. By Lemma 4.2, $\Gamma \vdash \mathbf{i} \subseteq \mathbf{m}$ can be made Ackermann-ready with respect to any occurring Ω-minimal variable. After applying the appropriate Ackermann-rule to eliminate this variable, the resulting quasi-inclusion is again (Ω, ϵ)-stripped, now containing one propositional variable fewer. Now, Lemma 4.2 can be applied again. This process is iterated until all occurring propositional variables are eliminated and a pure system is obtained.

The next theorem is a direct corollary of Theorems 4.1 and 4.2.

Theorem 4.3. *Every inductive* $\mathcal{L}_R$*-inclusion, hence, every inductive* $\mathcal{L}_R$*–formula, has an effectively computable first-order frame correspondent on Routley-Meyer frames.*

4.4.5 Comparing with other Sahlqvist classes in the literature

Here we compare the inductive and Sahlqvist $\mathcal{L}_R$ formulae, as we have defined them, with two other proposals in the literature. We first consider the Sahlqvist relevance formulae introduced by Guillermo Badia in [2]. It turns out that this class of formulae is incomparable with our Sahlqvist class, but forms a proper subclass of the inductive formulae. We next turn our attention to the Sahlqvist class for *modal* relevance logic introduced by Takahiro Seki in [27]. Since $\mathcal{L}_R$ is a strict sublanguage of the language of modal relevance logic, we only consider the class of formulae obtained by restricting the latter definition to the language $\mathcal{L}_R$. We will see that this class lies properly within the intersection of our Sahlqvist class with that of Badia.

We first review Badia's definition of Sahlqvist relevance formulae [2] together with all the necessary notions needed in the build-up to this definition, and subsequently compare it to our definitions of Sahlqvist and inductive $\mathcal{L}_R$-formulae. Badia defines positive and negative relevance formulae in terms of their standard translations, viz. a relevance formula A is called positive (which we will refer to as **B-positive**) if $ST_x(A)$ is a formula built up from atomic formulae involving only unary predicates and first order formulae where the only non-logical symbols are R and O, using the connectives $\exists$, $\forall$, $\wedge$ and $\vee$. Similarly, a relevance formula A is called negative (which we will refer to as **B-negative**) if $ST_x(A)$ is a formula built up from Boolean negations of atomic formulae involving only unary predicates and first-order formulae where the only non-logical symbols are R and O, using the connectives $\exists$, $\forall$, $\wedge$ and $\vee$.

It is easy to check that the B-positive $\mathcal{L}_R$ formulae are exactly those built up from propositional variables and constant formulae using $\wedge$, $\vee$ and $\circ$, while the B-negative $\mathcal{L}_R$ formulae are built up from constant formulae and negated propositional variables, using $\wedge$, $\vee$ and $\circ$. It is also easy to check that every B-positive (B-negative) formula positive (negative) in the sense of the present paper. However, reading the definitions of B-positive and B-negative formulae as purely *syntactic*, the converse would not hold. For example, $\sim(p \rightarrow \sim q)$ is positive but translates to $\neg\forall yz(Rx^*yz \wedge P(y) \rightarrow \neg Q(z^*))$ which is not B-positive, while $p \rightarrow \sim q$ is negative but translates to $\forall yz(Rxyz \wedge P(y) \rightarrow \neg Q(z*))$ which is not B-negative. We will therefore read the definition of B-positive and B-negative formulae *up to equivalence of the standard translations*. Under this assumption we can drive in negations and consider, for example, $\exists yz(Rx^*yz \wedge P(y) \wedge Q(z^*))$ instead of $\neg\forall yz(Rx^*yz \wedge P(y) \rightarrow \neg Q(z^*))$ as the translation of $\sim(p \rightarrow \sim q)$, thus rendering it B-positive. In general, it should be clear that, under this assumption, the definitions of B-positive (B-negative) and positive (negative) $\mathcal{L}_R$ formulae coincide.

We will need the notions of *relevance Sahlqvist implications* [2, Definition 3] and *dual relevance Sahlqvist implications* [2, Definition 4]. To avoid possible confusion with notions defined in the present paper, we will refer to the formulae in these classes as B-Sahlqvist implications and B-dual Sahlqvist implications, respectively.

Definition 4.12 ([2, Definitions 3 and 4]). A formula $A \rightarrow B$ is called a **B-Sahlqvist implication** if B is positive while A is a formula built up from propositional variables, double negated atoms (i.e., formulae of the form $\sim\sim p$), negative formulae, the constant $\mathbf{t}$ and implications of the form $\mathbf{t} \rightarrow p$ (for any propositional variable p) using only the connectives $\wedge$, $\vee$ and $\circ$.

A formula $A \rightarrow B$ is called a **dual B-Sahlqvist implication** if B is negative while A is a formula built up from negated propositional atoms $\sim p$, triple negated atoms $\sim\sim\sim p$, positive formulae, the constant $\sim\mathbf{t}$, and implications of the form $p \rightarrow \mathbf{t}$ (for any propositional variable p) using only the connectives $\wedge$, $\vee$ and $\circ$.

Example 4.9.

