COMPLETENESS AND INCOMPLETENESS IN THE BIMODAL BASE \( \mathcal{L}(R,-R) \)

Valentin Goranko

Sector of mathematical logic
Faculty of Mathematics and Computer Science, Sofia University
boul. Anton Ivanov 5, Sofia 1126, BULGARIA

INTRODUCTION

The paper deals with a modal language \( \mathcal{L}(R,-R) \), having an ordinary modality \( \Box \) (dual - \( \Diamond \)) with an usual Kripke-semantics \( \Box p \iff \forall y (R_{xy} \rightarrow y)p \) and an additional modality \( \Diamond \) (dual - \( \Box \)), with the same semantics however over the complement \( -R \): \( \Diamond p \iff \forall y (-R_{xy} \rightarrow y)p \). Such a modality has been considered by some authors in different contexts - see e.g. [Hum] and [GPT1], where the completeness theorems for the minimal normal \( \mathcal{L}(R,-R) \) logic are independently proved. This language appears as a special case of the notion polymodal base, introduced by the author in [Gor]. This notion combines a polymodal language \( \mathcal{L}(Q_1,\ldots,Q_n) \) with a set of formulae \( \mathcal{Q} \), having a usual relational semantics over structures \( \langle W,R_1,\ldots,R_n \rangle \) (frames) and a theory \( T \) in some language (for definiteness - first-order) for such structures. We shall denote such a base \( \mathcal{L}_T(R_1,\ldots,R_n) \). The models of the theory \( T \) will be called standard frames of this base. In particular, when the theory \( T \) determines some of the relations \( R_1,\ldots,R_n \) by means of the rest of them, the polymodal base becomes an enriched (polymodal) language. A typical example of it provides the modal language for tense logics - it is a bimodal base with a theory \( T' \) having a single axiom \( \langle - \rangle \forall xy(R_{1xy} \leftrightarrow R_{2xy}) \) and standard frames \( \langle W,R,R \rangle \) - it is an enriched modal language for \( \langle W,R \rangle \). Another example is the language in question \( \mathcal{L}(R,-R) \) being a bimodal base with theory \( T' \) with an axiom \( \langle - \rangle \forall xy(R_{1xy} \leftrightarrow -R_{2xy}) \) and standard frames \( \langle W,R,-R \rangle \). However, there exists an important distinction between the two

1 Research partially supported by the Committee for Science at the Council of Ministries of Bulgaria, Contract No. 933.
2 The author is grateful to the referees for the useful remarks.
bases: The axiom }\neg (\neg A) \text{ is modally definable while } (\neg A) \text{ is not. This creates considerable differences between the expressive possibilities } \Sigma(R, -R) \text{ is a strong, with respect to definability, modal language; e.g. each universal formula for } R \text{ and } = \text{ is definable in it (see [Gor])) and the axiomatizing procedures in them. The aim of this paper is to suggest a general technique for proving completeness and to apply it in concrete situations.}

PRELIMINARIES

We shall briefly recall or introduce some notions concerning modal logic and especially modal bases. The basic notions of modal logic (as given in the initial chapters of [Seg], [HCl], or [Bem2], in particular; valuation and model over a given frame; general (first-order) frame, forcing (**) and validity in a model (frame, general frame), canonical frame, model and general frame etc. will be expected to be familiar. Their generalizations in a modal language are trivial. We shall also deal with the natural generalizations of the basic frame constructions: generated subframes, p-endoframe, ultrafilter extensions (we) and Stone representation (SR) with a little specification: the notion 'generated subframe' of a given frame will be reserved for the ones generated from one world and the others will be called simply subframes. If } F \text{ and } B \text{ are frames and } F \equiv_B (G) \text{ then } B \text{ will be called an ultrafilter congruence of } F.

Let } C \text{ and } D \text{ be classes of frames for a given modal language with a set of formulae } \Sigma. \text{ Define:}

- modal theory of } C: \Theta^C \text{ (mod) } (C \equiv (\text{spec } C) \equiv Cp)\),
- modally definable closure of } C \text{ in } D: \Theta^D \text{ consists of all frames from } D, \text{ in which } \Theta^C \text{ is valid. When } D \text{ is the class of all frames of the given language, the closure will be denoted by } \Theta.

