Two-player preplay negotiation games with conditional offers

Valentin Goranko[1] and Paolo Turrini[2]

[1]Department of Philosophy, Stockholm University,
valentin.goranko@philosophy.su.se
[2]Department of Computing, Imperial College London,
p.turrini@imperial.ac.uk

October 29, 2015

Abstract

We consider an extension of strategic normal form games with a phase before the actual play of the game, where players can make binding offers for transfer of utilities to other players after the play of the game, contingent on the recipient playing the strategy indicated in the offer. Such offers transform the payoff matrix of the original game but preserve its non-cooperative nature. The type of offers we focus on here are conditional on a suggested matching offer of the same kind made in return by the receiver. Players can exchange a series of such offers, thus engaging in a bargaining process before a strategic normal form game is played. In this paper we study and analyse solution concepts for two-player normal form games with such preplay negotiation phase, under several assumptions for the bargaining power of the players, as well as the value of time for the players in such negotiations. We obtain results describing the possible solutions of such bargaining games and analyse the degrees of efficiency and fairness that can be achieved in such negotiation process. We show the similarities and the differences with a variety of frameworks in the literature of bargaining games and games with a preplay phase.

1 Introduction

It is well known that some normal form games have no pure strategy Nash equilibria, while others, like the Prisoner's Dilemma, have rather unsatisfactory – e.g., strongly Pareto dominated – ones. These inefficiencies are often attributed to lack of possibility for the players to communicate and reach agreement on a mutually beneficial joint course of action, before the actual play of the game. Indeed, undesirable outcomes could often be avoided if players were able to communicate and make binding agreements on the joint strategy to play before the game starts. However, even if players could freely communicate before the game, the enforcing of such coalitional contracts is often not possible in practice and, furthermore, it would change the nature of the game from non-cooperative to essentially cooperative.

Here we consider a weaker and generally more realistic assumption, viz.:

> *Before the actual game is played any player, say A, can make a binding offer to any other player, say B, to pay him*[1]*, after the game is played, an explicitly declared amount of utility δ if B plays a strategy s specified in the offer by A.*

Building up upon this basic, *unconditional*, form of offer, we also consider a more general setting, where players, without acting as a coalition, can propose a game transformation to their fellow players, by making an offer under the condition of receiving another offer in return, proposal that can, in turn, be accepted or rejected. This newly obtained game transformation can be further modified, with proposals made by other players, until an agreement is reached. When endowing players with the possibility of playing such extra preplay moves, a whole *bargaining phase* emerges before a normal form game is actually played. In other words, we can think of the normal form game that is eventually played as an outcome of another game, played beforehand, in which players engage in exchanging offers on strategies of other players until an agreement is reached on the game to play.

Introducing an extensive-form bargaining structure preceding the play of a normal form game is relevant and important for the analysis of a wide spectrum of economic, social and political situations, such as:

- political, labour-related or business *negotiations and compromises* between non-cooperative parties,
- *compensations, concessions, out-of-court settlements of legal cases*, etc.,
- *collusions* between two or more parties in an economic activity, by exchanging 'behind the curtain' agreements for mutual incentives,
- *kickback schemes* and other quasi-legal incentives,
- *corruption schemes* involving bribes in exchange of illegal favours.

Many situations such as the ones listed above are based on self-interested players trying to influence the decision making of each other by means of explicit offers of unilateral money transfers, should certain outcomes be realised, or of mutual exchange of favours. For further details and discussions of these kinds of scenarios see for instance in [Guttman [1978]], [Guttman [1987]], [Schelling [1960]] and [Segal [1999]]. The game-theoretic modelling and analysis of such scenarios is the main motivation of the present work.

