# LOGICAL TOPOLOGIES AND SEMANTIC COMPLETENESS

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**Abstract.** We study the generic problem of proving semantic completeness of a logical system with respect to a class of "standard models", provided a weaker completeness result with respect to a larger class of "general models" has been obtained. We propose a natural topological approach to this problem based on the notion of *logical topology* and the related concept of *logical approximation*. After some general results regarding these concepts we discuss them in the framework of first-order logic. The paper ends with an example of a particualr application of ideas developed here.

*To Johan van Benthem, on the occasion of his 50th birthday, with high respect.*

## §1. Introduction.

**1.1. The relative completeness problem.** The present study is motivated by the following, often arising in logical studies, generic problem. Suppose a deductive system **L** (of any nature) in a certain logical language is intended to axiomatize a class of *standard models* **SM**, and a completeness theorem has been established with respect to a larger class of *general models* **GM**, i.e. it has been proved that

$$\mathbf{L} \vdash \phi \text{ iff } \mathbf{L} \models_{GM} \phi$$

The goal is **to prove completeness of L with respect to the standard models**, i.e.

$$\mathbf{L} \vdash \phi \text{ iff } \mathbf{L} \models_{SM} \phi$$

Here are three illustrative cases:

1. *Finite model property* in classical, modal, etc. logics. The "general models" are all models for the logic **L**, and the "standard models" are the *finite* models. While completeness with respect to general models is a uniform result in classical logic, due to Gödel's completeness t heorem, completeness with respect to "standard" (i.e. finite) models is an essentially nontrivial property, as Trakhtenbrot's theorem testifies.
2. *Kripke-frame completeness* in modal logic. The "general models" are all Kripke models for the logic **L**, and the "standard models" are those Kripke models based on frames for **L**. Again, the completeness with respect to

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the class of general models is a general result in modal logic (based on the standard canonical construction) but the completeness with respect to the standard models, i.e. Kripke completeness, is the non-trivial and important one. For more details and sample results, see [van Benthem, 1983].

3. *First-order approximation of* $\Pi^1_1$*-theories.* Often a structure, or a class of structures, can be characterized by means of a $\Pi^1_1$-sentence, but not in first-order logic. Typical examples are: the ordering of natural numbers (with the induction axiom), the ordering of the reals (with the continuity axiom), the class of all well-orderings etc. On the other hand, every $\Pi^1_1$-sentence $\Phi$ can be 'approximated' by the first-order schema $\Phi_1$ obtained from $\Phi$ by restricting the universal second-order quantification to all instances of parametrically first-order definable relations and functions. Now, the models of $\Phi$ are the "standard models" and the models of $\Phi_1$ are the "general models". The question '*does the scheme* $\Phi_1$ *axiomatize the first-order fragment of* $\Phi$?' is essentially the question of relative completeness we discuss here. A number of elaborated positive completeness results of this type have been obtained in [Doets, 1987] and, using his techniques, in [Backofen, Rogers, and Vijay-Shanker, 1995] and [Venema, 1993].

There is no general method for solving the problem described above, but usually some specific model-theoretic constructions are applied which transform general into standard models while preserving satisfiability.

Here we do not offer a general solution to that problem either, but rather a *methodology* based on a natural topological approach which can be applied in various particular cases. Because of space limitations we only outline one example of a non-trivial application at the end of the paper.

**1.2. A topological approach.** The idea in a nutshell is to **find an appropriate topology $\mathcal{T}$ on the class**[1] **of general models GM such that**:

(i) **The class of standard models SM is dense in GM with respect to $\mathcal{T}$, i.e. the closure in $\mathcal{T}$ of SM is GM.**
(ii) **Validity is a closed property with respect to $\mathcal{T}$.**

"Closedness" here is used in the standard topological sense: a property is closed if the set (class) of points which satisfy it is closed. Since every closed and dense set in a topological space coincides with the whole space, the following observation is immediate.

> *If the conditions above hold for some topology $\mathcal{T}$ on* **GM** *then completeness with respect to* **GM** *implies completeness with respect to* **SM**.

Alternatively, one can associate with every general model its *theory*, i.e. the set of its valid formulae in the language under consideration, and look for an appropriate topology *on the set of all theories of general models,* for which an analogous result can be stated. This approach has some technical advantages, but the two approaches are essentially equivalent, as it will be shown further.

Although some intimate connections between logic and topology have been established and studied (see e.g. [Barwise and Feferman, 1985]), it seems that,

[1] Here we talk about topologies on (proper) classes, rather than on sets. This foundational issue will not affect what follows, as we will see later that every such topology is essentially equivalent to a "small", i.e. set-based, one.

except in abstract model theory, topological methods and results have so far been under-utilized for solving purely logical problems, and there are few publications which more explicitly pursue that direction.