Let } \Sigma = \Sigma(R_1, \ldots, R_n) \text{ be a fixed modal language with a class of standard frames } F(T). \text{ The models over standard frames will be called standard models. The frames from } F(T) \text{ will be called basic frames and their generated subframes - total frames. Basic } \Sigma \text{-logic is the logic } K_\Sigma = \Theta^C \text{ (mod) } (F(T)) \text{ each simple extension of } K_\Sigma \text{ is an } \Sigma \text{-logic. } \Sigma \text{-logic } L \text{ is complete with respect to a class of frames (models) } C \text{ iff } \Theta^C \equiv (\text{spec } C) \equiv Cp. \text{ A frame } F \text{ is a frame for } L \text{ (L-frame) iff } \Theta^C \equiv (\text{spec } C) \equiv Cp. \text{ An } L \text{-model is defined in the same manner.}

Some notions concerning completeness. An } \Sigma \text{-logic } L \text{ is:

trivially complete if } L \text{ is complete with respect to the class of all } L \text{-models; standardly complete if } L \text{ is complete with respect to the class of the standard } L \text{-models; basically complete if } L \text{ is complete with respect to the class of all } L \text{-frames; normally complete if } L \text{ is complete with respect to the class of the standard } L \text{-frames.}

to the class of the standard } L \text{-frames. Trivial completeness is provided by the canonical model and always holds, so it is not interesting as distinct from the proving method. The interesting notion of completeness in the classical modal logic is basic completeness. However in modal bases the goal is normal completeness, which guarantees the adequacy of the axiomatics with respect to the special semantics specified at the base. The purpose of the paper is to prove namely the normal completeness. So the adjective "normal" will be omitted and completeness will mean normal completeness from now on.

Some more notions: the logic } L' \text{ is a weak extension of } L \text{ if } L' \text{ is an extension of } L \text{ and has the same standard frames as } L. \text{ ( Obviously, } L \text{ has proper weak extensions iff } L \text{ is incomplete). The complete weak extension of } L \text{ will be called a completion of } L \text{ and denoted by } c(L). \text{ Let } \Sigma_1 \text{ and } \Sigma_2 \text{ be modal languages such that } \Sigma_1 \subseteq \Sigma_2 \text{ and } K_{\Sigma_1} \text{ and } K_{\Sigma_2} \text{ are their minimal logics, and } \Gamma \subseteq \Sigma_1. \text{ Then the } \Sigma_2 \text{-logic } K_{\Sigma_2} \text{ (axiomatized over } \Gamma_2 \text{ with } \Gamma) \text{ will be called a minimal extension of } K_{\Sigma_2} \Gamma \text{ in the language } \Sigma_2. \text{ It is easy to prove that the minimal extensions are conservative.}

Now, let us recall some notions from modal logic. A general frame } \text{ is } (W, R_1, \ldots, R_n, M) \text{ is descriptive iff for each ultrafilter } u \text{ in } M:

1) \exists \omega \in \omega \text{ with } u\omega = (\omega)

2) \forall (R, \omega) \in (u) \omega = (\omega R) u \omega \text{ for } i = 1, \ldots, n.

A set of formulae } \Gamma \text{ is canonical iff for each descriptive frame } \text{ is } (C, M): \Gamma \text{-frame } \text{ is } C \text{-frame. A logic } L \text{ is canonical if } L \text{ is axiomatized with a canonical set of formulae. The canonical logics are complete, since the canonical model generates a descriptive frame, but the canonicity is a condition stronger than truth in the canonical frame; the logics with the latter property will be called (following Segerberg) natural.}

Now our strategy will be the following. The proof of completeness will be split into two stages: first - basic completeness and second - (normal) completeness. The first stage is attacked with traditional methods of the modal logic - the canonical model method when the logic is natural, or with an appropriate supplementary construction otherwise (see [Seg], [HC]). The following lemma guarantees transferring of canonicity (hence the basic completeness) to minimal extensions.