Agreements in such economic and political negotiations are usually reached in dynamic bargaining processes made of offers and counteroffers, rather than a one-shot simultaneous proposal ending the talks. The literature in economic theory abounds with examples of parties entering negotiations to overcome inefficient resource allocation, as well as schemes

[1] We refer to player A as a female, while to B as a male. This choice is not for the sake of political correctness but to make it easier to distinguish the players from the context.

of side payments, compensatory mechanisms, etc., which we review in detail in Section 6. Here we only mention some more recent studies of preplay contracting in games that consider one-shot simultaneous, in [Jackson and Wilkie [2005]], [Elligsen and Paltseva[2011]], or two-step, in [Yamada [2005]], offers preceding the actual game play and conditional on the *entire strategy profile* (see discussion in Section 6), which makes an essential difference in the analysis. Somewhat surprisingly, however, a systematic study of the extensive-form negotiation process preceding the actual game play seems still to be missing in the literature. With this paper we initiate such systematic study purporting to fill this gap, by formalising and studying the negotiation process preceding the actual game play as a bargaining among the players on the game to play, thus drawing connections with modern bargaining theory, in particular, Rubinstein's model of bargaining games [Rubinstein [1982], Osborne and Rubinstein [1994]]. We discuss our framework in more detail in Section 2, illustrate and discuss preplay offers and offer-induced game transformations in Section 3 and introduce normal form games with preplay negotiations phase in Section 4. Then we analyse the case with conditional offers in Section 5, where we obtain results for existence of efficient negotiation strategies of both players, significantly extending our previous work in [Goranko and Turrini [2013]]. We end the paper with discussion of related work in Section 6 and concluding remarks and directions for further study in Section 7.

2 Non-cooperative games with preplay offers: the conceptual framework

In this section we provide a more detailed description of preplay offers, discuss some motivating examples, and lay down several extra conditions that play a role in determining the outcome of the negotiation phase.

2.1 Nature and structure of preplay offers

We assume that any preplay offer by A to B is *binding for* A, conditional on B playing the strategy s specified by A.

However, such offer *does not* create any obligation for B and therefore it does not transform the game into a cooperative one, for B is still at liberty to choose his strategy when the game is actually played. In particular, after her offer A does not know before the game is played whether B will play the desired by A strategy s, and will thus make use of the offer, or not. Furthermore, several such offers can be made, possibly by different players, so the possible rational behaviours of the payers game maintain, in principle, all their complexity. The key observation applying to this assumption, is that *after any binding preplay offer is made, the game remains a standard non-cooperative normal form game, only the payoff matrix changes according to the offer.*

2.2 Motivating examples

First, we introduce the following notation: $A \xrightarrow{\delta/\sigma_B} B$ denotes an offer made by player A to pay an amount δ to player B after the play of the game if player B plays strategy σ_B.

Prisoners' Dilemma 1 Consider a standard version of the Prisoner' s Dilemma (PD) game in Figure 1. The only Nash Equilibrium (NE) of the game is (D, D), yielding a

	C	D
C	$4, 4$	$0, 5$
D	$5, 0$	$1, 1$

Figure 1: Prisoner's Dilemma 1

payoff of $(1, 1)$. Now, suppose *Row* $\xrightarrow{2/C}$ *Column*, that is, player *Row* makes to the player *Column* a *binding offer* to pay her 2 units of utility (hereafter, *utils*) after the game if *Column* plays C. That offer transforms the game by transferring 2 utils from the payoff of *Row* to the payoff of *Column* in every entry of the column where *Column* plays C, as pictured in Figure 2.

	C	D
C	$2, 6$	$0, 5$
D	$3, 2$	$1, 1$

Figure 2: An offer to cooperate by player Row.

In this game player *Row* still has the incentive[2] to play D, which strictly dominates C for him, but the dominant strategy for *Column* now is C, and thus the only Nash equilibrium is (D, C) with payoff $(3, 2)$ – strictly dominating the original payoff $(1, 1)$.

Thus, even though player *Row* will still defect, the offer he has made to player *Column* makes it strictly better for *Column* to cooperate.

Of course, *Column* can now realize that if player *Row* is to cooperate, then *Column* would be even better off, but for that an extra incentive for *Row* is needed. That incentive can be created by an offer *Column* $\xrightarrow{2/C}$ *Row*, that is, if *Column*, too, makes an offer to *Row* to pay him 2 utils after the game, if player *Row* cooperates. Then the game transforms, as in Figure 3.