In this paper we suggest a more systematic exploration of the idea of using basic topological techniques and results to obtain relative completeness results in logic.

The preliminary section 1 contains some background from logic and topology. In section 2 we introduce the notion of *logical topology* and the related concept of *logical approximation,* and study their basic properties. In particular, as a direct consequence of the Baire's category theorem, we obtain a general relative completeness result (theorem 3.8, and theorem 4.9 as a particular case in first-order logic) which seemingly has so far not been explicitly noted, despite the well-known relationship of the Baire's theorem to logic (see [Rasiowa and Sikorski, 1963] and [Goldblatt, 1985]). In section 3 we discuss logical topologies and logical approximation in classical logic. We show that, not surprisingly, a topology on the set of all complete theories in a first-order language is logical iff it contains the Stone topology (proposition 4.1) and briefly study a simple and natural extension of the Stone topology in languages with infinite signature. In section 4 we mention a specific application to the first-order theory of trees, and outline a proof of completeness based on ideas and results from the paper. The last section 5 discusses a research agenda arising from this study.

**§2. Preliminaries.** Here we summarize some basic topological facts that will be used further. For details on definitions and related results, [Ebbinghaus, Flum, and Thomas, 1994] and [Hodges, 1993] are general references on the necessary logical background, and e.g. [Engelking, 1985] – on topology.

Let $L$ be a first-order language, $\mathrm{SEN}(L)$ be the set of sentences of $L$, and $\mathcal{C}(L)$ be the set of complete theories in $L$. The **Stone topology** $\mathcal{S}(L)$ is defined on the set $\mathcal{C}(L)$ by a base of clopen sets $\{[\phi] | \phi \in \mathrm{SEN}(L)\}$ where $[\phi] = \{T \in \mathcal{C}(L) | \phi \in T\}$. It is easy to see that the closed sets in $\mathcal{C}(L)$ are precisely the sets $\{T \in \mathcal{C}(L) | \Gamma \subseteq T\}$ where $\Gamma$ is a closed theory in $L$ and that $\mathcal{S}(L)$ is a totally disconnected compact Hausdorff space.

The topology $\mathcal{S}(L)$ determines a topology $\mathcal{S}_{\mathrm{STR}}(L)$ on the class of all $L$-structures $\mathrm{STR}(L)$, called by Tarski the **elementary topology**, where the closed subclasses are precisely the first-order axiomatizable classes. This topology is pseudo-metrizable when the language is countable and every first-order axiomatizable class, considered as a subspace of $\mathcal{S}_{\mathrm{STR}}(L)$ then becomes a complete pseudo-metric space.

Given a topology $\mathcal{T}$ and a set $A$ in $\mathcal{T}$, $\mathrm{Cl}_{\mathcal{T}}(A)$ is the closure of $A$ in $\mathcal{T}$.

Definition 2.1. A subset $A$ of a topological space $\mathcal{T}$ on a set $X$ is **dense** if $\mathrm{Cl}_{\mathcal{T}}(A) = X$. A subset $A$ of a set $B$ in a topological space $\mathcal{T}$ over a set $X$ is **dense in** $B$ if $\mathrm{Cl}_{\mathcal{T}|B}(A) = B$, where $\mathcal{T}|B$ is the topology on $B$ induced by $\mathcal{T}$.

Definition 2.2. A topological space $\mathcal{T}$ has the **Baire's property** if every countable intersection of dense open sets in $\mathcal{T}$ is dense in $\mathcal{T}$.

Two well known versions of the **Baire category theorem** state that *every complete pseudo-metric space, as well as every compact Hausdorff space has the Baire's property.*

A topology is first-countable if every point has a countable base of open neighbourhoods. It is easy to see that the Stone topology is first countable iff the language is at most countable. In first-countable topologies closed sets can be characterized in terms of closure under limits of convergent sequences, while in general, they are characterized in terms of convergent nets or clustering filters.

**§3. Logical topologies on theories and structures.** We fix an arbitrary logical language $L$ with specified semantics, i.e. a class of $L$-structures and a relation $\models$ of validity of $L$ -formulae in $L$-structures.

Let $\mathcal{T}$ be a topology on the class of $L$-structures.

DEFINITION 3.1. The topology $\mathcal{T}$ is **logical on a class of $L$ -structures M** if validity is a closed property with respect to the topology on $\mathbf{M}$ induced by $\mathcal{T}$. $\mathcal{T}$ is **logical** if it is logical on the class of all $L$-structures.

For every $L$-structure $A$, we denote by $\mathrm{TH}(A)$ the *theory of* $A$, i.e. the set of $L$-formulae valid in $A$.

Now, let $\mathcal{T}$ be a topology on a set $\mathbf{TH}$ of theories of $L$-structures and for every subset $\mathbf{S} \subseteq \mathbf{TH}$, $\mathrm{Cl}_{\mathcal{T}}(\mathbf{S})$ be the closure of $\mathbf{S}$ with respect to $\mathcal{T}$.

