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An Extended Branching-Time Ockhamist Temporal Logic

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Abstract. For branching-time temporal logic based on an Ockhamist semantics, we explore a temporal language extended with two additional syntactic tools. For reference to the set of *all possible futures* at a moment of time we use syntactically designated "restricted variables" called *fan-names*. For reference to all possible futures *alternative* to the actual one we use a modification of a *difference modality*, localized to the set of all possible futures at the actual moment of time.

We construct an axiomatic system for this extended branching-time logic and prove its soundness and completeness with respect to bundle tree semantics. Finally, we show how our axiomatic system can be extended with a variety of important additional operators, such as *Since* and *Until*, a global difference operator, operators for undivided and divided histories, reference pointers, etc.

Key words: Temporal logic, branching-time, Ockhamist, bundle tree semantics, fan-names, local difference operator, axiomatic system, completeness

1. Introduction

Ever since Prior's seminal modern treatise (Prior, 1967) on temporal logic, the study of branching-time temporal logics has formed one major portion of the subject. Branching-time logics have been found technically convenient for modeling and analyzing properties of non-deterministic programs and related applications in computer science (Emerson, 1990; Stirling, 1992), but also conceptually important in the philosophical discussion of time and tense (Thomason, 1984). There are several reasonable semantic approaches to branching-time logics, arising from different treatments of the non-deterministic future possibility and necessity operators. The most typical of these are the *Priorean, Peircean* and *Ockhamist* semantics (Prior, 1967). The last of these corresponds to the most expressive language of the

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three, and we shall concentrate on it. According to the Ockhamist (or, *indeterminist actualist*, see Zanardo, 1991, 1996, 1998) approach, the truth of a formula is evaluated relative to *possible future branches*, or equivalently, at pairs (*moment, history through that moment*), and the temporal operators are relativised to the actual history of the evaluation. To discuss historical necessity (i.e., truth in all possible futures) an additional modal operator $\Box$ is introduced.

The standard semantic structures for the Ockhamist branching-time logic are trees. These are naturally generalized to *bundle trees* (Burgess, 1979; Zanardo, 1996) which are technically equivalent to *Kamp frames*, as shown in the latter paper. A bundle in a tree is a set of branches in the tree, closed under subbranches and extensions, and covering all nodes in the tree. The bundle tree semantics is more general than the standard one: there are formulae valid in all trees but not in all bundle trees (Thomason, 1984). It is a matter of philosophical debate which of the two semantics should be regarded as the more natural one (ibid.). The answer to this question depends to a great extent on the ontological commitments made about the nature of the structure of time, and whether moments or histories, or even branches, should be accepted as primitive temporal entities. We shall not enter that debate, but shall adopt the bundle tree semantics as the basic one for our study.

The Ockhamist bundle tree semantics for the standard temporal language with an additional "historical necessity" operator has been axiomatized in Zanardo (1985), and for the Since-Until language in Zanardo (1991). Further completeness and definability results for some generalized Ockhamist semantics have been proved in Zanardo (1996). The related $T \times W$-validity (Thomason, 1984) has been finitely axiomatized in von Kutschera (1997) using a additional rule of a special type, often called Gabbay-style rule (see Burgess, 1980 and Gabbay, 1981, where according to our knowledge such rules were independently introduced in temporal logic), and in Di Maio and Zanardo (1996) without such a rule but with an infinite system of axioms. There are some completeness results for specific time structures, e.g., see Stirling (1992) for a complete axiomatization of the past-less temporal logic for discrete branching-time, and Zanardo and Carmo (1993) where the past operators have been added.

In this article we explore an extended temporal language for dealing with branching-time temporal logic based on Ockhamist semantics. We have two kinds of reasons for putting forward such an extension.

Conceptual reasons: If we commit ourselves to the idea of Ockhamist indeterminism, and in particular to the ontological primacy of possible futures and histories over moments of time, then talk about what will or might be, or could have been true in the future has to be handled more carefully than is customary in natural language. To express some of the subtleties involved, we need to enrich our formal language. We propose two syntactically different tools for doing so, both of which have already featured in a number of extended modal logics designed for various purposes, and have recently been actively discussed in the literature. For reference to the set of *all possible futures* at a given moment of time (cf. the "immediate possibilities" in Belnap, 1992) we use syntactically designated "restricted variables" (see below), called here *fan-names*. The idea and technology of using "names" for possible worlds in modal logic has been developed in various sources, e.g., Prior (1956), Bull (1970), Blackburn (1989), Passy and Tinchev (1991), Garvov and Goranko (1993). For reference to all possible futures *alternative* to the actual one we use a modification of a *difference modality*, localized to the set of all possible futures at the actual moment of time. The difference modality, too, has recently been extensively studied and applied in, e.g., de Rijke (1989) and Venema (1991).

Technical reasons: The idea of using names naturally generalizes to the idea of using *restricted variables* in modal logic, i.e., variables whose possible valuations in Kripke models are subject to specific constraints. A typical well-known, though somewhat isolated, example occurs in *intuitionistic logic* where all variables are evaluated in upwards closed subsets of an intuitionistic Kripke frame. In our present framework fan-names can be regarded as another example of restricted variables, which are subject to the semantic condition of being evaluated not just in any set of possible futures (branches) but only in sets (fans) of all possible futures stemming from single moments of time. In the former case, the restriction on the valuations does not lead to increased expressiveness of the language (as there are well known translations of the intuitionistic logic into modal logic), because it is *syntactically definable*. In cases, like names and fan-names, where the semantic restrictions are not syntactically definable, the use of restricted variables leads to expressively more powerful languages and that, in particular, usually makes it easier to axiomatize the logic under consideration. The introduction of the difference operator has a similar effect, as has been demonstrated in the sources mentioned above.

It may seem at first that we should be able to get by with just one of these two new syntactic tools, using it in combination with the usual syntactic resources of temporal logic to get the other. That, however, is not so: neither of them is expressible in terms of the other. Combining use of the fan-names and the difference operator in our system renders the language expressive enough to provide a relatively simple and perspicuous axiomatic system, and to enable an elegant development of the model theory of the logic we study, necessary for the proof of the completeness theorem. A technical advantage of our approach is that the canonical model construction is considerably simplified and much closer in style to the traditional one in modal logic (appropriately modified to deal with the additional rules), and eventually leads to a direct construction of the canonical frame, explicitly defining the set of moments of the tree, with the branches then defined in a standard manner. The canonical model technique so far developed for branching-time logics in traditional temporal languages (Zanardo, 1985) typically first constructs a bundle of branches, then defines the set of moments as certain equivalence classes of branches, which usually results in technical complications.

It should be noted that some extensions of the language for Ockhamist branching-time logic and related systems have already been proposed in the literature, e.g., in Di Maio and Zanardo (1994) and Zanardo (1996, 1998). Zanardo (1996) considers an extension of the temporal language for so called *Ockhamist frames* (which are essentially equivalent to bundle trees) with a *global* difference operator and gives a complete axiomatic system using Sahlqvist-type axioms and Burgess–Gabbay style rules. (In Section 6 we discuss the relations between global and local difference operators.) Further, Zanardo (1998) introduces the notions of *undivided* and *indistinguishable histories* and modal operators related to them, which produce extensions of the language some of which are in a way definable in ours (see Section 6).

Such examples illustrate the recent trend to develop strongly expressive extended modal languages, and in particular to strengthen the expressiveness of the temporal language for branching-time structures. We note that the class of Ockhamist frames introduced in Zanardo (1996) is definable in our language.

The use of Burgess–Gabbay style rules is not new, either, in the framework of branching-time logics (see, e.g., Zanardo, 1991, 1996; von Kutschera, 1997), and in languages with names or difference operators it is rather the rule than the exception. (In our concluding remarks we include a short discussion on the pros and cons of using such rules.)

We have, therefore, put together existing ideas in a way appropriate for our framework and we believe that the result is a natural, while quite powerful, language for formalizing Ockhamist branching-time logics. The major advantage that we see in our logic is that it is expressively powerful enough to lend itself readily to easily axiomatized extensions with a wide range of additional operators, as shown in Section 6.

The paper is organized as follows: in Section 2 we introduce the syntax, and in Section 3 the bundle trees semantics, of the extended branching-time Ockhamist temporal logic EBOTL. Section 4 presents the axiomatic system for EBOTL and some syntactic results that are needed later on. Section 5 is the main one, where the soundness and completeness of our axiomatic system are proved. In Section 6 we briefly discuss some extensions of EBOTL and their axiomatizations, viz. with Since and Until, operators for undivided and divided histories, reference pointers, etc. Section 7 contains some concluding remarks.