1. The formula $p \rightarrow (p \circ p)$ is a B-Sahlqvist implication. Note that it is ϵ-Sahlqvist for $\epsilon(p) = +$, but not for $\epsilon(p) = -$.
2. The formula $(p \rightarrow \mathbf{t}) \rightarrow (\sim p \circ \sim p)$ is a dual B-Sahlqvist implication. Note that it is $(\varnothing, \epsilon)$-inductive for $\epsilon(p) = -$, but not for $\epsilon(p) = +$.

The proofs of the following two lemmas can be found in Appendix 4.A.

Lemma 4.7. *Any B-Sahlqvist implication $A \rightarrow B$ is ϵ-Sahlqvist for $\epsilon(p) = +$ for all occurring propositional variables p.*

Lemma 4.8. *Any dual Sahlqvist B-implication $A \to B$ is (Ω, ϵ)-inductive for $\Omega = \varnothing$ and $\epsilon(p) = -$ for all occurring propositional variables p.*

Definition 4.13 ([2, Definitions 5]). A B-Sahlqvist formula is any formula built up from (dual) B-Sahlqvist implications, propositional variables, and negated propositional variables using $\wedge$, the operations Θ on formulae (for any propositional variable free relevance formula θ) defined by $\Theta(B) = \theta \to B$, and applications of $\vee$ where the disjuncts share no propositional variables in common.

Example 4.10. Consider the conjunction of the B-Sahlqvist implication and dual B-Sahlqvist implication considered in Example 4.9, namely $(p \to (p \circ p)) \wedge ((p \to \mathbf{t}) \to ({\sim} p \circ {\sim} p))$. By definition 4.13 this conjunction is a B-Sahlqvist formula. Note, however, that it is not inductive, as choosing $\epsilon(p) = +$ gives rise to a critical branch in the signed generation tree $-((p \to \mathbf{t}) \to ({\sim} p \circ {\sim} p))$ which is not good, while choosing $\epsilon(p) = -$ gives rise to a critical branch in the signed generation tree $-(p \to (p \circ p))$ which is not good, either.

To get us out of this impasse, it is sufficient to recall that the frame correspondent of a conjunction is the conjunction of the frame correspondents, and that we may therefore equivalently consider the formula $(p \to (p \circ p)) \wedge ((q \to \mathbf{t}) \to ({\sim} q \circ {\sim} q))$ in which the variable p has been renamed to q in the second conjunct. This formula necessarily defines the same first-order condition Routley-Meyer frames as the one we started with and, moreover, it is $(\varnothing, \epsilon)$-inductive for $\epsilon(p) = +$ and $\epsilon(q) = 1$.

Remark 4.3. We can use the equivalences $A \to (B \wedge C) \equiv (A \to B) \wedge (A \to C)$, $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ and $(B \wedge C) \vee A \equiv (B \vee A) \wedge (C \vee A)$ to equivalently reformulate Definition 4.13 as follows: a **pre-B-Sahlqvist formula** is any formula built up from (dual) B-Sahlqvist implications, propositional variables, and negated propositional variables using the operations on formulae Θ (for any propositional variable free relevance formula θ) defined by $\Theta(B) = \theta \to B$ and applications of $\vee$ where the disjuncts share no propositional variable in common. A **B-Sahlqvist formula** is any conjunction of pre-B-Sahlqvist formulae.

The following proposition shows that every B-Sahlqvist formula is an inductive formula with empty dependency order. For the sake of brevity, we will refer to inductive formulae with empty dependency order as $\varnothing$-inductive formulae.

Proposition 4.4. *Modulo renaming of variables, every B-Sahlqvist formula is an $\varnothing$-inductive formula.*

The proof is in Appendix 4.A.

Example 4.11. The (K) axiom $p \to (q \to p)$ was shown to be Sahlqvist in Example 4.5. However, note that it is not B-Sahlqvist: indeed, since the consequent $(p \to p)$ is neither positive nor negative, the formula as a whole is neither a B-Sahlqvist implication nor a B-Sahlqvist dual implication. The conclusion now follows by noting that, moreover, the main connective cannot be seen as one of the implications $\theta \to B$ allowed by Definition 4.13 where θ is a constant formula.

Example 4.12. The (WB) axiom $(p \to q) \wedge (q \to r) \to (p \to r)$ was shown to be inductive in Example 4.7. Considerations similarly to those employed in Example 4.11 can be used to see that it is not B-Sahlqvist.

In [27] Seki formulates and proves a Sahlqvist correspondence and completeness result for *modal* relevance logic. The language of modal relevance logic is obtained by adding the unary modalities $\Diamond$ and $\Box$ to $\mathcal{L}_R$. We will write $\mathcal{L}_{RM}$ to denote this language. The defined connectives $\Diamonddot$ and $\boxdot$ are abbreviations for $\sim\Box\sim$ and $\sim\Diamond\sim$, respectively. This language is interpreted on Routley-Meyer models enriched with two relations for interpreting the Diamond and Box.

In what follows we will project Seki's definition of Sahlqvist formulae for the language $\mathcal{L}_{RM}$, as well as all necessary notions leading up to that definition, onto $\mathcal{L}_R$. To prevent possible confusion with terminology, we will refer to Seki's class of Sahlqvist formulae as the S-Sahlqvist formulae and also refer to the classes of negative and positive formulae defined in [27] as S-positive and S-negative, respectively.