	C	D
C	$4, 4$	$2, 3$
D	$3, 2$	$1, 1$

Figure 3: A second offer, by player Column.

[2]Intuitively, *having the incentive* to play a strategy should be understood as realising that that strategy is not dominated. Later on we will provide a formal and abstract notion of equilibrium, which will rule out dominated strategies to be part of the solution of a game.

In this game, the only Nash equilibrium is (C, C) with payoff $(4, 4)$, which is also Pareto optimal. Note that this is the same payoff for (C, C) as in the original PD game, but now both players have created incentives for their opponents to cooperate, and have thus escaped from the trap of the original inefficient Nash equilibrium (D, D).

Remark 1 *Clearly, preplay offers can only work in case when at least part of the received payoff can actually be transferred from a player to another. They obviously cannot apply to scenarios such as the original PD, where one prisoner cannot offer to the other to stay in prison for him, even if they could communicate before the play.*

Prisoners' Dilemma 2 Consider another version of the Prisoner' s Dilemma game in Figure 4. The only Nash Equilibrium in this game is (D_{Row}, D_{Col}), yielding the Pareto

	C	D
C	$4, 4$	$0, 5$
D	$5, 0$	$3, 3$

Figure 4: Prisoner's Dilemma 2

dominated payoff of $(3, 3)$. Now, note that none of the players can make a feasible first offer to improve the outcome. Indeed, in order to provide a sufficient incentive for *Column* to play C_{Col}, *Row* would have to offer him more than 3, which is unfeasible for *Row* because it would put him in a disadvantaged position. Likewise for *Column*.

Thus, by consecutive exchange of unilateral preplay offers rational players cannot realise the opportunity to play the Pareto optimal outcome (C_{Row}, C_{Col}).

This problem can be avoided if we allow *conditional offers* as follows: *Row* can make an offer $Row \xrightarrow{3/C_{Col}} Column$, but now, *conditional on Column making to Row the matching counter-offer* $Column \xrightarrow{3/C_{Row}} Row$, which we hereafter denote as $Row \xrightarrow{3/C_{Col} \mid 3/C_{Row}} Row$. The idea is that, unlike the so far considered unconditional offers, *Row* 's conditional offer is only confirmed and enforced if *Row* does make the required counter-offer, else it is cancelled and nullified before the play of the game.

We will introduce formally and discuss conditional offers in detail further.

2.3 Additional optional assumptions

There are several important additional assumptions that, depending on the particular scenarios under investigation may, or may not, be realistically made. We therefore do not commit to any of them, but we acknowledge that each of them can make a significant difference in the behaviour and abilities of players to steer the game in the best possible direction for them. So, we consider the possible options for each of them separately and study the consequences under the various combinations of assumptions.

The nature and value of time. Time in the preplay negotiations is measured discretely as the number of explicitly defined steps/rounds of the negotiations, It may or may not have value, i.e. players may, or may not, strictly prefer a reward in the present to the same reward in the future. Moreover, time may have the same value for all players, or may be more, or less, valuable for each of them depending on their *patience*.

- In the case when time is of no value, players can keep making offers at no extra cost.
- In the case when time is of value, making unacceptable or suboptimal offers should intuitively lead to inefficient negotiation and, consequently, strategies involving such offers would not be subgame perfect equilibrium strategies. This intuition is confirmed by our technical results.

The order of making offers. The order in which offers are made by the different players can be essential, especially in case of irrevocable offers. In such cases we assume that the order in which players can make offers is set by a separate, exogenous protocol which is an added component of the preplay negotiations game; for instance, it can be strictly alternating or random. Alternatively, the offers may be required to be made simultaneously by all players, as in [Jackson and Wilkie [2005]] and [Elligsen and Paltseva[2011]], however we do not consider this option, as the aim of our paper is to make the structure of preplay negotiations explicit in the models.