The reader is expected to have some general technical background in classical modal and temporal logics.

2. Syntax

Language: A propositional language containing:

– logical connectives $\bot, \rightarrow$;
– temporal modalities F, P;
– a modality for *in some alternative future* $\langle \mathbf{a} \rangle$;
– a denumerable set $PV = \{p_0, p_1 \ldots\}$ of propositional variables;
– a denumerable set $FN = \{c_0, c_1 \ldots\}$ of *fan-names*.

The set FOR of formulae of this language is defined in the usual way, with both propositional variables and fan-names taken as atomic formulae.

We shall use metavariables $p, q, r, \ldots$ for propositional variables, $c, d, e, \ldots$ for fan-names, and $\phi, \chi, \psi, \theta, \ldots$ for formulae.

Defined Operators: $\top, \neg, \vee, \wedge, \leftrightarrow,$ as usual;
$\langle \Rightarrow \rangle\, \phi := \phi \vee F\phi$, (*sometime now or in this future*)
$\langle \Leftarrow \rangle\, \phi := \phi \vee P\phi$, (*sometime now or in the past*)
$\Diamond\phi := \phi \vee \langle \mathbf{a} \rangle\phi$, (*along some possible future*)
$E\phi := \langle \Leftarrow \rangle \Diamond \langle \Rightarrow \rangle\, \phi$, (*sometime*)
$\langle \mathbf{h} \rangle\, \phi := P\phi \vee \phi \vee F\phi$, (*sometime along this history*);
$G, H, [\mathbf{a}], [\Rightarrow], [\Leftarrow], \Box, A, [\mathbf{h}]$, as duals
respectively to $F, P, \langle \mathbf{a} \rangle, \langle \Rightarrow \rangle, \langle \Leftarrow \rangle, \Diamond, E, \langle \mathbf{h} \rangle$.

Abbreviations: For any sets Γ, Δ of formulae
and X any of the operators listed above:
$X(\Gamma) := \{X\phi : \phi \in \Gamma\}$, $X^{-}(\Gamma) := \{\phi : X\phi \in \Gamma\}$.

We let BOX be $\{G, H, [\mathbf{a}]\}$.

By $\phi[\psi/\chi]$ we denote the result of replacing all occurrences of formula χ in ϕ by occurrences of formula ψ;

3. Semantics

DEFINITION 3.1. A *tree* T is a strict partial ordering $(M, <)$ such that for every $t \in M$ the set $\{s : s < t\}$ of predecessors of t is a chain in T, i.e., a subset of M linearly ordered by $<$. The least element of a tree, if it exists, is called the *root* of the tree; a tree is *rooted* if it has a root. Henceforth we shall only deal with rooted trees (though the deductive system proposed below can be readily weakened to axiomatize all trees). Since trees will be used here to represent branching-time structures, the elements of a tree will be called *moments*.

Some terminology for discussing trees. Let $T = (M, <)$ be a tree.

history any maximal chain in T;
$\mathbf{H}(T)$ is the set of all histories in T;

branch for each history h and each moment $t \in h$,

	$h_t := \{s \in h : t \leq s\}$ is a branch (of the tree and, in particular, of h); $\mathbf{B}(T)$ is the set of all branches in T;
subbranch	for $b, b' \in \mathbf{B}(T)$, b' is a subbranch of b if $b' \subseteq b$; b' is a *proper* subbranch of b if $b' \subset b$;
extension	for $b, b' \in \mathbf{B}(T)$, b' is an extension of b iff b is a subbranch of b';
initial moment	for each branch b, the initial moment $I(b)$ is the unique moment $t \in b$ such that $b = h_t$; note that $I(h_t) = t$;
stem	for each moment t, the set $\{s : s \leq t\}$ is the stem of t;
fan	for each moment t, the set $\mathbf{F}(t) = \{b \in \mathbf{B}(T) : t = I(b)\}$ is the fan (of branches) at t. $\mathbf{F}(T)$ is the set of all fans in T.

DEFINITION 3.2. A *bundle tree* is a pair $(T, \mathcal{B})$ where T is a tree and $\mathcal{B}$ is a non-empty set of branches (called a *bundle*) in T closed under subbranches and extensions and such that every moment of T belongs to some branch from $\mathcal{B}$. A *complete tree* is any bundle tree $(T, \mathbf{B}(T))$.

By a history in $(T, \mathcal{B})$, we mean a history in T all of whose branches are in $\mathcal{B}$. (Note that if *some* branch from the history is in the bundle, then all of them are there). By a fan in $(T, \mathcal{B})$, we mean the intersection of a fan of T with the bundle $\mathcal{B}$. We let $\mathbf{H}(\mathcal{B})$ be the set of histories in $(T, \mathcal{B})$ and let $\mathbf{F}(\mathcal{B})$ be the set of fans in $(T, \mathcal{B})$, and when a bundle tree is considered $\mathbf{F}(t)$ will mean the fan of t in that bundle tree rather than in the whole tree T.

Valuations:
A *valuation* on a bundle tree $\mathcal{T} = ((M, <), \mathcal{B})$ is any function $V : (FN \cup PV) \to \mathcal{P}(\mathcal{B})$ such that for every $c \in FN$:

$$V(c) = \mathbf{F}(t) \quad \text{for some } t \in M,$$

i.e., every fan-name is evaluated into a fan of the bundle tree.

A *model* is a pair $(\mathcal{T}, V)$ where $\mathcal{T}$ is a bundle tree and V is a valuation in $\mathcal{T}$. A model on a complete tree will be called a *complete model*.

Truth/Satisfaction:
We define *(Ockhamist) truth* of a formula in a model *relative to a branch b* in that model; or equivalently, to a history h and a moment $t \in h$, since each branch is associated with a unique initial moment and a unique history and that moment and history in turn uniquely determine the branch. The inductive definition of the truth conditions for a fixed model $\mathcal{M}$ is given below.

(TCc) $\quad \mathcal{M}, b \models c \quad$ iff $\quad b \in V(c)$ (for any $c \in FN$)

(TCp)	$\mathcal{M}, b \models p$	iff	$b \in V(p)$ (for any $p \in PV$)
(TC⊥)	$\mathcal{M}, b \not\models \bot$;		
(TC→)	$\mathcal{M}, b \models \phi \to \psi$	iff	$\mathcal{M}, b \models \phi$ implies $\mathcal{M}, b \models \psi$;
(TC G)	$\mathcal{M}, b \models F\phi$	iff	for some $b' \subset b$, $\mathcal{M}, b' \models \phi$;
(TC H)	$\mathcal{M}, b \models P\phi$	iff	for some $b' \supset b$, $\mathcal{M}, b' \models \phi$;
(TC[**a**])	$\mathcal{M}, b \models \langle \mathbf{a} \rangle\phi$	iff	for some $b' \neq b$ such that $I(b) = I(b')$, $\mathcal{M}, b' \models \phi$.

A formula is *(Ockhamist)-valid in a model* if it is true at every branch of that model. A formula is *(Ockhamist)-valid in a (bundle) tree* if it is Ockhamist-valid in every model on that (bundle) tree.

Finally, throughout this paper we shall call a formula *Ockhamist-valid* if it is Ockhamist-valid in every bundle tree. Hereafter, by validity we shall mean Ockhamist validity and shall usually omit the word "Ockhamist."