An $\mathcal{L}_{RM}$-formula is **S-positive** if it is built from propositional variables and **t** using $\wedge$, $\vee$, $\Box$, $\boxdot$, $\Diamond$ and $\Diamonddot$. An $\mathcal{L}_{RM}$-formula is **strongly S-positive** if it is of the form $\blacksquare^{m_1} p_1 \wedge \cdots \wedge \blacksquare^{m_k} p_k$ where $\blacksquare^n$ denotes a sequence of n of $\Box$s and $\boxdot$s. An $\mathcal{L}_{RM}$-formula is **S-negative** if it is equivalent to $\sim B$ for some S-positive formula B. An $\mathcal{L}_{RM}$-formula is **untied** if it can be constructed from S-negative formulae and strongly S-positive formulae using $\wedge$, $\Diamond$ and $\Diamonddot$.

An $\mathcal{L}_{RM}$-formula is **S-Sahlqvist** if it is equivalent to a conjunction of formulae of the form $\blacksquare^k(B \to C)$, where $k \geq 0$, B is untied and C is positive. Projecting the definition of $\mathcal{L}_{RM}$ S-Sahlqvist formulae onto $\mathcal{L}_R$, we see that an $\mathcal{L}_R$-formula is **S-Sahlqvist** iff it is equivalent to a conjunction of formulae of the form $B \to C$ where B is constructed from propositional variables using $\wedge$, $\vee$ and $\sim$only, while C is constructed from propositional variables using $\wedge$ and $\vee$ only.

Note that, modulo driving negations inwards, every S-Sahlqvist formula is a conjunction of B-Sahlqvist implications. By Lemma 4.7, the following proposition is therefore immediate:

Proposition 4.5. *Every S-Sahlqvist formula is a Sahlqvist formula.*

4.5 Concluding remarks and directions for further work

The present paper is part I of a bigger project. The forthcoming part II will explore some of the most interesting (or, at least most natural) continuations and extensions of this work, including, roughly in increasing order of projected difficulty:

- Adding modal operators and extending the methods and results to modal relevance logics, in particular covering the full class of Seki's Sahlqvist formulae. That should be fairly straightforward.

- Extending the language with all residual connectives, corresponding to permutations of the arguments in the semantic definitions. All results should apply likewise.
- Re-defining the classes of inductive formulae for plain and modal relevance logics in an alternative style, using a “flat language" of relevance logic (following the idea of using polyadic modal languages in [19] and [21]) where the logical connectives can be composed into new polyadic connectives. That would enable defining the class of inductive formulae by means of their global structure when suitably re-written in the richer language, rather than locally (‘branch’-wise’), in terms of the patterns of occurrence of each variable, as in Section 4.4.3.
- Proving canonicity of all formulae on which PEARL succeeds (in particular, all inductive formulae) would require some non-trivial technical work but it is of predictable nature.
- Strengthening PEARL to cover some known cases of FO definable relevance formulae, such as the axiom ζ discussed in Section 4.3.5, and refining it to incorporate simplification steps taking into account the semantic monotonicity conditions.
- Developing algorithmic correspondence for other semantics for relevance logics, in particular for Urquhart’s semilattice semantics. This research problem is yet to be explored.

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4.A Appendix A: Some proofs

Proof (Lemma 4.3). Notice that the distribution during preprocessing only swaps the order of approximation nodes on (critical) paths, and hence does not affect the goodness of critical branches. Moreover, inner parts are entirely unaffected, and in particular the side conditions on residuation nodes of critical branches are maintained. Finally, notice that all occurrences of $+\vee$ or $-\wedge$ in the approximation parts of critical branches can be surfaced by applying the distribution laws forming part of preprocessing and then eliminated via splitting, thus producing definite inductive inclusions.

Proof (Lemma 4.4). By assumption, $+\phi$ and $-\psi$ are both definite (Ω, ϵ)-inductive. Hence, for any ϵ-critical branch in $+\phi$ or $-\psi$, we can apply approximation rules successively to the approximation nodes on that branch, until they have all been 'stripped off' and all remaining nodes on that branch are inner nodes. To conclude the proof it suffices to note that the inclusions in the conclusions of approximation rules are always pure on at least one side.

Proof (Lemma 4.5). If $\xi \subseteq \chi \in \Gamma$ and $-\xi$ and $+\chi$ contain no ϵ-critical p_i-nodes then this inclusion already satisfies condition 2 of Definition 4.11. So suppose that $-\xi$ and $+\chi$ contain some ϵ-critical p_i-node among them. This means $\xi \subseteq \chi$ is of the form $\alpha \subseteq \mathsf{Pure}$ with the ϵ-critical p_i-node in α and Pure pure, or of the form $\mathsf{Pure} \subseteq \delta$ with Pure pure and the ϵ-critical p_i-node in δ. We can now prove by simultaneous induction on α and δ that these inclusions can be transformed into the form specified by clause 1 of definition 4.11.