Conditionality of offers. As discussed earlier, offers may be *unconditional*, i.e., not subject to acceptance or rejection by the player to whom the offer is made, or *conditional* upon an expected (suggested or demanded) counter-offer by the player to whom the offer was made. Acceptance of a conditional offer means both acceptance of the offer and making the expected counter-offer. We emphasise that *after acceptance, a conditional offer does not constitute a contract between the players turning the game into a cooperative one, but only a pair of unilateral offers, each binding only its proposer. It therefore transforms the current game into another non-cooperative game.* Rejection of a conditional offer means cancellation of both of the unconditional offers of which it consists. The option of rejection of conditional offers can be reasonably assumed under some circumstances (e.g. possibility for extended communication and for a low-cost negotiations), but not in others. We will consider both cases separately.

3 Preplay offers and induced game transformations

In this section we describe the game transformations induced by preplay offers in a general and more technical fashion.

Before proceeding further, we need to make two important clarifying remarks.

1. Consistently with the approach taken in the literature on non-cooperative games with transferrable utility, the value that players attach to outcomes can be transferred among them. This, however, is not the same as saying that players are only dealing with money or with a unique currency. Yet, it does assume that players' utilities *can be transformed* into ones based on a common unique unit of measure that can be transferred at no cost.

2. We are assuming the existence of a mechanism which allows players to make their offers binding. Our analysis is therefore only applicable to scenarios in which this assumption can be met. It can be implemented in practice in various ways, e.g., by a legal contract (*but only stipulating the accepted offers, not prescribing the actions*

that the players will take in the actual game), a trusted third party, a mediator, or any other empowered institution. While assuming the existence of such mechanism, we abstract away from analysing its inner rules and procedures.

3.1 Transformations of normal form games by preplay offers

	B_1	$\cdots$	B_j	$\cdots$
A_1	$\cdots$	$\cdots$	a_{1j}, b_{1j}	$\cdots$
A_2	$\cdots$	$\cdots$	a_{2j}, b_{2j}	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
A_i	$\cdots$	$\cdots$	a_{ij}, b_{ij}	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Figure 5: A general 2-player game

Here we formally define the notion of transformation induced by a preplay offers. For technical convenience we consider general 2-player game with a payoff matrix given in Figure 5; the case of N-player games is a straightforward generalisation.

Suppose player A makes a preplay offer to player B to pay her additional utility[3] $\alpha \geq 0$ if B plays B_j. Recall that we denote such offer by $A \xrightarrow{\alpha/B_j} B$. It transforms the payoff matrix of the game as indicated in Figure 6.

	B_1	$\cdots$	B_j	$\cdots$
A_1	$\cdots$	$\cdots$	$a_{1j} - \alpha, b_{1j} + \alpha$	$\cdots$
A_2	$\cdots$	$\cdots$	$a_{2j} - \alpha, b_{2j} + \alpha$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
A_i	$\cdots$	$\cdots$	$a_{ij} - \alpha, b_{ij} + \alpha$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Figure 6: A general 2-player game with an offer.

We will call such transformation of a payoff matrix a **primitive offer-induced transformation**, or a POI-transformation, for short.

Several preplay offers can be made by each players. Clearly, the transformation of a payoff matrix induced by several preplay offers can be obtained by applying the POI-transformations corresponding to each of the offers consecutively, in any order. We will call such transformations **offer-induced transformations**, or OI-transformations, for short. Thus, every OI-transformation corresponds to a *set* of preplay offers, respectively a set of POI-transformations. Note that the set generating a given OI-transformation need not be unique, e.g., A can make two independent offers $A \xrightarrow{\alpha_1/B_j} B$ and $A \xrightarrow{\alpha_2/B_j} B$ equivalent to the single offer $A \xrightarrow{\alpha_1+\alpha_2/B_j} B$.

[3]The reason we allow vacuous offers with $\alpha = 0$ is not only to have an identity transformation at hand, but also because such offers can be used by players as signalling, to enable coordination.