4. Axiomatic System for the Extended Branching-Time Ockhamist Temporal Logic EBOTL

4.1. Axioms

AI. BASIC AXIOMS FOR THE MODALITIES:

(I.1) Enough propositional tautologies.
(I.2) $G(p \to q) \to (Gp \to Gq)$
(I.3) $H(p \to q) \to (Hp \to Hq)$
(I.4) $[\mathbf{a}](p \to q) \to ([\mathbf{a}]p \to [\mathbf{a}]q)$
(I.5) $p \to GPp$
(I.6) $p \to HFp$
(I.7) $p \to [\mathbf{a}]\langle \mathbf{a} \rangle p$
(I.8) $\Box p \to [\mathbf{a}][\mathbf{a}]p$
(I.9) $p \to AEp$
(I.10) $Ap \to AAp$
(I.11) $Ap \to \Box[\mathbf{h}]p$

AII. AXIOMS FOR THE FAN-NAMES:

(II.1)	Ec	(referentiality of names)
(II.2)	$\langle \mathbf{a} \rangle c \to c$	(non-historicity of names)
(II.3)	$(c \wedge p) \to A(c \to \Diamond p)$	(unique assignment of names)

AIII: AXIOMS FOR THE STRUCTURE

(III.1)	$\langle \Leftarrow \rangle H\bot$	(rootedness)
(III.2)	$c \to G\neg c$	(irreflexivity)
(III.3)	$PPc \to Pc$	(transitivity)
(III.4)	$Fp \to G\langle \mathbf{h} \rangle p$	(linearity of branches)

(III.5)	$Pp \to H\langle \mathbf{h}\rangle\, p$	(linearity of stems)
(III.6)	$Pc \to \Box Pc$	(determinacy of the past)
(III.7)	$(c \wedge p \wedge [\mathbf{a}]\neg p) \to [\mathbf{a}]F\Box H(c \to \neg p)$	(distinguishability of branches)

4.2. Rules of Inference

(MP) from $\vdash \phi$ and $\vdash \phi \to \psi$, infer $\vdash \psi$;

(RN$_B$) from $\vdash \phi$ infer $\vdash B\phi$,
where $B \in$ BOX;

(COV$_0$) from $\vdash c \to \phi$, infer $\vdash \phi$,
where $c \in FN$, provided c does not occur in ϕ;

(COV) from $\vdash \psi \to B(c \to \phi)$, infer $\vdash \psi \to B\phi$,
where $B \in$ BOX, $c \in FN$,
provided c does not occur in ψ or ϕ;

(DIF$_0$) from $\vdash (p \wedge [\mathbf{a}]\neg p) \to \phi$ infer $\vdash \phi$,
where $p \in PV$, provided p does not occur in ϕ;

(DIF) from $\vdash \psi \to B((p \wedge [\mathbf{a}]\neg p) \to \phi)$ infer $\vdash \psi \to B\phi$,
where $B \in$ BOX, $p \in PV$,
provided p does not occur in ψ or ϕ;

(SUBc) from $\vdash \phi$ infer $\vdash \phi[d/c]$, for any $c, d \in FN$,

(SUBp) from $\vdash \phi$ infer $\vdash \phi[\psi/p]$, for any $p \in PV$.

Notation: we write $\vdash \phi$ if ϕ is a theorem and $\nvdash \phi$ otherwise.

4.3. Some Syntactic Results

DEFINITION 4.1. We define *universal contexts of* $*$ (where $*$ marks a position for a formula) recursively as follows:

1. $*$ is a universal context of $*$, called the *empty* context.
2. If $U(*)$ is a universal context of $*$, ϕ is a formula and $B \in$ BOX, then $\phi \to U(*)$ and $BU(*)$ are universal contexts of $*$.

Note that every universal context of $*$ can be represented (up to equivalence) in a uniform way:

$$U(*) = \phi_0 \to B^1(\phi_1 \to \ldots B^n(\phi_n \to *)\ldots),$$

where $B^1, \ldots, B^n \in$ BOX and some of $\phi_0, \ldots, \phi_n$ may be $\top$ if necessary.

We consider each formula of the form $U(\phi)$ as arising by substituting ϕ for $*$ in the universal context $U(*)$.

LEMMA 4.1. *The following rules are derivable in EBOTL:*

1. (RN_A): *from* $\vdash \phi$ *infer* $\vdash A\phi$.
2. (COV_U): *for any universal context* $U(*)$,

$$\textit{from} \ \vdash U(c \to \phi) \ \textit{infer} \ \vdash U(\phi),$$

provided c *does not occur in* $U(\phi)$;

3. (DIF_U): *for any universal context* $U(*)$,

$$\textit{from} \ \vdash U((p \wedge [\mathbf{a}]\neg p) \to \phi) \ \textit{infer} \ \vdash U(\phi),$$

provided p *does not occur in* $U(\phi)$;

Proof. (1) This is an easy exercise in modal logic.

(2) For the empty context the results follows from the rule COV_0. For any non-empty context it follows from the fact that the language is *versatile* (Venema, 1993), i.e., every basic modality of the language has its inverse in that language, and therefore every universal context can be derivably converted from inside out, as observed in Gabbay and Hodkinson (1990).

(3) Likewise. □

LEMMA 4.2 . *The rules* (COV_0) + (COV) *and* (DIF_0) + (DIF) *are deductively equivalent respectively to:*

(COV*) *from* $\vdash U(c \to \phi)$ *for every* $c \in$ FN, *infer* $\vdash U(\phi)$;

(DIF*) *from* $\vdash U((p \wedge [\mathbf{a}]\neg p) \to \phi)$ *for every* $p \in PV$, *infer* $\vdash U(\phi)$.

Proof. By the previous lemma, (COV) and (DIF) are equivalent respectively to (COV_U) and (DIF_U). By the substitution rule (SUBc), the latter are equivalent respectively to (COV*) and (DIF*). □

LEMMA 4.3 (The Theorem Schemata Lemma). *The following are theorem schemata of EBOTL:*

1. $A(\phi \to \psi) \to (A\phi \to A\psi)$;
2. $A\phi \to B_1 \ldots B_n\phi$, *for any* $B_1 \ldots B_n \in BOX$;
3. $P(c \wedge \phi) \to H(c \to \phi)$;
4. $F(c \wedge \phi) \to G(c \to \phi)$;

5. $\Diamond\Box\phi \to \phi$;
6. $\Diamond Pc \to Pc$;
7. $(c \wedge Pd) \to A(c \to Pd)$;
8. $H\bot \to (c \vee \Diamond Fc)$;
9. $E(c \wedge d) \to (\langle\Rightarrow\rangle\, c \leftrightarrow \langle\Rightarrow\rangle\, d)$.

Proof. Most of the derivations above are standard exercises in classical modal logic and none of them makes use of the special rules. We give sample derivations modulo some easy modal logic steps.

3: By Axiom III.5, $\vdash P(c \wedge \phi) \to H(P(c \wedge \phi) \vee (c \wedge \phi) \vee F(c \wedge \phi))$. By contraposition to Axiom III.2 $\vdash Fc \to \neg c$, hence, by RN_H and Axiom I.3, $\vdash HFc \to H\neg c$, so by Axiom I.6, $\vdash c \to H\neg c$. Further, it follows from Axiom III.2 that $\vdash c \wedge F(c \wedge \phi) \to \bot$ hence $\vdash c \wedge F(c \wedge \phi) \to \phi$. Likewise, $\vdash c \wedge P(c \wedge \phi) \to \phi$. Also, $\vdash c \wedge (c \wedge \phi) \to \phi$. Therefore, $\vdash P(c \wedge \phi) \vee (c \wedge \phi) \vee F(c \wedge \phi)) \to (c \to \phi)$, whence $\vdash P(c \wedge \phi) \to H(c \to \phi)$.

4: Similarly.

6: From Axiom III.6 it is easy to derive that $\vdash \Diamond Pc \to \Diamond\Box Pc$, hence $\Diamond Pc \to Pc$ by theorem schema (5).

7: From Axioms I.2 and III.5 and rule RN_G we obtain $\vdash GPc \to GH\langle\mathbf{h}\rangle c$, whence $\vdash c \to GH\langle\mathbf{h}\rangle c$ by Axiom I.5.

8: Follows from (7) and Axiom II.3. □

5. Soundness and Completeness of EBOTL

5.1. Soundness

THEOREM 5.1. *Every theorem of EBOTL is Ockhamist-valid.*

Proof. It is a routine task to check that all axioms are valid. All rules preserve this validity. As an illustration, let us show that for the rule COV. Suppose $\psi \to G\phi$ is falsified at some branch b of a bundle-tree model $\mathcal{M}$. Then $\mathcal{M}, b \models \psi$ and $\mathcal{M}, b' \not\models \phi$ for some subbranch b' of b. Let $I(b') = t$ and let c be any fan-name not occurring in ϕ or ψ. We modify the valuation V of $\mathcal{M}$ into V' by putting $V'(c) = \mathbf{F}(t)$, and for all other atomic formulae let V' coincide with V. Let the resulting model be $\mathcal{M}'$. Then $\mathcal{M}', b' \not\models c \to \phi$ and $\mathcal{M}', b \models \psi$ hence $\mathcal{M}', b \not\models \psi \to G(c \to \phi)$, i.e., the premise of COV is not valid. Similarly, DIF preserves validity. For COV_0 and DIF_0 the argument is similar but simpler. □

5.2. Proper Theories and Complete Theories

DEFINITION 5.1. By a *proper theory*, we mean any set of formulae containing all theorems and closed under MP, COV*, and DIF*.

Since we shall only work with proper theories, hereafter we call them just *theories*.