The base cases are when $-\alpha = -p_i$ and $+\delta = +p_i$. Here the inclusions are in desired shape and no rules need be applied to them. We will only check here a few of the inductive cases. If $-\alpha = -(\alpha_1 \vee \alpha_2)$, then applying the $\vee$-adjunction rule we transform $\alpha_1 \vee \alpha_2 \subseteq \mathsf{Pure}$ into $\alpha_1 \subseteq \mathsf{Pure}$ and $\alpha_2 \subseteq \mathsf{Pure}$. The resulting system is clearly still (Ω, ϵ)-stripped, and we may apply the inductive hypothesis to $\alpha_1 \subseteq \mathsf{Pure}$ and $\alpha_2 \subseteq \mathsf{Pure}$.

If $-\alpha = -(\alpha_1 \circ \alpha_2)$, then, as per definition of inductive inclusions, exactly one of $-\alpha_1$ and $-\alpha_2$ contains an ϵ-critical node, and the other one is pure (here we use the Ω-minimality of p_i). Assume that the critical node is in $-\alpha_1$ and that α_2 is pure. Then, applying the $\circ$-residuation rule transforms $(\alpha_1 \circ \alpha_2) \subseteq \mathsf{Pure}$ into $\alpha_1 \subseteq \alpha_2 \rightarrow \mathsf{Pure}$, yielding an (Ω, ϵ)-stripped quasi-inclusion to which the inductive hypothesis is applicable. The cases for $-\alpha = -(\alpha_1 \wedge \alpha_2)$ and $-\alpha = - \sim \alpha_1$ can be are treated similarly with applications of the $\wedge$-residuation rule and the $\sim$-Left adjunction rule, respectively.

If $+\delta = + \sim \delta_1$, then $\mathsf{Pure} \subseteq \sim \delta_1$ becomes $\delta_1 \subseteq \sim^{\sharp}\mathsf{Pure}$ through the application of the ~Right-adjunction rule, where $-\delta_1$ is (Ω, ϵ)-inductive and $\sim^{\sharp}\mathsf{Pure}$ is pure, hence resulting in an (Ω, ϵ)-stripped quasi-inclusion to which the inductive hypothesis is applicable. The cases for $+\delta = +(\delta_1 \wedge \delta_2)$, $+\delta = +(\delta_1 \vee \delta_2)$ and $+\delta = +(\delta_1 \rightarrow \delta_2)$ are left to the reader.

Proof (Lemma 4.6). Let $\Gamma \vdash \mathbf{i} \subseteq \mathbf{m}$ be an (Ω, ϵ)-stripped quasi-inclusion which is Ackermann-ready with respect to p_i. We only consider the case in which the right Ackermann-rule is applied, the case for the left Ackermann-rule being dual. This means that $\Gamma = \{\alpha_k \subseteq p \mid 1 \subseteq k \subseteq n\} \cup \{\beta_j(p_i) \subseteq \gamma_j(p_i) \mid 1 \leq j \leq m\}$ where the αs are pure and the $-\beta$s and $+\gamma$s contain no $+p_i$ nodes. We denote the pure formula $\bigvee_{k=1}^{n} \alpha_k$ by α. It is sufficient to show that for each $1 \leq j \leq m$, the trees $-\beta(\alpha/p_i)$ and $+\gamma(\alpha/p_i)$ satisfy the conditions of Definition 4.10. Conditions 2 follows immediately once we notice that, since α is pure and is being substituted everywhere for variable occurrences corresponding to non-critical nodes, $-\beta(\alpha/p_i)$ and $+\gamma(\alpha/p_i)$ have exactly the same ϵ-critical paths as $-\beta(p_i)$ and $+\gamma(p_i)$, respectively. Condition 1, namely that $-\beta(\alpha/p_i)$ and $+\gamma(\alpha/p_i)$ are (Ω, ϵ)-inductive, also follows using additionally the observation that all new paths that arose from the substitution are variable free.

Proof (Lemma 4.7). In the negative generation tree of $A \rightarrow B$, all leaves in the signed subtree of $-B$ are either $+\mathbf{t}$ or signed negative (since B is positive) and hence there are no ϵ-critical leaves in that subtree. Note that, in A, propositional variable occurrences in the negative subformulae give rise to negatively signed leaves (and hence ϵ-non-critical branches) in $+A$, while propositional variables and double negated propositional variables (outside the negative subformulae) only occur in the scope of $\wedge$, $\vee$ and $\circ$, are therefore give rise to positively signed leaves on excellent, ϵ-critical branches (consisting of approximation nodes only). Thus, since all ϵ-critical branches in the signed generation tree $-(A \to B)$ are excellent, $A \to B$ is ϵ-Sahlqvist.

Proof (Lemma 4.8). In the negative generation tree of $A \to B$, all leaves in the signed subtree of $-B$ are either $+\mathbf{t}$ or signed positive (since B is negative) and hence there are no ϵ-critical leaves in that subtree. Note that, in A, propositional variable occurrences in the positive subformulae give rise to positively signed leaves (and hence ϵ-non-critical branches) in $+A$, while negated and triple negated propositional variables (outside the positive subformulae) only occur in the scope of $\wedge$, $\vee$ and $\circ$, are therefore give rise to positively negatively leaves on excellent, ϵ-critical branches (consisting of approximation nodes only). Subformulae of the form $p \to \mathbf{t}$ also only occur in the scope of $\wedge$, $\vee$ and $\circ$, hence the $\to$ gives rise to the residuation-node $+\to$ through which a good critical branch with leaf $-p$ runs. Moreover, since the other argument of $\to$ is the constant $\mathbf{t}$, the conditions of Definition 4.8 are trivially met.