The general mathematical theory of OI-transformations is studied in more detail in [Goranko[2012]]. Here we only mention some observations about the game-theoretic effects of OI-transformations, which will be useful later on.

1. An OI-transformation does not change the sum of the payoffs of all players in any outcome, only redistributes it. In particular, OI-transformations preserve the class of zero-sum games.

2. An OI-transformation induced by a preplay offer by player A does not change the preferences of A regarding her own strategies. In particular, (weak or strict) dominance between strategies of player A is invariant under OI-transformations induced by preplay offers of A, i.e.: a strategy A_i dominates (weakly, resp. strongly) a strategy A_j before a transformation induced by a preplay offer made by A if and only if A_i dominates (weakly, resp. strongly) A_j after the transformation.

3. The players can collude to make *any* designated outcome, with any redistribution of its payoffs, a dominant strategy equilibrium, by exchanging sufficiently high offers to make the strategies generating that outcome with that redistribution of the payoffs, strictly dominant.

Thus, preplay offers can transform the game matrix radically. However, we note that not every matrix transformation that preserves the sums of the payoffs in every outcome can be induced by preplay offers. In particular, this is the case if the transformed matrix differs from the original one in only one payoff. For general necessary and sufficient condition for a normal form game to be obtained from another by preplay offers see [Goranko[2012]].

A central question arising is what should be regarded as a *solution* of a strategic game allowing binding preplay offers. The possible answers to that question crucially depend on the additional assumptions discussed earlier and on the procedure of 'preplay negotiations'; these will be discussed further.

3.2 Extending preplay offers and OI-transformations

3.2.1 Conditional offers

Unconditional offers always decrease the proponent's payoff at some outcomes, and hence making an unconditional offer comes with a cost. As in the Prisoners' Dilemma 2 previously described, this can be a hindrance for making mutually beneficial offers. Furthermore, we can easily think of real life situations where players who make such preplay offers expect some form of reciprocity from their fellow players and make their offers conditional on an expected 'return of favour'.

For these reasons, we now extend the preplay offers framework to enable players to *suggest* a transformation of the starting game, by making a *conditional offer* to an opponent for payment subject to playing a certain strategy, in exchange for a similar 'counter-offer' from that opponent. More precisely, every conditional offer, denoted as $A \xrightarrow{\alpha/\sigma_B \ | \ \beta/\rho_A} B$ is associated with a **suggested transformation** of the starting game $\mathcal{G}$ into a game $\mathcal{G}(X)$ where $X = \{A \xrightarrow{\alpha/\sigma_B} B, B \xrightarrow{\beta/\rho_A} A\}$.

Two responses of the recipient of a conditional offer $A \xrightarrow{\alpha/\sigma_B \ | \ \beta/\rho_A} B$ are possible: it can be *accepted* or *rejected* by the player receiving it. If rejected, the offer is immediately cancelled and does not commit any of the players to any payment, and therefore it does not induce any transformation of the game matrix. If accepted, the **actual transformation** induced by the offer is the suggested transformation defined above. Two important observations:

- an unconditional offer has the same effect as an accepted conditional offer with a trivial counter-offer where $\beta = 0$.
- a conditional offer can be seen as the proposal of two separate unconditional offers *that can only be enforced together*.

Conditional offers can be made to different players. Multiple conditional offers can be made to the same player, contingent upon same or different strategies of the recipient and the proposer, too.

4 Normal form games with preplay negotiations phase

In this section we first give some technical preliminaries on normal form games and their solution. Then we introduce normal form games with preplay negotiations. We start out with an informal introduction, followed by a formal definition of their structure, together with the notion of efficiency negotiation strategies.

4.1 Preliminaries: solution concepts and values of normal form games

We will be using $i, j, \ldots$ for variables ranging over players, while $A, B, \ldots$ will denote individual players.

