SONE SYNTACTIC AND SEMANTIC RELATIONS BETWEEN
SUPERINTUITIONISTIC AND MULTIPLE-VALUED
PROPOSITIONAL LOGICS

Valentin F. Goranko

A reversible syntactic and semantic translation between \(m\)-valued propositional calculus on a superintuitionistic base and intuitionistic propositional calculus is constructed. Then various properties as completeness, decidability, tabularity, pretabularity, interpolation property, etc. are shown to be transferable in both directions.

The multiple-valued propositional calculus on an intuitionistic base and the pseudo-Post algebras as its algebraic semantics are introduced by Rousseau [5] and studied by Yankov [6]. The construction of a pseudo-Post algebra as a co-product of a pseudo-Boolean algebra with a chain (see [6]) gives a possibility for natural reversible syntactic and semantic translations between the above calculus and the intuitionistic propositional calculus. Various properties as completeness, decidability, tabularity, pretabularity, locally finiteness, interpolation property, etc. are transferable in a natural way in both directions which is done in this paper.

0. Preliminaries. The language of the \(m\)-valued Post propositional calculus (PPCm for short) \(\mathfrak{L}_m\) is an extension of the language \(\mathfrak{L}\) of the intuitionistic propositional calculus (IPC) with propositional constants \(p_1, \ldots, p_m\), and unary operations \(\mathfrak{B}_1, \ldots, \mathfrak{B}_m\). The set of formulas of \(\mathfrak{L}_m\) is denoted by \(\text{FOR}_m\) and of \(\mathfrak{L}\) by \(\text{FOR}\).

PPCm is an extension of IPC with the following axioms and rules of inference (see [6]):

Axioms:

\[
\begin{align*}
(\mathfrak{B}) & \quad \mathfrak{B}_i (\alpha \lor \beta) \iff \mathfrak{B}_i (\alpha) \lor \mathfrak{B}_i (\beta); \\
(\mathfrak{C}) & \quad \mathfrak{B}_i (\alpha \land \beta) \iff \mathfrak{B}_i (\alpha) \land \mathfrak{B}_i (\beta); \\
(\mathfrak{D}) & \quad \neg \mathfrak{B}_i (\beta) ; \\
(\mathfrak{F}) & \quad \mathfrak{B}_i (\alpha) \iff \mathfrak{B}_i (\alpha); \\
(\mathfrak{G}) & \quad \mathfrak{B}_i (\alpha) \iff \mathfrak{B}_j (\alpha) \quad \text{for} \quad i = j; \\
(\mathfrak{H}) & \quad \alpha \iff \mathfrak{B}_m (\alpha \lor \beta) ; \\
\end{align*}
\]

for \(i, j = 1, \ldots, m\):
Theorem 6.1 (representation of PFA). Let $\mathcal{H}$ be a PFA with a support $A$. Let $P(A) = \{\{a_1, ..., a_m\}, \ldots, \{b_1, ..., b_n\}\}$ be a set of operations defined as above. The system $(P(A), V, \cup, \land, \neg, \rightarrow, \forall, \exists)$ is a pseudo-Boolean algebra as above a PFA. So $P(A)$ is a PFA. The construction is by induction on the complexity of $\alpha$.

1. $\alpha \equiv \beta$, $\forall \alpha \in V$.
2. $\alpha \equiv \beta$, $\exists \alpha \in V$.
3. $\alpha \equiv \beta$, $\forall \alpha \in V$.
4. $\alpha \equiv \beta$, $\exists \alpha \in V$.

The proof of (1) is straightforward. For (2) and (3), note that the mapping $\forall \alpha \in V$ is a PFA.

| Theorem 6.1. Let $\mathcal{H}$ be a PFA. Then $P(A) = \{\{a_1, ..., a_m\}, \ldots, \{b_1, ..., b_n\}\}$ is a PFA if and only if $\mathcal{H}$ is a PFA.