Proof (Proposition 4.4). Let A be a B-Sahlqvist formula. We first show that any pre-B-Sahlqvist formula is a $\varnothing$-inductive formula. We do this recursively on the construction of pre-B-Sahlqvist formulae. By Lemmas 4.7 and 4.8 B-Sahlqvist implications and dual B-Sahlqvist implications are $\varnothing$-inductive formulae. Trivially, propositional variables and negations of propositional variables are $\varnothing$-inductive formulae. Suppose that B and C are $(\varnothing, \epsilon_B)$ and $(\varnothing, \epsilon_C)$-inductive, respectively and the they have no propositional variables in common, then the generation tree of $-(B \vee C)$ simply joins those $-B$ and $-C$ with a new root $-\vee$. Clearly, any good branch in $-B$ or $-C$ is now part of a longer good-branch in $-(B \vee C)$. Moreover, taking ϵ to be the union of ϵ_B and ϵ_B (since B and C have no variables in common this is a function), any residuation-node on a ϵ-critical branch is a residuation-node on an ϵ_B-critical branch in $-B$ or on an ϵ_C-critical branch in $-C$ and hence still satisfies the requirement of Definition 4.8. If follows that $B \vee C$ is $(\varnothing, \epsilon)$-inductive.

If θ is a constant formula, then the generation tree $-(\theta \to B)$ is the combination of the trees $+\theta$ and $-B$ with the approximation node $-\to$ as new root. Every good branch in the subtree $-B$ is now part of a longer good branch in $-(\theta \to B)$. Since θ contains no variables it follows that $\theta \to B$ is $(\varnothing, \epsilon_B)$-inductive.

We have thus established that every pre-B-Sahlqvist formula is a $\varnothing$-inductive.

Now, by definition, A is a conjunction of pre-B-Sahlqvist formulae $B_1, \ldots, B_n$. Assuming that the formulae $B_1, \ldots, B_n$ are pairwise variable-disjoint, it follows that $-A'(B_1, \ldots, B_n)$ is $(\varnothing, \epsilon)$-inductive when ϵ is simply the union of the ϵ_i. This assumption is justified by the fact that the frame correspondent of a conjunction is the conjunction of the frame correspondents (for relevance logic this is proved in [2, Lemma 9]) and that we may therefore, up to frame-equivalence, rename the propositional variables in the B_i to insure that they are variable-disjoint.

4.B Appendix B: Some axioms of relevance logics on which PEARL succeeds and their first-order conditions

Most of the axioms below are copied from [25]. Hereafter A, B, C are treated as variables. Following [25] we use $\neg$, &, $\vee$, * for the FO connectives in the correspondence language, as well as $\leq$ instead of $\preceq$, as in the main text. For classical implication we use $\Rightarrow$ instead of $>$.

4.B.1 Axioms and rules for the system B and extensions

4.B.1.1 Axioms and rules of the system B

A1. $A \to A$
A2. $A \wedge B \to A$
A3. $A \wedge B \to B$
A4. $A \to A \vee B$
A5. $B \to A \vee B$
A6. $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
A7. $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$
A8. $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
A9. $\sim\sim A \to A$

Proposition 4.6. *All axioms A1-A9 are Sahlqvist formulae.*

Proof. Routine. We illustrate the execution of PEARL on two of them, and leave the rest to the reader. We also sketch the post-processing for these, skipping some trivial steps.

A1. $A \to A$

PEARL begins with $\mathbf{t} \subseteq A \to A$,
transformed by (FA) to: $\mathbf{i} \subseteq \mathbf{t},\ A \to A \subseteq \mathbf{m} \ \Vdash\ \mathbf{i} \subseteq \mathbf{m}$,
followed by another approximation to: $\mathbf{i} \subseteq \mathbf{t},\ \mathbf{j} \subseteq A,\ \mathbf{j} \to A \subseteq \mathbf{m} \ \Vdash\ \mathbf{i} \subseteq \mathbf{m}$.
Now, applying the Ackermann-rule produces $\mathbf{i} \subseteq \mathbf{t},\ \mathbf{j} \to \mathbf{j} \subseteq \mathbf{m} \ \Vdash\ \mathbf{i} \subseteq \mathbf{m}$.
A post-processing reverse approximation eliminates $\mathbf{i}$: $\mathbf{j} \to \mathbf{j} \subseteq \mathbf{m} \ \Vdash\ \mathbf{t} \subseteq \mathbf{m}$.
(Hereafter we will omit the introduction and elimination of that initially used nominal.) A post-processing reverse approximation eliminates $\mathbf{m}$: $\mathbf{t} \subseteq \mathbf{j} \to \mathbf{j}$. Now, it is easy to see that the FO equivalent is a validity, as expected.