DEFINITION 5.2. A theory Γ is:

– *consistent* if $\perp \notin \Gamma$;
– *complete* if for each formula ϕ, either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$.

Some observations:

(1) The intersection of any family of theories is a theory.
(2) The set of all theorems is (by soundness) a consistent theory; the set of all formulae is an inconsistent theory.

Given any set Γ of formulae, we let $C(\Gamma)$, called *the closure of* Γ, be the smallest theory containing Γ, i.e., the intersection of all theories containing Γ. A set of formulae is consistent if its closure is a consistent theory.

THEOREM 5.2 (The Deduction Theorem). *If Γ is any theory and ϕ, ψ are formulae, then:*

$$\phi \to \psi \in \Gamma \ \textit{ iff } \ \psi \in C(\Gamma \cup \{\phi\}).$$

Proof. From left to right is trivial by MP. For the converse, suppose $\psi \in C(\Gamma \cup \{\phi\})$, and consider the set $\Delta = \{\theta : \phi \to \theta \in \Gamma\}$. By the usual arguments, every theorem is in Δ, every element of Γ is in Δ, ϕ is in Δ, and Δ is closed under MP. Moreover, Δ is COV*-closed, because if $U(c \to \theta) \in \Delta$ for each $c \in FN$, then $\phi \to U(c \to \theta) \in \Gamma$ for each $c \in FN$. Since $\phi \to U(*)$ is a universal context whenever $U(*)$ is one, and Γ is COV-closed, we then have $\phi \to U(\theta) \in \Gamma$, and hence $U(\theta) \in \Delta$. By a similar argument, Δ is DIF*-closed. Hence Δ is a theory containing $\Gamma \cup \{\phi\}$. Since $\psi \in C(\Gamma \cup \{\phi\})$, $\psi \in \Delta$, i.e., $\phi \to \psi \in \Gamma$. □

LEMMA 5.1 (The Consistent Theory Lemma). *If Γ is any consistent theory, then:*

(1) for at least one $c \in FN : \neg c \notin \Gamma$;
(2) for at least one $p \in PV : \neg(p \wedge [\mathbf{a}]\neg p) \notin \Gamma$.

Proof. (1) If $\neg c \in \Gamma$, i.e., $c \to \perp \in \Gamma$, for each $c \in FN$, then $\perp \in \Gamma$ by COV*-closedness.

(2) Likewise, by DIF*-closedness. □

LEMMA 5.2 (The Complete Theory Lemma). *If Γ is any complete theory, and ϕ, ψ any formulae, then:*

(1) for at least one $c_\Gamma \in FN$, $c_\Gamma \in \Gamma$;

(2) for at least one $p_\Gamma \in PV$, $(p_\Gamma \wedge [\mathbf{a}]\neg p_\Gamma) \in \Gamma$*;*
(3) $(\phi \to \psi) \in \Gamma$ *iff either* $\phi \notin \Gamma$ *or* $\psi \in \Gamma$.

Proof. (1) and (2): by completeness and the previous lemma; (3) is known from classical propositional logic. □

We let $\mathcal{C}$ be the set of all complete theories.

LEMMA 5.3 (The Reciprocality Lemma). *For any* $\Gamma, \Delta \in \mathcal{C}$*:*
(1) the following are equivalent:
(i) $G^-(\Gamma) \subseteq \Delta$,
(ii) $F(\Delta) \subseteq \Gamma$,
(iii) $P(\Gamma) \subseteq \Delta$,
(iv) $H^-(\Delta) \subseteq \Gamma$*;*

(2) the following are equivalent:
(i) $\Box^-(\Gamma) \subseteq \Delta$,
(ii) $\Diamond(\Delta) \subseteq \Gamma$,
(iii) $\Box^-(\Delta) \subseteq \Gamma$,
(iv) $\Diamond(\Gamma) \subseteq \Delta$.

Proof. As usual. □

LEMMA 5.4 (The Lindenbaum Lemma). *Every consistent set* Δ *can be extended to a complete theory.*

Proof. Let $U_1(\phi_1), \ldots, U_n(\phi_n), \ldots$ be an enumeration of all applications $U(\phi)$ of a universal context $U(*)$ to a formula ϕ, for all choices of context and formula. (This enumeration includes all formulae, because of the empty universal context; it also includes harmless repetitions of formulae.)

Now we define theories $\Gamma_0, \ldots, \Gamma_n, \ldots$ as follows: $\Gamma_0 = C(\Delta)$. Suppose Γ_k is defined. If $C(\Gamma_k \cup \{U_k(\phi_k)\})$ is consistent, then we set $\Gamma_{k+1} = C(\Gamma_k \cup \{U_k(\phi_k)\})$. Otherwise, by the deduction theorem, $\neg U_k(\phi_k) \in \Gamma_k$. Then, since Γ_k is closed under COV*, there must be at least one name c_i such that $U_k(c_i \to \phi_k) \notin \Gamma_k$. Let $\Gamma'_k = C(\Gamma_k \cup \{\neg U_k(c_i \to \phi_k)\})$ for the first appropriate index i. Then Γ'_k is consistent. Similarly, since Γ'_k is closed under DIF*, there must be at least one propositional variable p_j such that $U_k((p_j \wedge [\mathbf{a}]\neg p_j) \to \phi_k) \notin \Gamma'_k$. In that case, we set $\Gamma_{k+1} = C(\Gamma'_k \cup \{\neg U_k((p_j \wedge [\mathbf{a}]\neg p_j) \to \phi_k)\})$ for the first appropriate index j. Γ_{k+1} is then consistent.

We finally put Γ to be the union of all the Γ_n. By construction, Γ will be a complete theory extending Δ. □

5.3. Canonical Frames

For every complete theory Γ we define the *kernel* of Γ to be the set

$$K(\Gamma) = A^{-}(\Gamma) = \{\phi : A\phi \in \Gamma\}.$$

LEMMA 5.5 (The Kernel Lemma). *Let Γ be a complete theory and $B \in$ BOX. Then:*

$$K(\Gamma) \subseteq B^{-}(\Gamma).$$

Proof. This follows from the facts that Γ is closed under MP and that $A\phi \rightarrow B\phi$ is a theorem for each $B \in$ BOX. □

Let Θ be a fixed complete theory. Let $W = \{\Gamma \in \mathcal{C} : K(\Theta) \subseteq \Gamma\}$.

LEMMA 5.6. *For every $\Gamma \in W$, $K(\Gamma) = K(\Theta)$.*

Proof. This follows from the fact that A is an S5 modality. □

COROLLARY 5.1. *If $A\phi \in \Gamma$ for some $\Gamma \in W$, then $\phi \in \Delta$ for every $\Delta \in W$.*

LEMMA 5.7. *Let $\Gamma \in W$ and $B \in$ BOX. Then $B^{-}(\Gamma)$ is a theory.*

Proof. $B^{-}(\Gamma)$ contains all theorems since whenever ϕ is a theorem, $B\phi$ is one, hence it belongs to Γ. $B^{-}(\Gamma)$ is closed under MP due to the K-axiom for B. Moreover, it is closed under COV* and DIF* since Γ is closed and $BU(*)$ is a universal context whenever $U(*)$ is one. □

LEMMA 5.8 (The Transfer Lemma). *Let $\Gamma \in W$, $B \in \{G, H, [\mathbf{a}]\}$, B' be the dual of B and $B'\phi \in \Gamma$ for some formula ϕ. Then there is a complete theory $\Delta \in W$ such that $B^{-}(\Gamma) \cup \{\phi\} \subseteq \Delta$.*

Proof. By the previous lemma, $B^{-}(\Gamma)$ is a theory. Furthermore, $B^{-}(\Gamma) \cup \{\phi\}$ is consistent, otherwise $\neg\phi \in B^{-}(\Gamma)$, i.e., $B\neg\phi \in \Gamma$, which together with $B'\phi \in \Gamma$ contradicts the consistency of Γ. By the Lindenbaum Lemma 5.4, $B^{-}(\Gamma) \cup \{\phi\} \subseteq \Delta$ for some complete theory Δ. By the Kernel Lemma 5.5, $K(\Gamma) \subseteq B^{-}(\Gamma)$, hence $\Delta \in W$. □

For each $c \in FN$ let $[c] := \{d \in FN \mid (\exists\Gamma \in W)(c \in \Gamma \;\&\; d \in \Gamma)\}$.

Then we put $M = \{[c] \mid c \in FN\}$.