The following definition, though it may look somewhat unintuitive, will turn out to match the one of a logical topology on a class of structures given above.

DEFINITION 3.2. The topology $\mathcal{T}$ is **logical on TH** if for every subset $\mathbf{S} \subseteq \mathbf{TH}$, $\bigcap \mathbf{S} = \bigcap \mathrm{Cl}_{\mathcal{T}}(\mathbf{S})$; $\mathcal{T}$ is **logical** if it is logical on the set of all theories of $L$-structures.

There is a natural duality between the two notions of logical topologies. For every topology $\mathcal{T}$ on a class of structures $\mathbf{M}$ we can associate a topology $\mathcal{T}_{\mathrm{TH}}$ on the set of their theories, where the closed sets are of the type $\{\mathrm{TH}(A) | A \in \mathbf{C}\}$ for each closed set $\mathbf{C}$ in $\mathcal{T}$. Conversely, for every topology $\mathcal{T}$ on a set of theories $\mathbf{T}$ we can associate a topology $\mathcal{T}_{\mathrm{STR}}$ on the class of all models of theories from $\mathbf{T}$, with closed sets of the type $\{A | \mathrm{TH}(A) \in \mathbf{C}\}$ for each closed set $\mathbf{C}$ in $\mathcal{T}$.

PROPOSITION 3.3.

1. *If $\mathcal{T}$ is a logical topology on a class $\mathbf{M}$ of $L$-structures, then $\mathcal{T}_{\mathrm{TH}}$ is a logical topology on the set $\mathbf{T}$ of their theories.*
2. *If $\mathcal{T}$ is a logical topology on a set $\mathbf{T}$ of theories of $L$-structures, then $\mathcal{T}_{\mathrm{STR}}$ is a logical topology on the class $\mathbf{M}$ of their models.*

PROOF. 1. It is sufficient to note that for every $\mathbf{S} \subseteq \mathbf{T}_{\mathrm{TH}}$, the closure of $\mathbf{S}$ in $\mathcal{T}_{\mathrm{TH}}$ consists of all theories of structures which are in the closure of $\{A | \mathrm{TH}(A) \in \mathbf{S}\}$ in $\mathcal{T}$.
2. Likewise.

⊣

Thus, both notions are essentially equivalent. While most of the ideas and concepts discussed here look more natural when formulated in terms of structures, it is technically more convenient and elegant to state and prove many of the results in terms of theories, so we shall use interchangeably the two frameworks.

PROPOSITION 3.4. *If $\mathcal{T}, \mathcal{R}$ are topologies on a set of theories* **TH**, $\mathcal{T} \subseteq \mathcal{R}$, *and $\mathcal{T}$ is logical, then $\mathcal{R}$ is logical, too.*

PROOF. $\mathcal{T} \subseteq \mathcal{R}$ implies $\mathrm{Cl}_R(\mathbf{S}) \subseteq \mathrm{Cl}_T(\mathbf{S})$, so $\bigcap \mathbf{S} \subseteq \bigcap \mathrm{Cl}_{\mathcal{T}}(\mathbf{S}) \subseteq \bigcap \mathrm{Cl}_{\mathcal{R}}(\mathbf{S})$ for every $\mathbf{S} \subseteq \mathbf{TH}$. ⊣

DEFINITION 3.5. Let $\mathcal{T}$ be a logical topology on the class of $L$-structures. A structure $A$ is **logically approximated (with respect to $\mathcal{T}$)** in a class of structures $\mathbf{M}$ if $A$ belongs to the closure of $\mathbf{M}$ (with respect to $\mathcal{T}$). A class of structures $\mathbf{K}$ is **logically approximated (with respect to $\mathcal{T}$)** by $\mathbf{M}$ if every structure from $\mathbf{K}$ is logically approximated in $\mathbf{M}$. The closure $\mathrm{Cl}_{\mathcal{T}}(\mathbf{K})$ of $\mathbf{K}$, i.e. the class of all structures logically approximated in the class $\mathbf{K}$, will be called the **logical closure of K** (with respect to $\mathcal{T}$).

Note that if $\mathbf{M} \subseteq \mathbf{K}$ then $\mathbf{K}$ is logically approximated by $\mathbf{M}$ with respect to a topology $\mathcal{T}$ iff $\mathbf{M}$ is dense in $\mathbf{K}$ with respect to $\mathcal{T}$. Thus, the following statement formalizes the idea outlined in the introduction and provides formal grounds for applications of our approach to solving the relative completeness problem.