Proof. Let $\mathcal{H}$ be a PFA. Then $P(A) = \{\{a_1, ..., a_m\}, \ldots, \{b_1, ..., b_n\}\}$ is a PFA. Conversely, let $\mathcal{H}$ be a PFA. Then $P(A) = \{\{a_1, ..., a_m\}, \ldots, \{b_1, ..., b_n\}\}$ is a PFA. The proof is straightforward. For (2) and (3), note that the mapping $\forall \alpha \in V$ is a PFA.

| Theorem 6.2. Let $\mathcal{H}$ be a PFA. Then $P(A) = \{\{a_1, ..., a_m\}, \ldots, \{b_1, ..., b_n\}\}$ is a PFA if and only if $\mathcal{H}$ is a PFA.

Proof. Let $\mathcal{H}$ be a PFA. Then $P(A) = \{\{a_1, ..., a_m\}, \ldots, \{b_1, ..., b_n\}\}$ is a PFA. Conversely, let $\mathcal{H}$ be a PFA. Then $P(A) = \{\{a_1, ..., a_m\}, \ldots, \{b_1, ..., b_n\}\}$ is a PFA. The proof is straightforward. For (2) and (3), note that the mapping $\forall \alpha \in V$ is a PFA.

| Theorem 6.3. Let $\mathcal{H}$ be a PFA. Then $P(A) = \{\{a_1, ..., a_m\}, \ldots, \{b_1, ..., b_n\}\}$ is a PFA if and only if $\mathcal{H}$ is a PFA.

Proof. Let $\mathcal{H}$ be a PFA. Then $P(A) = \{\{a_1, ..., a_m\}, \ldots, \{b_1, ..., b_n\}\}$ is a PFA. Conversely, let $\mathcal{H}$ be a PFA. Then $P(A) = \{\{a_1, ..., a_m\}, \ldots, \{b_1, ..., b_n\}\}$ is a PFA. The proof is straightforward. For (2) and (3), note that the mapping $\forall \alpha \in V$ is a PFA.
\[ \alpha, \beta \in \mathcal{F} \text{, such that } \mathcal{F} = \{ \alpha, \beta \}. \]

Therefore by (1):

\[ \mathcal{A}(\alpha, \beta) \subseteq \mathcal{A}(\alpha, \beta). \]

But \[ \mathcal{A}(\alpha, \beta) = \mathcal{A}(\alpha, \beta) \text{, and } \mathcal{A}(\alpha, \beta) \subseteq \mathcal{A}(\alpha, \beta). \]

Let \[ \mathcal{A}(\alpha, \beta) \subseteq \mathcal{A}(\alpha, \beta) \text{, and } \mathcal{A}(\alpha, \beta) \subseteq \mathcal{A}(\alpha, \beta). \]

So any axiom of an extension of \( \mathcal{F} \) can be replaced by a formula from the language \( \mathcal{L} \) enriched with the operation \( \mathcal{D}_i \).

**Lemma 1.7.** \( \mathcal{A}(\mathcal{E})(\mathcal{F}) \subseteq \mathcal{A}(\mathcal{F}) \).

**Proof.** It is sufficient to prove that the mapping \( \mathcal{X}: \mathcal{A}(\mathcal{F}) \to \mathcal{A}(\mathcal{F}) \) defined by \( \mathcal{X}(\alpha) = \mathcal{F}(\alpha) \) and \( \mathcal{X}(\alpha) = \mathcal{X}(\alpha) \) is inverse isomorphic, which is clear.

Recall some definitions: A variety \( \mathcal{V} \) is a variety if it is modeled by a finite algebra. A variety \( \mathcal{V} \) is a variety if it is not trivial but any proper subvariety of \( \mathcal{V} \) is trivial.

**Corollary 1.8.** There exist continuous many varieties of PPA's, respectively continuous many pseudo-Post logics of order \( m \).

**Corollary 1.9.** If let \( \mathcal{F}^{(1)} \) and \( \mathcal{F}^{(2)} \) be PPA's, then \( \mathcal{F}^{(2)} \) is a variety as well.

**Lemma 1.11.** If \( \mathcal{V} \) is finite and \( \mathcal{V} \) is generated by \( \mathcal{E} \), then \( \mathcal{V} \) is generated by \( \mathcal{E} \).