4.1.1 Normal form games

Let $\mathcal{G} = (N, \{\Sigma_i\}_{i \in N}, u)$ be a normal form game (hereafter abbreviated as NFG), where $N = \{1, \ldots, n\}$ a finite set of players, $\{\Sigma_i\}_{i \in N}$ a family of strategies for each player and $u : N \times \prod_{i \in N} \Sigma_i \to \mathbb{R}$ is a **payoff function** assigning to each player a utility for each strategy profile. The game is played by each player i choosing a strategy from Σ_i. The resulting strategy profile σ is the **outcome** of the play and $u_i(\sigma) = u(i, \sigma)$ is the associated payoff for i. An outcome of a play of the game $\mathcal{G}$ is called **maximal** if it is a Pareto optimal outcome with the highest sum of the payoffs of all players.

4.1.2 Solution concepts and solutions of normal form games

Let $\mathbf{G_N}$ be the set of all normal form games for a set of players N. By **solution concept for $\mathbf{G_N}$** we mean a map $\mathfrak{S}$ that associates with each $\mathcal{G} \in \mathbf{G_N}$ a non-empty set $\mathfrak{S}(\mathcal{G})$ of outcomes of $\mathcal{G}$, called the **$\mathfrak{S}$-solution of the game**. At times we will talk about players' strategies that are consistent with some solution concept. For a player i, we denote $\mathfrak{S}_i$ to be the restriction of the mapping $\mathfrak{S}$ to i returning, instead of full outcomes, only strategies of player i consistent with $\mathfrak{S}$ in the sense that $\mathfrak{S}_i(\mathcal{G}) = \{\sigma_i \in \Sigma_i \mid \sigma \in \mathfrak{S}(\mathcal{G})\}$. Slightly abusing notation we will also consider mappings of the form $\mathfrak{S}_{-i}$ to indicate the mapping $\mathfrak{S}(\mathcal{G})$ restricted to player i's opponents. Solution concepts formalise the concepts of rationality of the players in the strategic games. A $\mathfrak{S}$-solution of a strategic game $\mathcal{G}$ basically tells us what outcomes of the game the players could, or should, select in an actual play of that game, if they adopt the solution concept $\mathfrak{S}$.

In this work we do not commit to a specific solution concept for the normal form games but we assume that the one adopted by the players satisfies the necessary condition that *every outcome in any solution prescribed by that solution concept must survive iterated elimination of strictly dominated strategies*. We will call such solution concepts **acceptable**. This condition reflects the assumption that players would never play strategies that are dominated, and that this exclusion is a common knowledge amongst them and can be used in their strategic reasoning. Thus, the weakest acceptable solution concept is the one that returns all outcomes surviving iterated elimination of strictly dominated strategies.

Games for which the solution concept $\mathfrak{S}$ returns a single outcome will be called **$\mathfrak{S}$-solved**. For instance, every game with a strongly dominating strategy profile is $\mathfrak{S}$-solved for any acceptable solution concept $\mathfrak{S}$. Games for which $\mathfrak{S}$ returns only maximal outcomes will be called **optimally $\mathfrak{S}$-solvable**. If for every player all these maximal outcomes provide the same payoffs, we call the game **perfectly $\mathfrak{S}$-solvable**. Games that are $\mathfrak{S}$-solved and perfectly $\mathfrak{S}$-solvable (i.e., $\mathfrak{S}$ returns one maximal outcome) will be called **$\mathfrak{S}$-perfectly solved**.

The ultimate objective of a preplay negotiation is to transform the starting NFG into a perfectly $\mathfrak{S}$-solvable one. Ideally, it should be a $\mathfrak{S}$-perfectly solved one, but this is not always possible: cf. any symmetric Coordination game.

4.1.3 Players' expected values of a game

It is necessary for the preplay negotiation phase that will be introduced later for each player to have an **expected value** of any NFG that can be played. Naturally, that expected value would depend not only on the game but also on the adopted solution concept and on the player's level of risk tolerance. A risk-averse player would assign as expected value the minimum of his payoffs over all outcomes in the respective solution, while a risk-neutral player could take the probabilistic expected value of these payoffs, etc. Note that the expected value of any $\mathfrak{S}$-solved game for any player i naturally should equal the payoff for i from the only outcome in the solution.