THEOREM 3.6. *Let $\mathcal{L}$ be a deductive system in the language $L$, complete for a class of structures* $\mathbf{K}$, *$\mathcal{T}$ be logical on* $\mathbf{K}$, *and* $\mathbf{M}$ *be a subclass of* $\mathbf{K}$ *which approximates logically* $\mathbf{K}$ *with respect to $\mathcal{T}$. Then $\mathcal{L}$ is complete for* $\mathbf{M}$.

A direct application of the Baire category theorem yields:

LEMMA 3.7. *Let* $\mathbf{K}$ *be a class of $L$-structures, $\mathcal{T}$ be a logical topology on* $\mathbf{K}$ *with the Baire's property, and $\{\mathbf{M}_k\}_{k\in\mathbf{N}}$ be a family of open subclasses of* $\mathbf{K}$ *such that* $\mathbf{K}$ *is logically approximated by each $\mathbf{M}_k$. Then* $\mathbf{K}$ *is logically approximated by $\mathbf{M} = \bigcap_{k\in\mathbf{N}} \mathbf{M}_k$.*

The following theorem is a combination of the previous two statements.

THEOREM 3.8. *Let $\mathcal{L}$ be a deductive system in $L$, complete with respect to a class of $L$-structures* $\mathbf{K}$, *$\mathcal{T}$ be a logical topology on* $\mathbf{K}$ *with the Baire's property and $\{\mathbf{M}_k\}_{k\in\mathbf{N}}$ be a family of open and dense subclasses of* $\mathbf{K}$. *Then* $\mathbf{L}$ *is complete with respect to $\mathbf{M} = \bigcap_{k\in\mathbf{N}} \mathbf{M}_k$.*

**§4. Logical topologies in first-order logic and elementary approximations of structures.** We now fix an arbitrary first-order language $L$. With no risk of confusion we shall denote both the Stone topology on $\mathcal{C}(L)$ and the elementary topology $\mathcal{S}_{\mathrm{STR}}(L)$ by $\mathcal{S}$, and the closure operator in both topologies by $\mathrm{Cl}_{\mathcal{S}}$.

Note that for every $\mathbf{S} \subseteq \mathcal{C}(L)$, $\mathrm{Cl}_{\mathcal{S}}(\mathbf{S}) = \{T \in \mathcal{C}(L) | \bigcap \mathbf{S} \subseteq T\}$. On the other hand, for every class of $L$-structures $\mathbf{K}$, $\mathrm{Cl}_{\mathcal{S}}(\mathbf{K})$ is the **elementary closure of K**, i.e. the smallest elementary class which contains $\mathbf{K}$. Thus, $\mathrm{Cl}_{\mathcal{S}}(\mathbf{K})$ is the class $\mathrm{MOD}(\mathrm{TH}(\mathbf{K}))$, of all models of the first-order theory of $\mathbf{K}$. Therefore, a theory $T$ is complete for a class $\mathbf{K}$ iff $\mathrm{TH}(\mathbf{K}) = T$ i.e. $\mathbf{K}$ is dense in $\mathrm{MOD}(T)$.

PROPOSITION 4.1. *A topology $\mathcal{T}$ on $\mathcal{C}(L)$ is logical iff it contains the Stone topology.*

PROOF. First, suppose $\mathcal{T}$ is logical and let $\mathbf{S} \subseteq \mathcal{C}(L)$ be closed in $\mathcal{S}(L)$, i.e. $\mathbf{S} = \{T \in \mathcal{C}(L) | \bigcap \mathbf{S} \subseteq T\}$. Then $\bigcap \mathbf{S} \subseteq \bigcap \mathrm{Cl}_{\mathcal{T}}(\mathbf{S})$, so $\mathrm{Cl}_{\mathcal{T}}(\mathbf{S}) \subseteq \mathcal{F}$, i.e. $\mathrm{Cl}_{\mathcal{T}}(\mathbf{S}) = \mathbf{S}$. For the converse, by proposition 3.4, it suffices to show that the Stone topology is logical. Indeed, for any $\mathbf{S} \subseteq \mathcal{C}(L)$, if $T \in \mathrm{Cl}_{\mathcal{S}}(\mathbf{S})$ then $\bigcap \mathbf{S} \subseteq T$, hence $\bigcap \mathcal{F} \subseteq \bigcap \mathrm{Cl}_{\mathcal{S}}(\mathbf{S})$. ⊣

Thus, the Stone topology is the weakest logical topology on the class of all $L$-structures, but there can be even weaker logical topologies suitable on some subclasses.

Sometimes it may easier to deal with logical topologies stronger than the Stone topology. A natural example of such a topology in first-order logic can be introduced by using an appropriate metric (which need not be inducing the Stone topology) on $\mathcal{C}(L)$.

The notion of *quantifier rank of a formula* is introduced as usual in languages with relational signatures, and appropriately modified for languages including constant and functional symbols, as in [Ebbinghaus, Flum, and Thomas, 1994].