**LEMMA 1.** A minimal extension of a canonical logic is canonical.

**PROOF.** Let } L' \text{ be a minimal extension of the canonical logic } L \text{ and, for convenience, the base of } L \text{ be } \Sigma = \Sigma(R_1, \ldots, R_n) \text{ and of } L' = \Sigma = \Sigma(R_1, R_2, \ldots, R_n) \text{, } n \text{-k. Then an } \Sigma \text{-frame } (W, R_1, \ldots, R_n) \text{ is an } L' \text{-frame iff } (W, R_1, \ldots, R_n) \text{ is}
L-frame. Let $\mathfrak{F} = \langle W, R_1, \ldots, R_n, U \rangle$ be a descriptive $L'$-frame. Then
$\mathfrak{F} = \langle W, R_1, \ldots, R_n, U \rangle$ is a descriptive $L'$-frame $\iff (W, R_1, \ldots, R_n)$ is an $L'$-frame $\implies (W, R_1, \ldots, R_n)$ is an $L'$-frame $\iff L'$ is canonical.

It is not clear when the naturalness is transferred into minimal extension, but, as Fine has proved (see [Fin]), basic completeness (in particular naturalness) $+$ first order definability implies canonicity.

The second stage, having a basic completeness result, is to prove (normal) completeness. So, let $B$ be a basically complete logic of some base and $\phi$ be a non-theorem of $L$, refuted in a basic $L'$-frame $F$. Let $F_{\phi} = \langle W, F_{\phi}(L), F_{\phi}(U) \rangle$ be the class of basic / total, standard $L'$-frames. If $F$ is complete, then $F \models [F_{\phi}(L)] = F_{\phi}(L)$. So if we know the description of the modally definable closure as a sequence of closure operations then the way back, starting from $F$, will bring us to a standard frame $G$, refuting $\phi$. The way can be shortened, e.g., if we start from a total frame, refuting $\phi$. Conversely if the procedure always goes through this proves the completeness of $L$; if not this, may show us the reasons for the incompleteness and show the missing axioms.

Completeness, incompleteness and completions in $\mathfrak{Z}(L')$.

The basic $\mathfrak{Z}(L')$-logic, denoted in [GPT] by $K'$, is axiomatized there with the 55-axioms for the modality $\Phi$ (dual $-$ $\phi$), defined by $D_{B} E_{B} E_{B} \Phi_{B} (r, t) = b$, and $B = \Phi_{B}$ and (s) $p \phi_{B}$. So the basic $\mathfrak{Z}(L')$-frames are those $\langle W, R, \eta, R >$ in which $R_{\eta} U_{\eta}$ is an equivalence relation and the total $\mathfrak{Z}(L')$-frames $-$ those in which $R_{\eta} U_{\eta}$ is an universal relation. Denote the class of all total $\mathfrak{Z}(L')$-frames by $D_{B}$. The proof of the standard completeness of $K'$, exposed in [GPT] is directly transferred to any simple extension. However the problem for completeness is rather more complicated. We shall apply the idea sketched above for both proving completeness and completing basically complete extensions of $K'$ in the case when the axioms are first-order definable formulae. Let $L$ be a basically complete simple extension of $K'$ with first-order definable axioms and $\phi$ be a non-theorem of $L$. Then $\phi$ is refuted in a total $L'$-frame $F$. If $L$ is complete then $F \models [F_{\phi}(L)] = C_{\phi}$ which consists of all ultrafilter contractions of $p$-morphic images of members of $\mathfrak{Z}(L')$ (see [Gor]). Hence the refuting standard $L'$-frame is to be constructed as a $p$-morphic inverse-image of an ultrafilter extension of $G$. Conversely, if for an arbitrary non-theorem $\phi$ and total frame $G$, refuting $\phi$, we succeed to find in such a way standard $L'$-frame $F$ then $F \models \phi$ since $F \models [F_{\phi}(L)] = C_{\phi}$. So the completeness of $L$ will be established. Really, as we shall see, the procedure establishing completeness is still simpler.

**LEMMA 2.**

If $\mathfrak{F} = \langle W, R, U \rangle$ is a descriptive frame then $F$ is a $p$-morphic image of $\mathfrak{F}(F_{\phi})$.