For sake of definiteness, unless otherwise specified further, we adopt here the conservative, risk-averse approach and will define for every acceptable solution concept $\mathfrak{S}$, game $\mathcal{G}$ and a player i, the expected value of $\mathcal{G}$ for i relative to the solution concept $\mathfrak{S}$ to be:

$$\mathsf{v}_i^{\mathfrak{S}}(\mathcal{G}) = \max_{\sigma_i \in \mathfrak{S}_i(\mathcal{G})} \min_{\sigma_{-i} \in \mathfrak{S}_{-i}(\mathcal{G})} u_i(\sigma)$$

4.2 Normal form games with preplay negotiations phase: informal introduction

Our setting for normal form games with preplay offers begins with a given 'starting' normal form game $\mathcal{G}$ and consists of two phases:

- A *preplay negotiation phase*, where players negotiate on how to transform the game $\mathcal{G}$ by making unconditional offers, accepting or rejecting conditional offers they receive. This phase constitutes an extensive form game, which we call a *preplay negotiation game* (PNG).
- An *actual play* phase where, after having agreed on some OI-transformation X in the previous phase, the players play the resulting game $\mathcal{G}(X)$.

Players' moves in the preplay negotiation phase should intuitively be understood as based on *strategies* that players follow in order to modify the starting normal form game in the best for them possible way, before they get to play it.

Players engage in preplay negotiations with the purpose of reaching a best for them possible agreement based on OI transformation of the original game $\mathcal{G}$. Major questions that we set out to study are:

- *What constitutes an optimal/rational/efficient negotiation strategy and what are the expected outcome(s) when players follow such strategies?*
- *In particular, when can players agree upon Pareto optimal outcomes in their preplay negotiations if playing rationally?*
- *What can, or should, players agree upon in the preplay negotiations phase when the original game has several Pareto optimal outcomes?*

Further we introduce, first informally and then fully formally, the setup of PNGs as extensive-form bargaining games, including the concepts of moves and histories, the order of moves, the possibility of players come to a disagreement, and finally a notion of solution for these games.

4.3 Moves, histories and preplay negotiations games

Depending on some of the optional assumptions, the players can have several possible moves in the preplay negotiations phase. Let us consider the most general case, based on conditional offers. Then the moves available to the player whose turn is to play depend on whether or not he has received any conditional offers since his previous move. If so, we say that the player has **pending conditional offers**. The possible moves of the player in turn are as follows.

1. If the player has no pending conditional offers, he can:
   (a) *Make an offer* (conditional or not).
   (b) *Pass*.

(c) (Optional) *Opt out* (see Section 4.5).

2. If the player has pending conditional offers, for each of them he can:

   (a) *Accept the pending offer* by making the requested counter-offer to the player who has made the conditional offer, and then make an offer of his/her own or pass or opt out (when available).

   (b) *Reject the pending offer*, and then make an offer of his/her own or pass or opt out (when available).

If *all players have passed* at their last move, or *any player has opted out*, the preplay negotiations game is over.

We say that an offer of the game is **passing** if its acceptance by the opponents is followed by a pass of the proponent. In other words, the one making the offer would be happy to end the game with the suggested transformation. Likewise, an acceptance is passing if, once declared, it is followed by a pass move of the same player. In other words, with a passing acceptance a player declares agreement to terminate the game with the proposed transformation. When opting out is not allowed, passing moves (i.e. offers or acceptances that are passing), are the only way for players to terminate the game in agreement and the only way to effectively deviate from undesired outcomes.

We now define the notion of a **history** in the preplay negotiations phase as a finite or infinite sequence of admissible moves by the players who take their turns according to an externally set protocol (see further). Every finite history in such a game is associated with the **current NFG**: the result of the OI-transformation of the starting game by all offers that are so far made and accepted. The current NFG of the empty history is the input NFG of the preplay negotiations game.

A **play** of a preplay negotiations game is any finite history at the end of which the preplay negotiations game is over, or any infinite history.