Let $\mathrm{SEN}^{(n)}(L)$ be the set of $L$-sentences of (modified) rank $\leq n$ and for every $\Gamma \subseteq \mathrm{SEN}$, $\Gamma^{(n)} = \Gamma \cap \mathrm{SEN}^{(n)}(L)$.

First, we define *distance* in $\mathcal{C}(L)$ as follows:

$$\mathbf{d}(T_1, T_2) = \begin{cases} 0 & \text{if } T_1 = T_2, \\ \frac{1}{n+1} & \text{if } n \text{ is the least integer such that } T_1^{(n)} \neq T_2^{(n)}. \end{cases}$$

PROPOSITION 4.2.

1. $\langle \mathcal{C}(L), \mathbf{d} \rangle$ is a bounded and complete metric space.
2. The topology $\mathcal{C}_{\mathbf{d}}(L)$ on $\mathcal{C}(L)$ induced by $\mathbf{d}$ is logical.

PROOF. 1. To see that $\mathbf{d}$ is a metric it is sufficient to note that $\mathbf{d}(T_1, T_3) \leq \max(\mathbf{d}(T_1, T_2), \mathbf{d}(T_2, T_3))$ for any $T_1, T_2, T_3 \in \mathcal{T}(L)$. Boundedness is obvious. For completeness[2], let $T_1, T_2, \ldots, T_n \ldots$ be a Cauchy sequence in $\mathcal{T}(L)$. Then, for each $n \in \mathbf{N}$ there is $N \in \mathbf{N}$ such that for all $p, q > N$, $T_p^{(n)} = T_q^{(n)}$. Let us denote the latter by $\Gamma_n$. Thus we obtain a chain of theories $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$. Let $\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n$. $\Gamma$ is a complete theory. Indeed, $\Gamma$ is closed. For, let $\Gamma \models \phi$ where $\phi \in \mathrm{SEN}^{(m)}(L)$ for some $m$. Then $\Gamma_k \models \phi$ for some $k$. Hence $\phi \in \Gamma_{\max(k,m)}$, so $\phi \in \Gamma$. Furthermore, for every $\phi \in \mathrm{SEN}^{(m)}(L)$ for some $m$, either $\phi$ or $\neg\phi$ is in $\Gamma_k$ for every $k \geq m$. Finally, it is clear that $\lim_{n\to\infty} T_n = \Gamma$.

2. We shall prove that $\mathcal{C}_{\mathbf{d}}(L)$ contains the Stone topology. Let $\mathbf{S}$ be a closed set in $\mathcal{S}(L)$. Then $\mathbf{S} = \{T \in \mathcal{C}(L) | \bigcap \mathbf{S} \subseteq T\}$. Since every metric space is first-countable, it is sufficient to show that the limit $T$ in $\mathcal{C}_{\mathbf{d}}(L)$ of any sequence $T_0, T_1, \ldots$ from $\mathbf{S}$ is in $\mathbf{S}$. Indeed, $\bigcap \mathbf{S} \subseteq T$ since every sentence from $\bigcap \mathbf{S}$ with a rank $n$ will belong to all complete theories in the open $\frac{1}{n+1}$-neighbourhood of each theory from $\mathbf{S}$. Thus, $\mathbf{S}$ is closed in $\mathcal{C}(L)$.

⊣

[2] A stronger result has been proved in [Cifuentes, Sette, and Mundici, 1996]: for any first-order language $L$, the elementary topology $\mathcal{S}_{\mathrm{STR}}(L)$ is Cauchy complete, i.e. every Cauchy net converges.

PROPOSITION 4.3. *For every first-order language $L$ the following are equivalent:*

1. The language $L$ has finitely many non-logical symbols.
2. The topology $\mathcal{C}_{\mathbf{d}}(L)$ coincides with the Stone topology.
3. The space $\mathcal{C}_{\mathbf{d}}(L)$ is compact.
4. $\mathcal{C}_{\mathbf{d}}(L)$ is totally bounded.
5. For every $n \in \mathbf{N}$, the set $\{T^{(n)} | T \in \mathcal{C}(L)\}$ is finite.
6. For every $n \in \mathbf{N}$ there is no infinite independent subset of $\mathrm{SEN}^{(n)}(L)$.

PROOF. (1) $\Rightarrow$ (2): Let $\mathbf{S} \subseteq \mathcal{C}(L)$ be closed in $\mathcal{C}_{\mathbf{d}}(L)$ and $\Gamma = \bigcap \mathbf{S}$. We shall prove that $\mathbf{S} = \{T \in \mathcal{C}(L) | \Gamma \subseteq T\}$. We only need to show that $T \in \mathbf{S}$ whenever $\Gamma \subseteq T$. Indeed, for every $n \in \mathbf{N}$, $T^{(n)}$ is finite, so it is included in some $T_n \in \mathbf{S}$, otherwise $\neg \bigwedge T^{(n)} \in \Gamma$, so $T$ would be inconsistent. Thus, $T$ is the limit in $\mathcal{C}_{\mathbf{d}}(L)$ of a sequence $T_1, T_2, \ldots$ in $\mathbf{S}$, hence $T \in \mathbf{S}$.