**Proof.** Let \( \mathcal{V} \) be generated from the elements \( \delta_1, \ldots, \delta_m \). Then it follows from the identity \( \mathcal{E}(\alpha, \beta) \subseteq \mathcal{V} \), that \( \mathcal{V} \) is generated by \( \delta_1, \ldots, \delta_m \).

**Corollary 1.12.** A variety of PPA's \( \mathcal{V} \) is locally finite if the variety \( \mathcal{V} \) is locally finite.

**Proof.** It is clear that \( \mathcal{V} \) is finite if it is finitely generated as a PPA.

For a given class \( \mathcal{K} \) of algebras of an arbitrary signature we obtain a category \( \mathcal{K} \) in the following way: the class of objects is obtained from the class \( \mathcal{K} \) identifying the isomorphic algebras and the class of morphisms is obtained from the class of homomorphisms in \( \mathcal{K} \) identifying the corresponding pairs of homomorphisms \( g, h \) for which the diagram \( a = b \) is commutative.

The subcategories of \( \mathcal{K} \) obtained taking the class of: monomorphic - \( \mathcal{M}_{\mathcal{K}} \), epimorphisms - \( \mathcal{E}_{\mathcal{K}} \), isomorphisms - \( \mathcal{I}_{\mathcal{K}} \), instead of \( \mathcal{M}_{\mathcal{K}} \) will be denoted respectively by \( \mathcal{M}_{\mathcal{M}_{\mathcal{K}}} \), \( \mathcal{M}_{\mathcal{E}_{\mathcal{K}}} \), \( \mathcal{M}_{\mathcal{I}_{\mathcal{K}}} \).

**Lemma 1.13.** If \( \mathcal{H} \subseteq \mathcal{H} \) is a homeomorphic - \( \mathcal{M}_{\mathcal{H}} \), \( \mathcal{E}_{\mathcal{H}} \), \( \mathcal{I}_{\mathcal{H}} \) - morphism of PPA's, then the mapping \( \mathcal{H}: \mathcal{H}(\mathcal{H}) \to \mathcal{H}(\mathcal{H}) \), defined by \( \mathcal{H}(\mathcal{H}(\mathcal{H})) = \mathcal{H}(\mathcal{H}(\mathcal{H})) \), is a homeomorphic - \( \mathcal{M}_{\mathcal{H}} \), \( \mathcal{E}_{\mathcal{H}} \), \( \mathcal{I}_{\mathcal{H}} \) - morphism of PPA's.

**Proof.** It is an easy verification to prove that \( \mathcal{H} \) is a PPA-homomorphism.
There are no difficulties to prove that $\mathbf{F}$ is a bijective functor.

Note that in virtue of L.1.13 the respective restrictions of the above functor $\mathbf{F}$ realize the equivalences $A_1' \equiv P(A_1)$, $A_2' \equiv P(A_2)$, $A_3' \equiv P(A_3)$.

Recall that the class $\mathcal{K}$ of algebras has the amalgamation property (AP) if for any three algebras $A_1, A_2, A_3 \in \mathcal{K}$ and monomorphisms $f_1: A_1 \rightarrow A_2$ and $f_2: A_2 \rightarrow A_3$ there exist an algebra $A \in \mathcal{K}$ and monomorphisms $g_1: A_1 \rightarrow A$ and $g_2: A_2 \rightarrow A$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

**Lemma 1.19.** Let $\mathcal{K}$ and $\mathcal{S}$ be classes of algebras of arbitrary signatures and $A_1' \equiv S_{A_1}$. Then $\mathcal{K}$ has the AP iff $\mathcal{S}$ has the AP.