LEMMA 5.9 (The Partition Lemma). *M is a partition on FN.*

Proof. We shall prove that $c \in [d]$ is an equivalence relation in FN. First, for each $c \in FN$, there is some $\Gamma \in W$ such that $c \in \Gamma$ by the Transfer Lemma, since Ec is an axiom and hence $Ec \in \Theta$. Therefore, $c \in [c]$. Symmetry is trivial. Finally, suppose $c \in [d]$ and $d \in [e]$. Then $\{c, d\} \subseteq \Gamma_1$ and $\{d, e\} \subseteq \Gamma_2$ for some

$\Gamma_1, \Gamma_2 \in W$. By Axiom II.3 $\vdash (d \wedge e) \rightarrow A(d \rightarrow \Diamond e)$, so $A(d \rightarrow \Diamond e) \in \Gamma_2$. Then $(d \rightarrow \Diamond e) \in \Gamma_1$, so $\Diamond e \in \Gamma_1$, hence $e \in \Gamma_1$ by Axiom II.2, i.e., $c \in [e]$. □

Now we are about to construct a frame as follows.

Let M be as above.

For each $c \in FN$, let $S_{[c]} := \{\Gamma \in W : [c] \subseteq \Gamma\}$.

(Equivalently: $S_{[c]} = \{\Gamma \in W : c \in \Gamma\}$.)

Note that $\{S_{[c]} : c \in FN\}$ is a partition of W.

Now, for $\Gamma, \Delta \in W$ we define

$$\Gamma \prec \Delta \text{ iff } F(\Delta) \subseteq \Gamma.$$

Then we define

$$S_{[c]} \prec S_{[d]} \text{ iff } (\exists \Gamma \in S_{[c]})(\exists \Delta \in S_{[d]})(\Gamma \prec \Delta).$$

Finally, let $[c] \prec [d]$ iff $S_{[c]} \prec S_{[d]}$.

DEFINITION 5.3. The tree $T_\Theta = \langle M, \prec \rangle$ constructed as above is called the *canonical frame* for Θ.

For $\Gamma \in S_{[c]}$, $n(\Gamma) := [c]$, i.e., $n(\Gamma)$ is the set of fan-names in Γ. Note that for $\Gamma, \Delta \in W$, if $\Gamma \prec \Delta$ then $n(\Gamma) \prec n(\Delta)$.

5.4. Structure of the Canonical Frames

Throughout this section $T = \langle M, \prec \rangle$ is an arbitrarily fixed canonical frame.

LEMMA 5.10 (The Visibility Lemma). *Let* $\Gamma \prec \Delta$. *Then* $F(n(\Delta)) \subseteq \Gamma$ *and* $P(n(\Gamma)) \subseteq \Delta$.

Proof. From the definition and the Reciprocality Lemma 5.3. □

LEMMA 5.11. *Let* $[c] \prec [d]$. *Then:*

(i) [Pastwards Transfer]: *for every* $\Delta \in S_{[d]}$ *there is a unique* $\Gamma \in S_{[c]}$ *such that* $\Gamma \prec \Delta$*; and*

(ii) [Futurewards Transfer]: *for every* $\Gamma \in S_{[c]}$, *if* $Fd \in \Gamma$ *then there is a unique* $\Delta \in S_{[d]}$ *such that* $\Gamma \prec \Delta$.

Proof. Assume $[c] \prec [d]$. Then for some $\Delta_0 \in S_{[d]}$ and some $\Gamma_0 \in S_{[c]}$, we have $F(\Delta_0) \subseteq \Gamma_0$. By the Reciprocality Lemma, we also have $P(\Gamma_0) \subseteq \Delta_0$.

(i) Since $c \in \Gamma_0$, we have $Pc \in \Delta_0$. Also, since $d \in \Delta_0$, we have $(d \wedge Pc) \in \Delta_0$. Then, since $\vdash (d \wedge Pc) \rightarrow A(d \rightarrow Pc)$ (see the Theorem Schemata Lemma 4.3), we have $A(d \rightarrow Pc) \in \Delta_0$. Suppose $\Delta \in S_{[d]}$. Then $d \rightarrow Pc \in \Delta$, hence $Pc \in \Delta$. By the Transfer Lemma 5.8, it follows that there is a complete theory $\Gamma \prec \Delta$ with

$c \in \Gamma \in W$. We shall prove that $\Gamma = \{\phi : H(c \to \phi) \in \Delta\}$, hence it is unique. If $H(c \to \phi) \in \Delta$ then $(c \to \phi) \in \Gamma$, hence $\phi \in \Gamma$. Conversely, if $\phi \in \Gamma$ then $P(c \wedge \phi) \in \Delta$, hence $H(c \to \phi) \in \Delta$ by the Theorem Schemata Lemma 4.3.

(ii) The proof is analogous. □

COROLLARY 5.2 (The Momentary Ordering Corollary). *For* $\Gamma \in S_{[c]}$*:*
(i) If $d \in \Gamma$ *then* $[d] = [c]$*;*
(ii) If $Fd \in \Gamma$ *then* $[c] \prec [d]$*;*
(iii) If $Pd \in \Gamma$ *then* $[d] \prec [c]$.

Proof. By the Partition Lemma 5.9 and Pastwards and Futurewards Transfer Lemmata 5.11. □

LEMMA 5.12 (The Momentary Association Lemma). *If* $\Delta_1, \Delta_2 \in S_{[d]}$ *then* $\Box^-(\Delta_1) \subseteq \Delta_2$ *and* $\Diamond(\Delta_2) \subseteq \Delta_1$*. Moreover, if* $\Delta_1 \neq \Delta_2$ *then* $[\mathbf{a}]^-(\Delta_1) \subseteq \Delta_2$.

Proof. If $\Delta_1, \Delta_2 \in S_{[d]}$ then $d \in \Delta_1, \Delta_2$. Suppose $\phi \in \Delta_2$. By axiom II.3, $\vdash (d \wedge \phi) \to A(d \to \Diamond\phi)$, hence $A(d \to \Diamond\phi) \in \Delta_2$, so $d \to \Diamond\phi \in \Delta_1$, and hence $\Diamond\phi \in \Delta_1$. Then $\Box^-(\Delta_1) \subseteq \Delta_2$ follows by the Reciprocality Lemma.

Now, suppose $\Delta_1 \neq \Delta_2$. Then there is some $\psi \in \Delta_1 - \Delta_2$. Let $[\mathbf{a}]\phi \in \Delta_1$. Suppose $\phi \notin \Delta_2$. Let $\theta = \phi \vee \psi$. Then $\theta \notin \Delta_2$. On the other hand $\theta \in \Delta_1$ and $[\mathbf{a}]\theta \in \Delta_1$, hence $\Box\theta \in \Delta_1$ – a contradiction with $\Box^-(\Delta_1) \subseteq \Delta_2$. □

THEOREM 5.3 (The Tree Theorem). $T = \langle M, \prec\rangle$ *is a rooted tree.*

Proof. Irreflexivity: Suppose $[c] \prec [c] \in M$. Then $(\exists\Delta \in S_{[c]})(\exists\Gamma \in S_{[c]})[F(\Delta) \subseteq \Gamma]$. Therefore $c \in \Gamma$, and $Fc \in \Gamma$ since $c \in \Delta$. Hence $c \wedge Fc \in \Gamma$ which contradicts the consistency of Γ by the irreflexivity axiom.

Transitivity: Suppose $[c] \prec [d]$ and $[d] \prec [e]$. Then $(\exists\Gamma \in S_{[c]})(\exists\Delta \in S_{[d]})[P(\Gamma) \subseteq \Delta]$ and $(\exists\Sigma \in S_{[d]})(\exists\Pi \in S_{[e]})[F(\Sigma) \subseteq \Pi]$. Then $Pd \in \Sigma$ and $(d \wedge Pc) \in \Delta$, hence $A(d \to Pc) \in \Delta$ (see The Theorem Schemata Lemma 4.3) so $A(d \to Pc) \in \Pi$. Therefore $H(d \to Pc) \in \Pi$, and since $Pd \in \Pi$, by a simple modal logic argument, $PPc \in \Pi$ whence, by Axiom III.3, $Pc \in \Pi$. Therefore, $[c] \prec [e]$ by the Momentary Ordering Corollary 5.2.

Rootedness: By Axiom III.1, we have $\vdash \langle\Leftarrow\rangle H\bot$, i.e., $H\bot \vee PH\bot$. Let $\Gamma \in W$ and $n(\Gamma) = [c]$. $H\bot \vee PH\bot \in \Gamma$ hence $H\bot \in \Gamma$ or $PH\bot \in \Gamma$.