(2)$\Rightarrow$(3) Follows from compactness of the Stone topology.

(3)$\Leftrightarrow$ (4): Every complete metric space is compact iff it is totally bounded.

(4)$\Leftrightarrow$ (5): Since the complete theories in every $\frac{1}{n+1}$-neighbourhood of $\mathcal{T}(L)$ share the same $\mathrm{SEN}^{(n)}(L)$-fragment, $\mathcal{T}(L)$ is covered by finitely many open balls of radius $\frac{1}{n+1}$ iff there are finitely many $\mathrm{SEN}^{(n)}(L)$-fragments of theories from $\mathcal{T}(L)$.

(5) $\Rightarrow$ (6): Suppose $\Gamma$ is an infinite independent subset of $\mathrm{SEN}^{(n)}(L)$ for some natural $n$. For each $\delta \in \Gamma$ we consider the consistent theory $\Gamma_\delta = \Gamma - \{\delta\} \cup \{\neg\delta\}$. All these theories have different $\mathrm{SEN}^{(n)}(L)$ -fragments.

(6) $\Rightarrow$ (1): If the language has infinitely many non-logical symbols, then there are infinitely many atomic formulae of (at most) one variable and rank not greater than 1, no two of which share non-logical symbols, hence there is an infinite independent set of sentences in $\mathrm{SEN}^{(2)}(L)$. $\dashv$

Thus, we see that, $\mathcal{C}_{\mathbf{d}}(L)$ is simpler and easier to deal with in case of infinite languages, where it is stronger than the Stone topology, and especially in uncountable languages where the latter is not first-countable.

Logical approximation with respect to $\mathcal{S}(L)$ will be called *elementary approximation,* and the approximation with respect to $\mathcal{C}_{\mathbf{d}}(L)$ — *strong elementary approximation.* Note that a structure $A$ is *strongly elementarily approximated in a class* $\mathbf{K}$ iff for every $n \in \mathbf{N}$ there is $A_n \in \mathbf{K}$ such that $A \equiv_n \mathcal{A}_n$. The class of all structures which are strongly elementarily approximated in $\mathbf{K}$ will be called the *strong elementary closure* of $\mathbf{K}$. Thus, every structure, strongly elementarily approximated in a class $\mathbf{K}$, is elementarily approximated in $\mathbf{K}$, but the converse need not hold in a language with an infinite signature. Respectively, every elementary closure is a strong elementary closure, but not conversely, and the elementary closure of any class $\mathbf{K}$ contains its strong elementary closure.

Elementary approximation and closure are already well-understood from various classical model-theoretics results, and we shall only mention just two characterizations of elementary approximation. The first one, in $\mathcal{S}(L)$ is essentially equivalent to the compactness theorem (see [Hodges, 1993]): *a theory $T \in \mathcal{C}(L)$ is elementarily approximated by a set $\mathbf{S} \subseteq \mathbf{C}(L)$ iff every finite subset of $T$ is* *included in some complete theory from* **S**. The second one, in $\mathcal{S}_{\mathrm{STR}}(L)$, is a well-known preservation result: *a structure $A$ is elementarily approximated by a class of structures* **K** *iff $A$ elementarily equivalent to an ultraproduct of structures from* **K**.

Here are two easy characterizations of strong elementary approximations.

DEFINITION 4.4. A net of $L$-structures $\langle A_i \rangle_{i \in D}$, where $D$ is a directed indexing family, is **strongly convergent** if it is convergent in $\mathcal{C}_{\mathbf{d}}(L)$.

THEOREM 4.5. *A class* **K** *of $L$-structures is a strong elementary closure iff it is closed under elementary equivalence and ultraproducts of strongly convergent nets.*

PROOF. If **K** satisfies the closure conditions, then every structure strongly elementarily approximated in **K** belongs to **K** since the ultraproduct of $\langle A_i \rangle_{i \in D}$ over any free ultrafilter on $D$ is elementarily equivalent to the limit of that net. Conversely, every strong elementary closure is closed under elementary equivalence and therefore, under ultraproducts of strongly convergent nets. ⊣

In the case of a countable language, the result above can be stated in terms of converging sequences, rather than nets.

A simple game-theoretic characterization of strong elementary approximations exists, too.

DEFINITION 4.6. **Ehrenfeucht game with choice of a companion:** Given a structure $A$, and a class of structures **K**, the game goes between two players as follows. In his first move Player I selects a natural number $n$. Then Player II selects a structure $\mathcal{B}$ from **K**. Then the game continues as the usual Ehrenfeucht game for $A$ and $\mathcal{B}$ and ends after $n$ more moves. The winning conditions are the same as for the usual Ehrenfeucht games.