**PROOF.** $\mathfrak{F} = \langle W, R, U \rangle$ is a $p$-morphic image of $\mathfrak{F}(F_{\phi})$ (see [Gor]).

**LEMMA 3.** Each generated sub-frame of a canonical general frame for $\mathfrak{Z}(L')$-logic is descriptive.

**PROOF.** Let $L$ be an $\mathfrak{Z}(L')$-logic, $F$ be a maximal $L$-consistent set (L-CS) and $F = \langle W, R, U \rangle$ be the subframe of the canonical general $L'$-frame $\mathfrak{F} = \langle W, R, U \rangle$ that is generated from $F$, i.e., $W = \langle W, U \rangle$.

**LEMMA 3.** Each generated sub-frame of a canonical general frame for $\mathfrak{Z}(L')$-logic is descriptive.

**PROOF.** Let $L$ be an $\mathfrak{Z}(L')$-logic, $F$ be a maximal $L$-consistent set (L-CS) and $F = \langle W, R, U \rangle$ be the subframe of the canonical general $L'$-frame $\mathfrak{F} = \langle W, R, U \rangle$ that is generated from $F$, i.e., $W = \langle W, U \rangle$.

**LEMMA 3.** Each generated sub-frame of a canonical general frame for $\mathfrak{Z}(L')$-logic is descriptive.

**PROOF.** Let $F$ be an $\mathfrak{Z}(L')$-logic, $F$ be a maximal $L$-consistent set (L-CS) and $F = \langle W, R, U \rangle$ be the subframe of the canonical general $L'$-frame $\mathfrak{F} = \langle W, R, U \rangle$ that is generated from $F$, i.e., $W = \langle W, U \rangle$.

**LEMMA 3.** Each generated sub-frame of a canonical general frame for $\mathfrak{Z}(L')$-logic is descriptive.

**PROOF.** Let $F$ be an $\mathfrak{Z}(L')$-logic, $F$ be a maximal $L$-consistent set (L-CS) and $F = \langle W, R, U \rangle$ be the subframe of the canonical general $L'$-frame $\mathfrak{F} = \langle W, R, U \rangle$ that is generated from $F$, i.e., $W = \langle W, U \rangle$.
Note. It immediately follows from the definition of $R$ that $-R(x,i)(y,j)$ iff $R_y(x,y,j)$. #

**LEMMA 4.** The standard frame $F$ is isomorphic to a standard extension of $F'$ iff $F'$ is a $p$-morphic image of $F$. #

**PROOF**

1) If $F$ is a standard extension of $F'$ then we shall prove that the mapping $f:F\to F'$, defined by $f(x,i)=x$ is a $p$-morphism:

i) Let $R(x,i)(y,j)\Rightarrow R_y(x,y,j)$ by the definition.

ii) If $f(R(x,i)(y,j))$, then according to (8) there exists $y\in Y$ such that $R(x,i)(y,j)$. Analogously it follows from $f^{-1}(y)$ that $-R(x,i)(y,j)$.

2) Conversely, let $F=\langle W,R_1,R_2 \rangle$ and $f:F\to F'$ be a $p$-morphism. We shall prove that $F$ is isomorphic to a standard extension of $F'$ of $F$. Let $x\in W$.

Put $y=f^{-1}(x)$ and define predicate $P(x,i)(y,j)$ iff $R_{ij}$. The condition (8): Since $f$ is a $p$-morphism, then $R_{ij}$ implies that for every $i\in I_x$ there exists $j\in J_x$ such that $R_{ij}$; $R_{xy}$ implies that for every $j\in J_x$ there exists $i\in I_x$ such that $-R_{ij}$. The mapping $f':F'\to F$, defined by $f'(x,i)=x$ is an isomorphism. #

**COROLLARY 5.** If $F'$ is and $F$ is a standard extension of $F'$, then $F'$ is also.

So if $L$ is a logic with a natural first-order axiomatization in order to prove the completeness of $L$ it is sufficient to find a standard extension, which is an $L$-frame, for each total $L$-frame. To this aim we have to construct the carrier of the extension and the predicate $P$, satisfying the condition (8). Actually the standard extensions are natural generalizations of the construction 'copying' proposed by Vakarelov and applied in [GP] (for proving the completeness of $L^+$). [[Segl]]

**Theorem 6**

The logics

i) Ver$_n^L=K^n+\Box 1$

ii) Seq$_n^L=K^n+\Box^n T_i$

iii) Triv$_n^L=K^n+\Box^n 1^n$ are complete.