A9. $\sim\sim A \to A$

PEARL begins with $\mathbf{t} \subseteq \sim\sim A \to A$,
transformed by the reduced (FA) to: $\sim\sim A \to A \subseteq \mathbf{m} \ \Vdash\ \mathbf{t} \subseteq \mathbf{m}$,
then by approximation: $\mathbf{j} \subseteq \sim\sim A,\ \mathbf{j} \to A \subseteq \mathbf{m} \ \Vdash\ \mathbf{t} \subseteq \mathbf{m}$,
followed by a residuation step: $\sim A \subseteq \sim^{\sharp}\mathbf{j},\ \mathbf{j} \to A \subseteq \mathbf{m} \ \Vdash\ \mathbf{t} \subseteq \mathbf{m}$,
and then another residuation step: $\sim^{\flat} \sim^{\sharp}\mathbf{j} \subseteq A,\ \mathbf{j} \to A \subseteq \mathbf{m} \ \Vdash\ \mathbf{t} \subseteq \mathbf{m}$.
Now, applying the Ackermann-rule produces $\mathbf{j} \to \sim^{\flat} \sim^{\sharp}\mathbf{j} \subseteq \mathbf{m} \ \Vdash\ \mathbf{t} \subseteq \mathbf{m}$.
A post-processing reverse approximation eliminates $\mathbf{m}$: $\mathbf{t} \subseteq \mathbf{j} \to \sim^{\flat} \sim^{\sharp}\mathbf{j}$,
finally simplified to $\mathbf{t} \circ \mathbf{j} \subseteq \sim^{\flat} \sim^{\sharp}\mathbf{j}$.

The FO condition can now be easily computed:
$\forall w \forall x_1 (x_{\mathbf{i}} \leq w \Rightarrow \exists u (u^{**} = w \ \&\ x_1 \leq u))$
When * is an involution, this is clearly a validity.

4.B.1.2 Additional schemata that can be added to the system B

B1. $A \wedge (A \to B) \to B$
B2. $(A \to B) \wedge (B \to C) \to (A \to C)$
B3. $(A \to B) \to ((B \to C) \to (A \to C))$
B4. $(A \to B) \to ((C \to A) \to (C \to B))$
B5. $(A \to (A \to B) \to (A \to B)$ (or $(A \to (B \to C)) \to (A \wedge B \to C)$)
B6. $A \to ((A \to B) \to B)$
B7. $(A \to (B \to C)) \to (B \to (A \to C))$
B8. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
B9. $(A \to B) \to ((A \to (B \to C)) \to (A \to C))$
B10. $A \to (B \to B)$
B11. $B \to (A \to B)$

B12. $A \to (B \to (C \to A))$
B13. $A \to (B \to A \wedge B)$
B14. $(A \to B) \to ((A \to C) \to (A \to B \wedge C))$
B14'. $(A \to C) \to ((B \to C) \to (A \vee B \to C))$
B15. $(A \wedge B \to C) \to (A \to (B \to C))$
B16. $A \vee (A \to B)$
B17. $(A \to B) \vee (B \to A)$
B17'. $(A \to (A \wedge B)) \vee (B \to (A \wedge B))$
B18. $A \to (A \to A)$
B19. $A \vee B \to ((A \to B) \to B)$
B20. $(A \wedge B \to C) \to (A \to C) \vee (B \to C)$

Proposition 4.7. *All axioms B1-B20 are inductive formulae. Moreover, B1, B5–B7, and B10–B20 are Sahlqvist.*

Proof. Routine. Sketches of the execution of PEARL on a selection of these axioms are given below, omitting some elimination and post-processing steps:

B1. $A \wedge (A \to B) \to B$
Elimination steps: $A \wedge (A \to B) \to B \subseteq \mathbf{m} \ \Vdash \ \mathbf{t} \subseteq \mathbf{m}$,
$\mathbf{j} \subseteq A \wedge (A \to B),\ \mathbf{j} \to B \subseteq \mathbf{m} \ \Vdash \ \mathbf{t} \subseteq \mathbf{m}$,
$\mathbf{j} \subseteq A,\ \mathbf{j} \subseteq A \to B,\ \mathbf{j} \to B \subseteq \mathbf{m} \ \Vdash \ \mathbf{t} \subseteq \mathbf{m}$,
Elimination of A:
$\mathbf{j} \subseteq \mathbf{j} \to B,\ \mathbf{j} \to B \subseteq \mathbf{m} \ \Vdash \ \mathbf{t} \subseteq \mathbf{m}$,
$\mathbf{j} \circ \mathbf{j} \subseteq B,\ \mathbf{j} \to B \subseteq \mathbf{m} \ \Vdash \ \mathbf{t} \subseteq \mathbf{m}$,
Elimination of B:
$\mathbf{j} \to \mathbf{j} \circ \mathbf{j} \subseteq \mathbf{m} \ \Vdash \ \mathbf{t} \subseteq \mathbf{m}$,
Simplification: $\mathbf{t} \subseteq \mathbf{j} \to (\mathbf{j} \circ \mathbf{j})$, equiv. $\mathbf{t} \circ \mathbf{j} \subseteq \mathbf{j} \circ \mathbf{j}$,
FO condition: $\forall w \forall x_{\mathbf{j}} (x_{\mathbf{j}} \leq w \Rightarrow R x_{\mathbf{j}} x_{\mathbf{j}} w)$.
Equivalently, $\forall w Rwww$.