PROPOSITION 4.7. *A structure $A$ is strongly elementarily approximated in a class* **K** *iff Player II as a winning strategy for every game with choice of a companion.*

Finally, we state a useful result on relative completeness, which follows from theorem 3.8. Recall that a first-order theory $T$ is complete with respect to a class **K** iff **K** is dense in $\mathrm{MOD}(T)$, and that every elementary class is itself a compact and Hausdorff space with the induced elementary topology.

DEFINITION 4.8. A class of first-order structures $\mathcal{M}$ is *co-elementary* (in a class of structures $\mathcal{K}$ if its complement (in $\mathcal{K}$) is elementary (in $\mathcal{K}$).

THEOREM 4.9. *Let $\{\mathbf{M}_k\}_{k \in \mathbf{N}}$ be a family of classes of $L$-structures, each of them co-elementary in an elementary class $\mathcal{K}$, and let $T = \mathrm{TH}(\mathcal{K})$ be complete with respect to each $\mathbf{M}_k$. Then $T$ is complete with respect to $\mathbf{M} = \cap_{k \in \mathbf{N}} \mathbf{M}_k$.*

**§5. An application: a relative completeness result of the first-order theory of coloured $\omega$-trees.** In this section we outline a sample completeness result obtained using ideas and results presented here. We give this result just as an illustration, rather than for its own sake, as the idea of the proof can be used to establish a more general fact. For a detailed proof and related results see [Goranko, 1999].

First, we need some definitions. By a *tree* we mean any (strictly) partially ordered set with a least element (root), in which every element has a linearly ordered set of predecessors. The elements of a tree are called *nodes*. A *path* in a tree is any maximal linearly ordered subset. A tree in which every path has the order type of $\omega$ will be called an $\omega$*-tree*. The set of predecessors of a node $a$ will be called the *stem of* $a$. A *sibling* of a node $a$ in a tree $T$ is any node in $T$ with the same stem as $a$. The *level* $k$ of a tree consists of the nodes which have $k$-element stems. (Thus, the 0-level consists of the root of the tree). The *finite levels* in a tree are all $k$-levels for $k \in \omega$. A tree is *finitely branching (on a level* $k$*)* if every node (on a level $k$) has finitely many siblings.

Moreover, we can consider trees enriched with finitely many additional unary predicates which will be called *colours*, and the resulting structures *coloured trees*.

THEOREM 5.1. *The first-order theory* $CT_\omega$ *of all (coloured)* $\omega$*-trees is complete with respect to the class of finitely branching (coloured)* $\omega$*-trees.*

PROOF. (Sketch: )

Let $\mathcal{M}_\omega$ be the class of all models of $CT_\omega$ and $\mathcal{M}^f$ consist of all trees from $\mathcal{M}_\omega$ which are finitely branching on all finite levels and in which every satisfiable formula of one variable is satisfiable on a finite level.

The proof consists of two major steps. The first step is to prove that $CT_\omega$ is complete with respect to $\mathcal{M}^f$. For this we show that $\mathcal{M}^f$ is dense in $\mathcal{M}_\omega$ with respect to the elementary topology. Indeed, $\mathcal{M}^f$ can be represented as an intersection of the family of classes $\{\mathcal{M}_k\}_{k\in\mathbf{N}}$, where $\mathcal{M}_k$ consists of the models $M$ of $CT_\omega$ finitely branching at the first $k$ levels and satisfying on finite levels the first $k$ formulae of some fixed enumeration of the formulae satisfiable in $M$. Note that each $\mathcal{M}_k$ is co-elementary in $\mathcal{M}_\omega$. Furthermore, it can be proved, using Ehrenfeucht's theorem, that each $\mathcal{M}_k$ is dense in $\mathcal{M}_\omega$ because every tree from $\mathcal{M}_\omega$ is $n$-equivalent to a tree from $\mathcal{M}_k$. Now the claim follows by theorem 4.9.

The second step then is to prove completeness of $CT_\omega$ with respect to the class of finitely branching $\omega$-trees. For that it suffices to show that every tree $T$ from $\mathcal{M}^f$ is elementarily equivalent to a finitely branching $\omega$-tree. We shall use the omitting types theorem. Consider the 1-type

$$\tau(x) = \{\neg l_k(x) | k \in \mathbf{N}\}$$

where $l_k(x)$ says that $x$ is on a level $k$. Note that $\tau$ is not principal in $T$ since any generator of that type would be satisfiable in $T$ by a node which belongs to some finite level. Hence, $\tau$ is omitted in a countable model $T'$ such that $T' \equiv T$, and hence $T' \models CT_\omega$. Then $T'$ is an $\omega$-tree. Furthermore, $T'$ is finitely branching at every level because it satisfies the same formulae $\chi_{k,m}$ saying that every node on level $k$ has no more that $m$ siblings, as $T$ does. Thus, $T'$ is a finitely branching $\omega$-tree, whence the completeness. ⊣

**§6. Concluding remarks.** In this paper we have only outlined some basic ideas of using topological methods to prove relative completeness and have discussed some rather immediate results regarding logical topologies. This approach can be further developed both from logical and topological perspectives.