**PROOF**

Ver$_n^L$, Seq$_n^L$ and Triv$_n^L$ are canonical [Seg11], hence their minimal extensions are basically complete. It only remains to construct the corresponding standard extensions.

i) Let us note that the $\text{Ver}_n^L$-frames are those in which there are no $R_i$-chains with a length $n$ ($\text{HC}_n$). Let $F'=\langle W, R_1, R_2 \rangle$ be a total Ver$_n^L$-frame. Put $W_1=(0,1)$ and $P(x,i)(y,j)$ iff $i=y$. The condition (8) holds: if $i\in(0,1)$ then $P(x,i)(y,j)$.

The obtained standard frame $F$ is a Ver$_n^L$-frame: if $P(x,i)(y,j)$ then $P(x,y)$. #

ii) The Seq$_n^L$-frames are those in which from every point an $R_1$-chain with a length $n$ starts. Let $F'=\langle W, R_1, R_2 \rangle$ be a total Seq$_n^L$-frame. We shall use the construction from i). Then $F$ will be a Seq$_n^L$-frame: if $i\in(0,1)$ then there exists an $R_i$-chain with a length $n$ and first point $x$ in $F'$; $x_{i,1} \Rightarrow x_{i,2} \Rightarrow \cdots$, $i=1 \Leftrightarrow \cdots$, $(x_{i,1},x_{i,2},x_{i,3},x_{i,4},x_{i,5},x_{i,6},x_{i,7})$ is an $R$-chain with a length $n$ in $F$.

iii) The Triv$_n^L$-frames are $\langle W, R_1, R_2 \rangle$. Now the above construction does not apply. Put $W_1=(x,y)/R_1(x,y)$ and $W_2=(x,y)/R_2(x,y)$. The predicate $P(x,i)(y,j)$ is defined as before and a $\text{Triv}_n^L$-frame $F$ is obtained: $R_1(x,i)(y,j)$ and if $x_{i,1}(x,i)<y_{i,1}(x,i)$, then $x_{i,1}(x,i)$ if $P(x,i)(y,j)$ then $i=j \neq 0$; if $P(x,i)(y,j)$ then it follows from the definition of $R_i$ that $i=j$. In both cases $P(x,i)(y,j)$. #

Note that Triv$_n^L$ axiomatizes in essence the modality $[x]$ (which is $\Box x$).

The impression that completeness is almost directly transferred into minimal extensions is deceptive. On the contrary, as we shall see here the incompleteness events are quite usual and this is because the language $R_i(R_i-R)$ can express the same things in essentially different manners (with non-equivalent formulae).

**Example:** The logic KB$^{\Delta}$ is incomplete. Indeed, the KB$^{\Delta}$-frames are <W, R, R> in which R is a symmetric relation. In the standard KB$^{\Delta}$-frames it automatically follows that $R_1$ is also symmetric, which is expressed by the formula $p+\Box p$. However this formula is not true in all KB$^{\Delta}$-frames, hence is not provable in KB$^{\Delta}$. #

A great number of interesting modal logics are axiomatized by so called modal reduction principles (see [Ben]). A typical example is the formula $\Box^n \phi^{\Delta} (\phi^{\Delta})$ which expresses the condition $C_{ij}^k(x): \forall x,y,z (R^n(x,y)R^n(x,z) \to R^n(x,y)R^n(x,z))$ (LSJ). The formula $A_{i,j,k}^{n,\Delta}$ is canonical and the logic $L_{i,j,k}^{n,\Delta}$ is complete ($\text{HC},3,13$).

We shall introduce some denotations which will be used further on. Let $\langle W, R_1, R_2 \rangle$ be a total frame and $x\in W$. We shall say that $x$ has:

- entry defect (denoted $d_1(x)$) if $\exists y\in W(R_1(x,y)\Box^n R_2 x)$;
- exit defect (denoted $d(x)$) if $\exists y\in W(R_1(x,y)\Box^n R_2 y)$.