B3. $(A \to B) \to ((B \to C) \to (A \to C))$
Simplifying the pure quasi-inclusion after elimination:
$\mathbf{j}_2 \to \mathbf{n}_2 \subseteq \mathbf{n}_1,\ \mathbf{j}_3 \to (\mathbf{j}_2 \circ (\mathbf{j}_1 \circ \mathbf{j}_3)) \subseteq \mathbf{n}_2,\ \mathbf{j}_1 \to \mathbf{n}_1 \subseteq \mathbf{m} \ \vdash \ \mathbf{t} \subseteq \mathbf{m}$ iff
$\mathbf{j}_2 \to \mathbf{n}_2 \subseteq \mathbf{n}_1,\ \mathbf{j}_3 \to (\mathbf{j}_2 \circ (\mathbf{j}_1 \circ \mathbf{j}_3)) \subseteq \mathbf{n}_2 \ \vdash \ \mathbf{t} \subseteq \mathbf{j}_1 \to \mathbf{n}_1$ iff
$\mathbf{j}_2 \to \mathbf{n}_2 \subseteq \mathbf{n}_1,\ \mathbf{j}_3 \to (\mathbf{j}_2 \circ (\mathbf{j}_1 \circ \mathbf{j}_3)) \subseteq \mathbf{n}_2 \ \vdash \ \mathbf{t} \circ \mathbf{j}_1 \subseteq \mathbf{n}_1$ iff
$\mathbf{j}_3 \to (\mathbf{j}_2 \circ (\mathbf{j}_1 \circ \mathbf{j}_3)) \subseteq \mathbf{n}_2 \ \vdash \ \mathbf{t} \circ \mathbf{j}_1 \subseteq \mathbf{j}_2 \to \mathbf{n}_2$ iff
$\mathbf{j}_3 \to (\mathbf{j}_2 \circ (\mathbf{j}_1 \circ \mathbf{j}_3)) \subseteq \mathbf{n}_2 \ \vdash \ (\mathbf{t} \circ \mathbf{j}_1) \circ \mathbf{j}_2 \subseteq \mathbf{n}_2$ iff
$(\mathbf{t} \circ \mathbf{j}_1) \circ \mathbf{j}_2 \ \subseteq \ \mathbf{j}_3 \to (\mathbf{j}_2 \circ (\mathbf{j}_1 \circ \mathbf{j}_3))$.

B6. $A \to ((A \to B) \to B)$
Simplifying the pure quasi-inclusion after elimination:
$\mathbf{j}_2 \to (\mathbf{j}_2 \circ \mathbf{j}_1) \subseteq \mathbf{n}_1 \mathbf{j}_1 \to \mathbf{n}_1 \subseteq \mathbf{m} \ \vdash \ \mathbf{t} \subseteq \mathbf{m}$ iff
$\mathbf{j}_2 \to (\mathbf{j}_2 \circ \mathbf{j}_1) \subseteq \mathbf{n}_1 \ \vdash \ \mathbf{t} \subseteq \mathbf{j}_1 \to \mathbf{n}_1$ iff
$\mathbf{j}_2 \to (\mathbf{j}_2 \circ \mathbf{j}_1) \subseteq \mathbf{n}_1 \ \vdash \ \mathbf{t} \circ \mathbf{j}_1 \subseteq \mathbf{n}_1$ iff
$\mathbf{t} \circ \mathbf{j}_1 \subseteq \mathbf{j}_2 \to (\mathbf{j}_2 \circ \mathbf{j}_1)$.

Computing the FO condition:
$\forall w \forall o \in O \forall x_1 \forall x_2 (w \vdash \mathbf{t} \circ \mathbf{j}_1 \Rightarrow w \vdash \mathbf{j}_2 \to (\mathbf{j}_2 \circ \mathbf{j}_1))$ iff
$\forall w \forall x_1 \forall x_2 (x_1 \leq w \Rightarrow \forall z (Rwx_2z \Rightarrow z \Vdash (\mathbf{j}_2 \circ \mathbf{j}_1))$ iff
$\forall w \forall x_1 \forall x_2 \big(x_1 \leq w \Rightarrow \forall z (Rwx_2z \Rightarrow Rx_2x_1z)\big)$ iff
$\forall x_1 \forall x_2 \forall z (Rx_1x_2z \Rightarrow Rx_2x_1z)$.

B7. $(A \to (B \to C)) \to (B \to (A \to C))$
Simplifying the pure quasi-inclusion after elimination:
$\mathbf{j}_2 \to \mathbf{n}_2 \subseteq \mathbf{n}_1,\ \mathbf{j}_3 \to ((\mathbf{j}_1 \circ \mathbf{j}_3) \circ \mathbf{j}_2) \subseteq \mathbf{n}_2,\ \mathbf{j}_1 \to \mathbf{n}_1 \subseteq \mathbf{m} \ \vdash \ \mathbf{t} \subseteq \mathbf{m}$ iff
...
$(\mathbf{t} \circ \mathbf{j}_1) \circ \mathbf{j}_2 \ \subseteq \ \mathbf{j}_3 \to ((\mathbf{j}_1 \circ \mathbf{j}_3) \circ \mathbf{j}_2)$.