Case 1: $H\bot \in \Gamma$. Then we show that $[c]$ is a root in $\mathcal{F}$, i.e., for every $d \in FN$, either $[c] = [d]$ or $[c] \succ [d]$: since $\vdash H\bot \to (d \vee \Diamond Fd)$ (see the Theorem Schemata Lemma 4.3) it follows that $(d \vee \Diamond Fd) \in \Gamma$. If $d \in \Gamma$ then $[d] = [c]$. Otherwise, $Fd \in \Gamma$ or $\langle\mathbf{a}\rangle Fd \in \Gamma$. In either case, $Fd \in \Gamma'$ for some $\Gamma' \in S_{[c]}$ and then the Momentary Ordering Corollary 5.2 applies.

Case 2: $PH\bot \in \Gamma$. By the Transfer Lemma there is some $\Delta \in W$ with $H\bot \in \Delta$. Let $[d]$ be the set of names in Δ. Then, by Case 1, $[d]$ is a root.

Clearly, the root is unique, since $\prec$ is a strict partial ordering.

Treehood: Suppose $[c] \prec [e]$ and $[d] \prec [e]$. Then there are $\Gamma, \Delta, \Sigma, \Pi \in W$, with $c \in \Gamma, d \in \Delta, e \in \Sigma, e \in \Pi$, such that $P(\Gamma) \subseteq \Sigma$ and $F(\Pi) \subseteq \Delta$. Then $Pc \in \Sigma$, hence $\Box Pc \in \Sigma$ by Axiom III.6, so $Pc \in \Pi$, whence $H[\mathbf{h}]c \in \Pi$ by Axiom III.5, so $[\mathbf{h}]c \in \Delta$. Therefore, by the Momentary Ordering Corollary 5.2, either $[c] \prec [d]$ or $[c] = [e]$ or $[d] \prec [c]$. $\Box$

LEMMA 5.13 (The Extension Lemma). *$S_{[c]} \prec S_{[d]}$ iff $(\forall\Delta \in S_{[d]})(\exists\Gamma \in S_{[c]})[\{F(\Delta) \subseteq \Gamma]$.*

Proof. Let $[c] \prec [d]$ and $\Gamma \prec \Delta$ for some $\Gamma \in S_{[c]}$ and $\Delta \in S_{[d]}$. Take any $\Delta' \in S_{[d]}$. $Pc \in \Delta$ by the Reciprocality Lemma, so $\Box Pc \in \Delta$ by Axiom III.6, hence $Pc \in \Delta'$, by the Momentary Association Lemma 5.12. Therefore, by Lemma 5.11 (Pastwards Transfer) there is some $\Gamma' \in S_{[c]}$ such that $\Gamma' \prec \Delta'$, i.e., (by the Reciprocality Lemma 5.3) $F(\Delta') \subseteq \Gamma'$. $\Box$

LEMMA 5.14 (The Branch Definition Lemma). *If $d \in [c]$, then for any $\Gamma \in W$, $\langle\Rightarrow\rangle\, c \in \Gamma$ iff $\langle\Rightarrow\rangle\, d \in \Gamma$.*

Proof. Because $\vdash E(c \wedge d) \rightarrow (\langle\Rightarrow\rangle\, c \leftrightarrow \langle\Rightarrow\rangle\, d)$, by the Theorem Schemata Lemma 4.3. $\Box$

DEFINITION 5.4. For $\Gamma \in W$, let $b(\Gamma) = \{[c] \in M : \langle\Rightarrow\rangle\, c \in \Gamma\}$.

Note that, by the Branch Definition Lemma, the definition above is correct, i.e., does not depend on the choice of c in any cluster $[c]$.

LEMMA 5.15 (The Branch Ordering Lemma). *Let $\Gamma \in S_{[c]}$. Then:*

(i) $b(\Gamma)$ is a branch in $T =< M, \prec>$, with $I(b(\Gamma)) = [c]$;
(ii) For every $[d] \in b(\Gamma)$ such that $\Gamma \notin S_{[d]}$, there is a unique $\Gamma_{[d]} \in S_{[d]}$ such that $\Gamma \prec \Gamma_{[d]}$.
(iii) For any $[d], [e] \in b(\Gamma)$, $[d] \prec [e]$ iff $\Gamma_{[d]} \prec \Gamma_{[e]}$.

Proof. (i) Suppose $\langle\Rightarrow\rangle\, d \in \Gamma$ and $\langle\Rightarrow\rangle\, e \in \Gamma$. If $d, e \in [c]$ then $[d] = [e]$. Suppose $e \notin [c]$. Then $Fe \in \Gamma$. Therefore by the Momentary Ordering Corollary 5.2, $[c] \prec [e]$. If $d \in [c]$ then $[d] = [c]$, hence $[d] \prec [e]$. Otherwise, $Fd \in \Gamma$, hence by Axiom III.4 $G(Pd \vee d \vee Fd) \in \Gamma$. Let $\Delta \in S_{[e]}$ such that $\Gamma \prec \Delta$ (by the Pastwards and Futurewards Transfer Lemmata 5.11). Then $Pd \vee d \vee Pd \in \Delta$, hence $[d] \prec [e]$ or $[e] \prec [d]$ by the Momentary Ordering Corollary 5.2. Thus, $[d]$ and $[e]$ are comparable, so $b(\Gamma)$ is linearly ordered. Suppose $b(\Gamma)$ is not maximal, so it can be extended with some $[d]$. Then $[c] \prec [d]$, hence $Fd \in \Gamma$, so $[d] \in b(\Gamma)$ which is a contradiction.

(ii) Suppose $[d] \in b(\Gamma)$ and $\Gamma \notin S_{[d]}$. Then $\langle\Rightarrow\rangle$ $d \in \Gamma$, so either $d \in \Gamma$ or $Fd \in \Gamma$. The former is ruled out by the fact that $\Gamma \notin S_{[d]}$. Then, by the Futurewards Transfer Lemma 5.11 there is a unique $\Gamma_{[d]} \in W$ such that $d \in \Gamma_{[d]}$ and $\Gamma \prec \Gamma_{[d]}$.

(iii) If $\Gamma_{[d]} \prec \Gamma_{[e]}$ then $[d] \prec [e]$ by definition. Conversely, let $[d] \prec [e]$, for $[d], [e] \in b(\Gamma)$. By the Extension Lemma 5.13 there is $\Delta \in S_{[e]}$ such that $\Gamma_{[d]} \prec \Delta$. By transitivity, $\Gamma \prec \Delta$, hence $\Delta = \Gamma_{[e]}$, so $\Gamma_{[d]} \prec \Gamma_{[e]}$. □

LEMMA 5.16 (The Branch Witness Lemma). *Let* $\Gamma, \Delta \in W$ *and* $b(\Gamma) = b(\Delta)$. *Then* $\Gamma = \Delta$.

Proof. $n(\Gamma) = n(\Delta)$ by the Branch Ordering Lemma 5.15, part (i). Let $n(\Gamma) = [d]$. Then $d \in \Gamma$ and $d \in \Delta$. By the Complete Theory Lemma 5.2, there is some propositional variable p_Γ such that $p_\Gamma \wedge [\mathbf{a}]\neg p_\Gamma \in \Gamma$. Then, $[\mathbf{a}]F\Box H(d \rightarrow \neg p_\Gamma)) \in \Gamma$ by Axiom III.7.

Suppose $\Gamma \neq \Delta$. Then $F\Box H(d \rightarrow \neg p_\Gamma)) \in \Delta$ by the Momentary Association Lemma 5.12. By the Transfer Lemma 5.8, there is some Σ such that $\Delta \prec \Sigma$ and $\Box H(d \rightarrow \neg p_\Gamma) \in \Sigma$. Let $[e] = n(\Sigma)$. Then $Fe \in \Delta$, so $[e] \in b(\Delta) = b(\Gamma)$, hence $Fe \in \Gamma$, and by the Transfer Lemma again there is some $\Sigma' \in S_{[e]}$ such that $\Gamma \prec \Sigma'$. By the Momentary Association Lemma, $H(d \rightarrow \neg p_\Gamma) \in \Sigma'$ hence $(d \rightarrow \neg p_\Gamma) \in \Gamma$, so $\neg p_\Gamma \in \Gamma$, which contradicts the consistency of Γ. □

Thus, to summarize, each $\Gamma \in W$ determines a unique branch $b(\Gamma)$ in T which we shall call *the branch recognized by* Γ. Furthermore, every recognized branch b in T is recognized by a unique complete theory $Th(b) \in W$, which we shall call *the witness of* b.