Put $D_1(W)=d_1(x)$ $D_2(W)=d(x)$.

**Theorem 7**

i) The logic $L_{i,j,k}^{n,\Delta}=K^n+\Box^n 1^n+p+\Box^{n,\Delta} p$ is complete in the following cases:
Let \( n \geq 1 \).

ii. In all remaining cases \( m,n \) is incomplete and is completed by an additional axiom \( \text{ax}^{m,n} \) as follows:

\[
\begin{align*}
\text{ax}^{0,0} & : K = \phi \land \psi \lor \phi \land \neg \psi \land \phi \land \psi,
\text{ax}^{1,j} & : \neg K = \phi \land \psi \lor \phi \land \neg \psi \land \phi \land \psi,
\text{ax}^{j,1} & : K = \phi \land \psi \lor \phi \land \neg \psi \land \phi \land \psi.
\end{align*}
\]

The \( \text{ax}^{m,n} \) are equivalent to the completeness of \( \text{ax}^{1,1} \).

PROOF

Let us note that \( \text{ax}^{m,n} \) and \( \text{ax}^{1,1} \) are equivalent. The \( \text{ax}^{m,n} \) are those, for which \( R_1 \) satisfies the condition \( \text{ax}^{1,1} \). \( \text{ax}^{m,n} \) is basically complete since \( \text{ax}^{1,1} \) is canonical. Let us construct the corresponding standard extensions.

i. the formulae \( \text{ax}^{1,1} \) and \( \text{ax}^{0,0} \) are tautologies and the assertion follows from the completeness of \( K \).

ii. a) \( \text{ax}^{1,0} \) holds in \( \mathcal{L}_j \) by definition.

The condition \( b \) holds in \( \mathcal{L}_j \).

Let \( \mathcal{W} = (W, R_1, R_2) \) be a total \( \text{ax}^{1,0} \)-frame. Put \( W = W \times \{(0,0)\} \) and \( \text{ax}^{1,0} \) iff \( w \in W \) for every \( u,v \in W \). The condition \( a \) holds in \( \mathcal{L}_j \).

Let \( \mathcal{W} = (W, R_1, R_2) \) be a total \( \text{ax}^{1,0} \)-frame. Put \( W = W \times \{(0,0)\} \) and \( \text{ax}^{1,0} \) iff \( w \in W \) for every \( u,v \in W \). The condition \( b \) holds in \( \mathcal{L}_j \).

Let \( \mathcal{W} = (W, R_1, R_2) \) be a total \( \text{ax}^{1,0} \)-frame. Put \( W = W \times \{(0,0)\} \) and \( \text{ax}^{1,0} \) iff \( w \in W \) for every \( u,v \in W \). The condition \( b \) holds in \( \mathcal{L}_j \).

Finally if \( k > 2 \) put \( t \equiv (r, 0, 0, 0) \). According to the observation \( R^P \) and \( R^k \).

The proof is complete.

\( \square \)
Now we shall proof the completeness of 
$\rho_{yx} + \rho_{xy}$.

In all cases put $W = W(0) \cup D_1(0)$.

1. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

2. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

3. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

4. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

5. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

6. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

7. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

8. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

9. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

10. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

11. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

12. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

13. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

14. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

15. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

16. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

17. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

18. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

19. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

20. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

21. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

22. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

23. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

24. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

25. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

26. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

27. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

28. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

29. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

30. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

31. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

32. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

33. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

34. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

35. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

36. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

37. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

38. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

39. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

40. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

41. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

42. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

43. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

44. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

45. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

46. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

47. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

48. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

49. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

50. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

51. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

52. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

53. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

54. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

55. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

56. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

57. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

58. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

59. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

60. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

61. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

62. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

63. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

64. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

65. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

66. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

67. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

68. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

69. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

70. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

71. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

72. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

73. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

74. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

75. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

76. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

77. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

78. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

79. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

80. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

81. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

82. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

83. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

84. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

85. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

86. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

87. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

88. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

89. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

90. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

91. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

92. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

93. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

94. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

95. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

96. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

97. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

98. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.