From the topological perspective, there is much more to be done, as there is a number of non-trivial topological results which can be usefully reformulated in logical terms and applied for solving relative completeness (and other) problems. For instance, it is known (see [Fraïssé, 1967]) that $\mathcal{S}_{\mathrm{STR}}(L)$ are *uniform spaces*, which brings additional useful properties, little explored and used in logic so far.

A major logical perspective is to study logical topologies in second-order, infinitary, modal, etc. logics and to apply them to non-trivial completeness problems in these logics. Some of the results in first order logic easily generalize to a wide variety of other logical languages and systems. For instance, an analogue of the Stone topology can be introduced in every logical language with a disjunction, over the class of theories which are consistent and *prime* in sense that $\alpha \vee \beta \in T$ iff $\alpha \in T$ or $\beta \in T$. Then it is easy to check that $\mathrm{Cl}(\mathbf{S}) = \{T \in \mathbf{TH} | \bigcap \mathbf{S} \subseteq T\}$ defines a topological closure, and the logical topologies in that language are precisely the extensions of the resulting topology. Still, one can search for other useful constructions of topologies, logical *on a class of structures.*

One of the problems mentioned in the introduction can be re-phrased more generally as *elementary approximation of second-order properties*: *given a second-order theory $T$, and a first-order fragment $T_1$ of $T$, is the class of models of $T_1$ elementarily approximated in the class of models of $T$ ?* In other words: *is $T_1$ complete for the class of models of $T$?* Equivalently: *is $T_1$ the full first-order fragment of $T$?* It seems natural to explore this problem using logical topologies.

Finally, there is a number of basic model-theoretic constructions used in modal logic to transform Kripke models into 'standard' ones for the logic under consideration, such as *filtration* and *bisimulation*, (introduced in modal logic by van Benthem (see [van Benthem, 1984]) under the name of *zig-zag relation*). We hope that these constructions can be linked with the topological framework discussed here and thus the toolkit for proving completeness in modal logic can be strengthened and expanded.

## REFERENCES

[1995] R. Backofen, J. Rogers, and K. Vijay-Shanker, *A first-order axiomatization of the theory of finite trees*, ***Journal of Logic, Language and Information***, vol. 4 (1995), pp. 5–39.

[1985] J. Barwise and S. Feferman (editors), ***Model-theoretic logics***, Springer-Verlag, New York, 1985.

[1996] J. Cifuentes, A. Sette, and D. Mundici, *Cauchy completeness in elementary logic*, ***The Journal of Symbolic Logic***, vol. 61 (1996), no. 4, pp. 1153–1157.

[1987] H.C. Doets, *Completeness and definability: Applications of the Ehrenfeucht game in intensional and second-order logic*, ***Ph.D. thesis***, Department of Mathematics and Computer Science, University of Amsterdam, 1987.

[1994] H.-D. Ebbinghaus, J. Flum, and W. Thomas, ***Mathematical logic, 2nd ed.***, Springer-Verlag, Berlin, 1994.

[1985] R. Engelking, ***General topology, 2nd ed.***, PWN, Warsaw, 1985.

[1967] R. Fraïssé, ***Course de logique mathématique***, vol. 1, Gauthier-Villars, Paris, 1967.

[1985] R. Goldblatt, *On the role of the Baire category theorem and dependent choice in the foundations of logic*, ***The Journal of Symbolic Logic***, vol. 50 (1985), pp. 412–422.

[1999] V. Goranko, *Trees and finite branching*, ***Proc. of the second Panhellenic logic symposium*** (P. Kolaitis and G. Koletsos, editors), 1999, pp. 107–111.

[1993] W. Hodges, ***Model theory***, Cambridge University Press, Cambridge, 1993.

[1963] H. Rasiowa and R. Sikorski, ***The mathematics of metamathematics***, PWN, Warsaw, 1963.

[1983] J. van Benthem, ***Modal logic and classical logic***, Bibliopolis, Naples, 1983.

[1984] ———, *Correspondence theory*, (D.M. Gabbay and F. Guenthner, editors), vol. 2, Reidel, Dordrecht, 1984, pp. 167–247.

[1993] Y. Venema, *Completeness via completeness: Since and Until*, ***Diamonds and defaults*** (Dordrecht) (M. de Rijke, editor), Kluwer Academic Publishers, 1993, pp. 279–286.

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