DEFINITION 5.5. Given any branch b in the canonical frame and any $[d] \in b$, we let $b|_{[d]}$ be the subbranch of b with an initial moment $[d]$.

LEMMA 5.17 (The Subbranch Lemma). *Let* $\Gamma, \Delta \in W$. *Then* $\Gamma \prec \Delta$ *iff* $b(\Delta)$ *is a proper subbranch of* $b(\Gamma)$.

Proof. Let $\Gamma \prec \Delta$. Then $n(\Gamma) \prec n(\Delta)$, hence $b(\Delta)$ is a proper subbranch of $b(\Gamma)$ by transitivity and irreflexivity of $\prec$ in M. Conversely, let $b(\Delta) \subset b(\Gamma)$, with $\Gamma \in S_{[c]}$ and $\Delta \in S_{[d]}$. Then $[d] \in b(\Gamma)$, so by the Branch Ordering Lemma 5.15 there is a unique $\Delta' \in S_{[d]}$ such that $\Gamma \prec \Delta'$ and hence Δ' recognizes$b(\Gamma)|_{[d]} = b(\Delta)$. Therefore, by the Branch Witness Lemma 5.16, $\Delta = \Delta'$, hence $\Gamma \prec \Delta$. □

5.5. The Completeness Theorem

Let $R(T)$ be the set of all recognized branches in T.

THEOREM 5.4. $(T, R(T))$ *is a bundle tree.*

Proof. (i) Let $[c] \in T$ and $\Gamma \in S_{[c]}$. Then $b(\Gamma)$ stems from $[c]$.

(ii) Let $b = b(\Gamma)$ for some Γ and $[d] \in b$. Then there is a $\Delta \in S_{[d]}$ such that $\Gamma \prec \Delta$, hence $b|_{[d]}$ is recognized by Δ.

(iii) Let $b = b(\Gamma)$ for some $\Gamma \in S_{[c]}$ and $[d] \prec [c]$. Then by the Extension Lemma 5.13 there is some $\Delta \in S_{[d]}$ such that $\Delta \prec \Gamma$. Therefore, $b(\Gamma)$ is a subbranch of $b(\Delta)$ by the Subbranch Lemma 5.17, hence the (unique) extension $b|_{[d]}$ of b stemming from $[d]$ is recognized by Δ. Thus $R(T)$ is closed under subbranches and extensions, and every moment is on some branch in $R(T)$. □

Now we define the canonical valuation on $R(T)$ as follows:

$$V(c) = \{b : c \in Th(b)\},$$

$$V(p) = \{b : p \in Th(b)\}.$$

We now consider the bundle tree model $\mathcal{M} = ((T, R(T)), V)$, called a *canonical model* on $(T, R(T))$.

LEMMA 5.18 (The Truth Lemma). *For every formula ϕ and every branch $b \in R(T)$,*

$$\mathcal{M}, b \models \phi \ \textit{iff}\ \phi \in Th(b).$$

Proof. (i) The cases $\phi = c, p, \perp$ are trivial, and the case $\psi \rightarrow \chi$ comes from the Complete Theory Lemma.

(ii) Let $\phi = F\psi$. If $\mathcal{M}, b \models F\psi$ then $\mathcal{M}, b' \models \psi$ for some subbranch b' of b, hence $\psi \in Th(b')$. Then $Th(b) \prec Th(b')$ by Lemma 5.17, hence $F\psi \in Th(b)$. Conversely, if $F\psi \in Th(b)$ then, by the Transfer Lemma 5.8, $Th(b) \prec \Delta$ for some Δ such that $\psi \in \Delta$. Then $b(\Delta)$ is a subbranch of b, and by the induction hypothesis we have $\mathcal{M}, b(\Delta) \models \psi$, hence $\mathcal{M}, b \models F\psi$.

(iii) The case $\phi = P\psi$ is analogous.

(iv) Let $\phi = \langle \mathbf{a} \rangle \psi$. If $\mathcal{M}, b \models \langle \mathbf{a} \rangle \psi$, then $\mathcal{M}, b' \models \psi$ for some $b' \neq b$ from the fan of b, i.e., with the same initial moment. Therefore $\psi \in Th(b')$, hence by the Momentary Association Lemma 5.12, $\langle \mathbf{a} \rangle \psi \in Th(b)$.

Conversely, let $\langle \mathbf{a} \rangle \psi \in Th(b)$ and let $(p \wedge [\mathbf{a}]\neg p) \in Th(b)$. Then, by the Transfer Lemma 5.8, there is some Δ such that $[\mathbf{a}]^{-}(Th(b)) \in \Delta$ and $\psi \in \Delta$. Then $n(\Delta) = n(Th(b))$ and $\Delta \neq Th(b)$ since $p \in Th(b) - \Delta$. By the induction hypothesis $\mathcal{M}, b(\Delta) \models \psi$, hence $\mathcal{M}, b \models \langle \mathbf{a} \rangle \psi$. □

THEOREM 5.5 (Completeness). *Every Ockhamist-valid formula ϕ is a theorem of EBOTL.*

Proof. If ϕ is not a theorem then $\neg\phi$ is consistent, hence it can be extended to a complete theory Θ. Then ϕ is falsified by the canonical model on the bundle tree $(T_\Theta, R(T_\Theta))$. □

6. Further Extensions

6.1. Since and Until

The system EBOTL is easily extended to axiomatize the operators *Since* and *Until*. The Ockhamist semantics for these operators is as follows:

(TC S) $\mathcal{M}, b \models S(\phi, \psi)$ iff for some $b' \supset b$, $\mathcal{M}, b' \models \phi$ and for every b'' such that $b \subset b'' \subset b'$, $\mathcal{M}, b'' \models \psi$.

(TC U) $\mathcal{M}, b \models U(\phi, \psi)$ iff for some $b' \subset b$, $\mathcal{M}, b' \models \phi$ and for every b'' such that $b' \subset b'' \subset b$, $\mathcal{M}, b'' \models \psi$.

In the language of EBOTL these operators are *locally* definable (Goranko, 1996) by means of the following axioms:

$$c \to (S(p, q) \leftrightarrow P(p \wedge G(Fc \to q))$$

and

$$c \to (U(p, q) \leftrightarrow F(p \wedge H(Pc \to q)).$$

These completely axiomatize *Since* and *Until* over EBOTL. Indeed, the proof of completeness repeats the one for EBOTL up to the Truth Lemma, the proof of which goes by induction on the number of occurrences of S and U in the formula. The basic case is dealt with as before; the inductive steps for *Since* and *Until* are proved using the fact that every moment of the canonical frame has a name, so the axioms above eliminate the outer-most occurrence of S or U.

6.2. Global Difference Operator

As mentioned in the Introduction, Zanardo (1996) uses a global difference operator D meaning, in bundle trees: “at some other branch.” Our *local* difference operator is (almost certainly) not definable in terms of the global one, but in the presence of fan-names it is locally definable by the axiom

$$c \to (\langle \mathbf{a} \rangle p \leftrightarrow D(p \wedge c)).$$

On the other hand, the global difference operator is explicitly definable in terms of the local one in connected bundle trees:

$$Dp = \langle \mathbf{a} \rangle (p \vee Fp) \vee Fp \vee Pp \vee P\langle \mathbf{a} \rangle Fp.$$

It is questionable, though, whether the language with local difference alone is more expressive in terms of definability of properties than the one with a global difference.

6.3. Indistinguishable and Undivided Histories

Zanardo (1998) has proposed extending branching-time logics with the notion of *indistinguishable histories*. This provides a framework that unifies different semantics for these logics into a generalized semantics based on I*-trees*, which are trees endowed with a family of equivalence relations on histories, one such relation for each moment of the tree (the idea being that two histories are indistinguishable at a moment if they belong to the same equivalence class at that moment), where these equivalence relations satisfy a natural past-agreement condition. The two typical semantics, Peircean and Ockhamist, can be regarded as the two extreme cases of I-trees, viz. in the former all histories passing through a moment are indistinguishable, while in the latter they are all distinguishable. There is an important intermediate case where two histories are indistinguishable at a moment if they are *undivided* at that moment, i.e., they split at a later moment. Thus, in particular, the branching-time language can be extended with an operator $\Diamond^u$, where $\Diamond^u \phi$ means "at some history through this moment, undivided from the actual one, ϕ is true." As Zanardo proves, this extension is essential, i.e., the new operator is not definable in the old language. It seems that it is not definable even in our extended language (though we have not searched for a proof of that) but it is *locally* definable there, which is sufficient to extend our system EBOTL to completely axiomatize the new operator as well. The following additional axiom defines $\Diamond^u$ locally:

$$(c \wedge p \wedge [\mathbf{a}]\neg p) \rightarrow (\Diamond^u q \leftrightarrow \Diamond(q \wedge F\Diamond P(c \wedge p)).$$

Again, the completeness proof presented above is easily extended here.