(see [1])

**Corollary 1.17.** A class $\mathcal{A}$ of PBAs has the AP iff the class $P(\mathcal{A})$ has the AP.

**Theorem 1.18.** There exist just 8 varieties of PBAs of order $\mathfrak{m}$ having the AP and they are defined as subvarieties of $\mathcal{P}_{\mathfrak{m}}$ with additional identities as follows:

\[
\begin{align*}
\mathcal{P}_{\mathfrak{m}}^{(0)} &= \mathcal{P}_{\mathfrak{m}}, \\
\mathcal{P}_{\mathfrak{m}}^{(1)} &= \mathcal{P}_{\mathfrak{m}} + \mathcal{P}_{\mathfrak{m}} \cdot \mathcal{T} \leftrightarrow \mathcal{T}, \\
\mathcal{P}_{\mathfrak{m}}^{(2)} &= \mathcal{P}_{\mathfrak{m}} + \mathcal{P}_{\mathfrak{m}} \cdot \mathcal{T} \leftrightarrow \mathcal{T}, \\
\mathcal{P}_{\mathfrak{m}}^{(3)} &= \mathcal{P}_{\mathfrak{m}} + \mathcal{P}_{\mathfrak{m}} \cdot \mathcal{T} \leftrightarrow \mathcal{T}, \\
\mathcal{P}_{\mathfrak{m}}^{(4)} &= \mathcal{P}_{\mathfrak{m}} + \mathcal{P}_{\mathfrak{m}} \cdot \mathcal{T} \leftrightarrow \mathcal{T}, \\
\mathcal{P}_{\mathfrak{m}}^{(5)} &= \mathcal{P}_{\mathfrak{m}} + \mathcal{P}_{\mathfrak{m}} \cdot \mathcal{T} \leftrightarrow \mathcal{T}, \\
\mathcal{P}_{\mathfrak{m}}^{(6)} &= \mathcal{P}_{\mathfrak{m}} + \mathcal{P}_{\mathfrak{m}} \cdot \mathcal{T} \leftrightarrow \mathcal{T}, \\
\mathcal{P}_{\mathfrak{m}}^{(7)} &= \mathcal{P}_{\mathfrak{m}} + \mathcal{P}_{\mathfrak{m}} \cdot \mathcal{T} \leftrightarrow \mathcal{T}.
\end{align*}
\]

The proof follows from the result of [18] describing all varieties of PBAs with AP and from Cor. 1.17.

**Corollary 1.19.** There exist just 8 pseudo-Post logics of order $\mathfrak{m}$ in which the Craig interpolation theorem (CIT) holds and they are the corresponding logics of the above 8 varieties of PBAs.

The proof follows from the result of [1] where the equivalence of CIT in a large class equational logics containing the PPLs and AP in the corresponding varieties of algebras is ascertained.

2. If we remove the constants $e_1, \ldots, e_{m-1}$ from the language and replace the axioms $C_1, C_2$ and (MP) with the following:

\[
\begin{align*}
(MP) \quad &D_{j, k}(\alpha) \Rightarrow D_{i, k}(\alpha), \ i \neq j, m-2; \\
(C_1') \quad &A_{m-1}(\alpha) \Rightarrow e, \\
(C_2') \quad &D_{m-1}(\alpha) \Rightarrow D_{m-m}(\alpha), \ i \neq 1, m-2; \\
(D_{j, k}(\alpha) \Rightarrow \beta \Rightarrow D_{j, k}(\alpha), \ i \neq 1, m-1.
\end{align*}
\]

we shall obtain the so-called quasi-Post propositional calculus QPP$\mathcal{E}$ (see [3]). The corresponding algebraic systems are quasi-Post algebras (QPP$\mathcal{E}$) of order $\mathfrak{m}$. A representation theorem, analogous to Theorem 6.1, holds about QPP$\mathcal{E}$ but here the embedding is in general strict. By virtue of Corollary 1.6, (any identity in a QPA can be written down equivalently without constants $e_1, \ldots, e_{m-1}$) there exist a natural bijection between the lattice of varieties of QPP$\mathcal{E}$ of order $\mathfrak{m}$ and the lattice of varieties of QPP$\mathcal{E}$ of order $\mathfrak{m}$ in which no identity holds in QPP$\mathcal{E}$, then we can consider it as a QPP$\mathcal{E}$ of a lower order without such identities). So the most part of the results of this paper are easily transferred in the case of QPP$\mathcal{E}$ (and the corresponding quasi-pseudo Post logics).

**References**

7. A. B. Kuznetsov, Some properties of the structure of the pseudobialgebras of semantical $\mathcal{C}$. University of Catholic University, Rome, in press.