4.B.1.3 Axioms for negation

D1. $(A \wedge B \to C) \to (A \wedge \sim C \to \sim B)$
D2. $A \vee \sim A$
D3. $(A \to \sim A) \to \sim A$
D3'. $(A \to B) \to (\sim A \vee B)$
D4. $(A \to \sim B) \to (B \to \sim A)$
D5. $B \to (A \vee \sim A)$
D5'. $(A \wedge (\sim A \vee B)) \to B$
D6. $A \to (\sim A \to B)$
D6'. $\sim (A \to B) \to A$
D7. $\sim (A \to B) \to (B \to A)$
D8. $(A \to \sim (B \to C)) \to (\sim B \to \sim A)$

We state the following without proof.

Proposition 4.8. *All axioms D1-D7 are Sahlqvist, while D8 is properly inductive.*

4.B.1.4 An example of a Sahlqvist axiom with fusion

The following formula is Sahlqvist, but neither B-Sahlqvist nor S-Sahlqvist.

AF2. (MR2 in [25, p.377]) $(A \circ B \to C) \to (A \to (B \to C))$.
Pure quasi-inclusion after elimination:
$\mathbf{j}_1 \to \mathbf{n}_1 \subseteq \mathbf{m},\ \mathbf{j}_2 \to \mathbf{n}_2 \subseteq \mathbf{n}_1,\ \mathbf{j}_3 \to \mathbf{j}_1 \circ (\mathbf{j}_2 \circ \mathbf{j}_3) \subseteq \mathbf{n}_2 \ \vdash\ \mathbf{t} \subseteq \mathbf{m}$.
Simplified pure inclusion: $((\mathbf{t} \circ \mathbf{j}_1) \circ \mathbf{j}_2) \circ \mathbf{j}_3 \subseteq \mathbf{j}_1 \circ (\mathbf{j}_2 \circ \mathbf{j}_3)$.

4.B.2 Semantic conditions for axiomatic extensions of B

Following [25], we use $\Rightarrow$ (instead of $>$), $\neg$, &, $\vee$, * for the FO connectives in the correspondence language, as well as the following abbreviations:

$R^2abcd := \exists x(Rabx \,\&\, Rxcd)$, $R^2a(bc)d := \exists x(Rbcx \,\&\, Raxd)$,
$R^3ab(cd)e := \exists x(R^2abxe \,\&\, Rcdx)$.

We present here the frame conditions for axiomatic extensions of the System B (with the axioms listed on the right), copied from [25], where a, b, c, d, e are universally quantified variables. We invite the reader to verify whether the FO conditions computed by PEARL are respectively equivalent to those listed here.

q1.	$Raaa$	B1
q2.	$Rabc \Rightarrow R^2a(ab)c$	B2
q3.	$R^2abcd \Rightarrow R^2b(ac)d$	B3
q4.	$R^2abcd \Rightarrow R^2a(bc)d$	B4
q5.	$Rabc \Rightarrow R^2abbc$	B5
q6.	$Rabc \Rightarrow Rbac$	B6
q7.	$R^2abcd \Rightarrow R^2acbd$	B7
q8.	$R^2abcd \Rightarrow R^3ac(bc)d$	B8
q9.	$R^2abcd \Rightarrow R^3bc(ac)d$	B9
q10.	$Rabc \Rightarrow b \leq c$	B10
q11.	$Rabc \Rightarrow a \leq c$ (equivalently, given $s2$ below this reduces to $0 \leq a$)	B11
q12.	$R^2abcd \Rightarrow a \leq d$	B12
q13.	$Rabc \Rightarrow a \leq c \,\&\, b \leq c$	B13
q14.	$R^2abcd \Rightarrow Racd \,\&\, Rbcd$	B14
q15.	$R^2abcd \Rightarrow \exists x(b \leq x \,\&\, c \leq x \,\&\, Raxd)$	B15

q16. $a \leq b \;\&\; 0x \Rightarrow a \leq x$ B16
q17. $a \leq b \vee b \leq a$ B17
q18. $Rabc \Rightarrow a \leq c \vee b \leq c$ B18
q19. $Rabc \Rightarrow (Rbac \;\&\; a \leq c)$ B19
q20. $(Rabc \;\&\; Rade) \Rightarrow \exists x(b \leq x \;\&\; d \leq x \;\&\; (Raxc \vee Raxe)$ B20

Some conditions involving negation:

s1. $Rabc \Rightarrow \exists x(b \leq x \;\&\; c^* \leq x \;\&\; Raxb^*$ D1
s2. $x^* \leq x$ for all x such that $0x$ (reduces to $0^* \leq 0$) D2
s3. Raa^*a D3
s4. $Rabc \Rightarrow Rac^*b^*$ D4
s5. $a^* \leq a$ D5
s6. $Rabc \Rightarrow a \leq b^*$ D6
s7. $(Rabc \;\&\; Ra^*de) \Rightarrow (d \leq c \vee b \leq e)$ D7
s8. $(Rabc \Rightarrow \exists y(Rac^*y \;\&\; (\forall d, e)(Ry^*de \Rightarrow d \leq b^*))$ D8