This axiomatization can be similarly generalized to any type of indistinguishability operator which is (at least) locally definable in our language.

6.4. Distinguishable and Divided Histories

Not less (and, for some applications, more) interesting and natural operators seem to be the "local complements" of the ones discussed above, viz. "at some history *distinguishable* from the actual one," and in particular, "at some history *divided* (i.e., splitting at this moment) from the actual one." These are (again, almost certainly) not definable by means of Zanardo's operators. The latter one, $\Diamond^d$, is locally definable in our language by the axiom

$$(c \wedge p \wedge [\mathbf{a}]\neg p) \rightarrow (\Diamond^d q \leftrightarrow \Diamond(q \wedge G\Box H(c \rightarrow \neg p))$$

and therefore readily axiomatized over EBOTL.

6.5. Reference Pointers over Paths

Another very expressive extension of the temporal language is obtained by adding *reference pointers* (Goranko, 1996). These are syntactic devices that make it possible for formulas to make semantic references, and thus significantly increase the expressive power of the language. In the framework of Ockhamist branching-time logic it is natural to consider pointers referring to branches, rather than moments. To be specific, we introduce a pair of new syntactic symbols, a *point of reference* $\downarrow$ and a *reference pointer* $\uparrow$ (or, possibly, a whole indexed set of such pairs). The symbol $\uparrow$ behaves syntactically like a propositional variable, while $\downarrow$ acts like a quantifier, binding all free (i.e., not yet bound) occurrences of $\uparrow$ within its scope. We shall give the Ockhamist semantics of these only intuitively; for a formal semantics, see Goranko (1996). The formula $\downarrow \phi$ is true at a branch b of a model $\mathcal{M}$ if the formula $\phi(q/\uparrow)$ obtained from ϕ by replacing all free occurrences of $\uparrow$ with a fresh variable q is true at b in the model $\mathcal{M}_b^q$ obtained from $\mathcal{M}$ by modifying (if necessary) the valuation V at q as follows: $V(q) := \{b\}$. Intuitively, the first occurrence of the point of reference $\downarrow$ in the formula $\downarrow \phi$ refers all occurrences of $\uparrow$ in ϕ which it binds to the actual branch.

For instance, the formula $\downarrow \Diamond(\phi \wedge \neg \uparrow)$ says that ϕ is true at some *other* branch stemming from this moment, i.e., it expresses the operator $\langle \mathbf{a} \rangle$; the formula $\downarrow \Diamond(\phi \wedge F\Diamond P \uparrow)$ says that ϕ is true at some branch which has a common future moment with the actual one, i.e., which belongs to a history *undivided* presently from the actual one, so it expresses the operator $\Diamond^u$; and likewise the formula $\downarrow \Diamond(\phi \wedge G\Box H \neg \uparrow)$ expresses the operator $\Diamond^d$. Since and Until are easily expressible by means of pointers, too (Goranko, 1996). Furthermore, reference pointers allow for simulation of fan-names: the antecedent of the formula $E \downarrow A(p \leftrightarrow \Diamond \uparrow) \rightarrow \phi$ forces the variable p to behave like a fan-name in the consequent ϕ. Therefore, the language with branch-reference pointers is stronger than any of those discussed above.

The axiomatic system EBOTL and its completeness proof can be extended for branch-reference pointers, using the axiomatization of temporal logics with reference pointers from Goranko (1996).

Note that all extended branching-time logics discussed here are translatable into fragments of the monadic second-order logic of trees with quantifiers over histories. Because, as is proved in Gurevich and Shelah (1985), this logic is decidable, it follows that validity with respect to complete trees is decidable for each of the logics discussed above. Decidability of validity with respect to bundle trees (which is strictly weaker; see the last remark below) for each of them, however, is still an unsolved problem.

7. Concluding Remarks

This article deals with an extension of Prior's temporal language suitable for Ockhamist branching-time logic. As mentioned in the introduction, we justify that extension as both conceptually natural and technically convenient, and believe that the results presented here support our choice. However, the present work raises several problematic issues, too, of which we shall discuss two.

The first one concerns the use of non-Hilbert style rules, such as COV and DIF, in our system. General schemata of such rules have been studies and used for axiomatizations in Venema (1993), and Goranko (1998), called in the later paper *context rules*. Although proliferating in the recent design of extended modal and temporal logics, these rules are still regarded by many people working in the field rather as a drawback than as a virtue of the axiomatic system, and alternative axiomatizations which eliminate them are sought (see, e.g., Di Maio and Zanardo, 1996). While we agree that using such rules is a deviation from the traditional format of the Hilbert style calculi, we would like to make a few points in defense of them.

– Context rules are introduced for the sake of constructing a complete axiomatization of modal (temporal, multimodal etc.) logics, which are usually based on extended, strongly expressive languages, or are designed for particular classes of models. Without using such rules, the complete axiomatic system becomes much more complicated, sometimes infinitary, as in Di Maio and Zanardo (1996), and often is still unknown. Therefore, the use of such rules is harmonious with the *spirit* of Hilbert-style axiomatic systems, which are designed for the sake of analyzing and proving meta-logical properties, rather than as practically applicable deductive systems.
– Moreover, in many cases context rules have better structural properties and more standardized behavior as rules of inference than Modus Ponens, as they formalize a clear and specific semantic idea (e.g., the rule COV is meant to enable naming any fan (equivalently, any moment) in the tree-like model; the rule DIF enforces separation of the actual branch from all others in the fan, and thus makes it possible to refer to that particular branch) which suggests how they should be used in the derivation. The idea behind using such rules can be explicated for the rules COV and DIF as follows.

> Deriving a formula within the formal system amounts to establishing its validity. Validity means truth at every branch of every model. Given a branch in a model, we want to demonstrate the truth of the formula at that particular branch. In the process of that demonstration we would like to be able to refer to the fan to which the branch belongs; we can introduce a name for it by using a fresh fan-name symbol. We would also like to be able to distinguish the particular branch from the others in the fan; for this we can introduce a fresh propositional variable which is true at that branch while false at all others in the fan. With those at hand, we have much more expressive power for formal argumentation (derivation). Once done, we apply the rules COV and DIF to dispose of the introduced symbols and obtain the derivation of the original formula.

This, of course, resembles the argument in use when proving universal statements in natural deduction (or, informal mathematical) reasoning. Examples of derivations which make use of a COV rule (for names for possible worlds) can be found in, e.g., Gargov and Goranko (1993).

– Finally, context rules can be used to express semantic conditions (e.g., irreflexivity) which are otherwise not definable by means of formulae of the language, and thus to enhance the language's expressiveness. This is effected by eliminating the unwanted models as ones which do not preserve validity of the additional rule(s). For detailed discussion and many examples on that, see Venema (1993) and Goranko (1998).

Using non-Hilbert-style rules raises the question of their eliminability. There are cases where they are known to be redundant (for instance adding Gabbay's irreflexivity rule to the basic modal logic K does not add any more theorems), and there are other cases where they are not (see, e.g., Gargov and Goranko, 1993), but there is a large grey area in between (into which our system falls), where it is not known if these rules are redundant, or if not, whether they can be eliminated at the expense of adding finitely many axioms (as is the case with $T \times W$ completeness, see Di Maio and Zanardo, 1996, and von Kutschera, 1997). Another related generic problem is to incorporate such rules into natural deduction or semantic tableau systems. Thus, the proof theory of context rules still awaits development.

The second issue which needs to be mentioned here concerns validity with respect to complete trees. Irrespective of the choice we make (branches vs. moments) for the ontological basis of the structure of time, the semantics based on complete trees remains the technically most natural one, and it is therefore important to find a complete axiomatization of its validity, moreover one that is known to be recursive (Gurevich and Shelah, 1985). It is known that the set of valid formulae in all bundle trees is strictly included in the set of valid formulae in all complete trees (see, e.g., Thomason, 1984), but a complete finite axiomatization of the latter in the standard branching-time temporal logic has not yet been found. (However, recently Reynolds has announced a solution to this problem.) This is essentially the problem to find a complete finitary axiomatization of the logic CTL* (Emerson, 1990). Similarly, it is still an unsolved problem for us to extend the system EBOTL to a complete axiomatic system for validity in all trees.

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