99. Define $\rho_{yx}$. If $i = 1$, then $\rho_{yx}$.

100. Define $\rho_{xy}$. If $i = 1$, then $\rho_{xy}$.
ii) c(S4") is basically complete, according to lemma 1, since S4 is canonical. Let \( F' = \langle W', R_1, R_2 \rangle \) be a total c(S4")-frame, i.e., \( R_1 \) is reflexive and transitive and \( R_1 \) and \( R_2 \). The standard extension: \( W \equiv W'^{(0)} \cup D \) \((W'^{(1)})^{(1,2,3)}; \) Put \( \langle x, i > < y, j \rangle \) iff \( \langle x, i \rangle < < y, j \rangle \) or \( j \) is an even number. The condition (\@): Let \( (R_1 \upharpoonright R_2) \). Then \( d_1(y) \equiv \langle x, i > < y, o \rangle \). The proof of weak completeness.

Theorem 9

1) The completion of SS" is \( c(SS") = SS" + p \Rightarrow E \Rightarrow p + \Rightarrow E \Rightarrow p \).

Proof

1) \( SS" \Rightarrow p \Rightarrow E \Rightarrow p \).

Theorem 10

1) The logic of weak linearity \( S4.3" = S4" + \Rightarrow E \Rightarrow p \Rightarrow E \Rightarrow p \) is incomplete.

ii) Its completion \( c(S4.3") \) is axiomatized as follows:

**Axioms:**

- (ref) \( \Rightarrow p \Rightarrow p \)
- (trans) \( \Rightarrow E \Rightarrow p \Rightarrow E \Rightarrow p \)
- (trans') \( \Rightarrow E \Rightarrow p \Rightarrow E \Rightarrow p \)
- (left) \( \Rightarrow E \Rightarrow p \Rightarrow E \Rightarrow p \)
- (right) \( \Rightarrow E \Rightarrow p \Rightarrow E \Rightarrow p \)

**Corresponding conditions in total frames:**

- \( R_1 \)
- \( R_1 \Rightarrow E \Rightarrow R_2 \)
- \( R_1 \Rightarrow E \Rightarrow R_2 \)
- \( R_1 \Rightarrow E \Rightarrow R_2 \)
- \( R_1 \Rightarrow E \Rightarrow R_2 \)

**Proof**

The correctness in total frames and naturalness of the proposed axioms are directly verified. So \( c(S4.3") \) is basically complete. Now let \( F'' = \langle W'', R_1, R_2 \rangle \) be a total c(S4.3")-frame. In order to construct the corresponding standard extension we shall define relations \( \rho \) and \( \eta \) in \( W'' \):

- \( \rho \) is an equivalence relation in \( W'' \) since \( R_1 \) is a quasi-ordering.

**Proof**

The correctness in total frames and naturalness of the proposed axioms are directly verified. So \( c(S4.3") \) is basically complete. Now let \( F'' = \langle W'', R_1, R_2 \rangle \) be a total c(S4.3")-frame. In order to construct the corresponding standard extension we shall define relations \( \rho \) and \( \eta \) in \( W'' \):

- \( \rho \) is an equivalence relation in \( W'' \) since \( R_1 \) is a quasi-ordering.

**Proof**

The correctness in total frames and naturalness of the proposed axioms are directly verified. So \( c(S4.3") \) is basically complete. Now let \( F'' = \langle W'', R_1, R_2 \rangle \) be a total c(S4.3")-frame. In order to construct the corresponding standard extension we shall define relations \( \rho \) and \( \eta \) in \( W'' \):

- \( \rho \) is an equivalence relation in \( W'' \) since \( R_1 \) is a quasi-ordering.
The obtained frame is an interval ordering.

Finally let us note that the exposed technique can be adapted in appropriate manners in other bases, e.g. \((R_1,R^{-1},R^{-1})\) etc.

I am grateful to Dimiter Vakarelov, George Gargov and Solomon Passy for fruitful discussions on the problems of the paper. Also I thank Mark Brown from the Syracuse University for some stylistic remarks.

REFERENCES


[GP] Gargov G., S. Passy, A Note on Boolean Modal Logic, this volume


[Vak1] Vakarelov D., S4 + S5 together - S4+5, LMPS'87, Vol. 5, Moskow, 1987