In order to eventually define realistic solution concepts for preplay negotiations games we need to endow every history in such games with value for every player. Intuitively, **the value of a history** is the value for the player of the current NFG associated with that history in the case of non-valuable time, and the same value accordingly discounted in the case of valuable time.

Now, a **preplay negotiation game (PNG)** can be defined generically as a turn-based, possibly infinite, extensive form game that starts with an input NFG $\mathcal{G}$ and either ends with a transformed game $\mathcal{G}'$ or goes on forever, which we discuss further. The **outcome of a play of the PNG** is the resulting transformed game $\mathcal{G}'$ in the former case and 'Disagreement' (briefly D) in the latter case.

The fact that we are using the term *preplay* to denote the negotiation phase, as well as the term *actual game play* to denote strategy selection in the normal form game, should not be interpreted as the negotiation phase being "virtual" or not taking place in reality. Just like in a bargaining game ([Osborne and Rubinstein [1994]]), players play a *real* negotiation game before making their final decisions, according to an exogenously determined protocol. As noted above, the preplay negotiations games take place in discrete time, measured not as "calendar" time but as the number of moves in the negotiation game, where every move is assumed to take one time unit. The time duration of the preplay negotiation is not set in advance, but, as it will become clear further, the players have no incentive to delay forever, and when time is valuable for them they have the incentive to complete the negotiation as quickly as possible. This is different from other treatments of virtual and endogenously determined time, as in e.g., [Hamilton and Slutsky [1990]].

4.4 Preplay negotiation games formally

Here we provide a formal definition for the general N-player case of preplay negotiation games.

Definition 2 (Preplay negotiation game) *A* preplay negotiation game *is a tuple* $\mathcal{E} = (\mathrm{N}, \mathcal{G}, \mathfrak{S}, \mathcal{A}, \mathcal{H}, \mathsf{turn}, \{\Sigma_i\}_{i\in\mathrm{N}}, \mathbf{g}, \mathsf{out}, \mathbf{u})$*, where:*

- N *is the set of players.*
- $\mathcal{G}$ *is the starting normal form game.*
- $\mathfrak{S}$ *is an acceptable solution concept for normal form games.*
- $\mathcal{A}$ *is a set of* **actions**, *or* **moves** *of types as discussed earlier.*
- $\mathcal{H}$ *is a non-empty set of finite or infinite sequences of actions, called* **histories**, *that includes the empty sequence* ϵ *and is* prefix-closed, *meaning that every prefix of a history in* $\mathcal{H}$ *belongs to* $\mathcal{H}$, *and* limit-closed, *meaning that the infinite union of a chain by extension of finite histories in* $\mathcal{H}$ *belongs to* $\mathcal{H}$, *too.*

  A history $h \in \mathcal{H}$ *is* **terminal in** $\mathcal{H}$ *if it is infinite or there is no history in* $\mathcal{H}$ *extending it. The set of terminal histories in* $\mathcal{H}$ *is denoted by* $\mathcal{H}^t$ *and the set of finite histories in* $\mathcal{H}$ *by* $\mathcal{H}^f$.

  For $h, h' \in \mathcal{H}^f$ *and* $o \in \mathcal{A}$ *we denote by* $h; o$ *the extension of* h *with the action* o *and by* $h; h'$ *the concatenation of* h *with* h'. *XX*
- $\mathsf{turn} : \mathcal{H} \setminus \mathcal{H}^t \to N$ *is the* **turn function**, *assigning the players who are to move at non-terminal histories. We denote* $\mathcal{H}_i := \mathsf{turn}^{-1}(i)$ *for each* $i \in N$ *the set of histories where it is* i*'s turn to play.*

  Here we assume that the turned function is exogenously defined, e.g. in some fixed cyclic order or depending on the last move made.
- Σ_i, *for each* $i \in N$, *is a non-empty set of* **strategies** $\sigma_i : \mathcal{H}_i \to \mathcal{A}$ *that assigns an action for* i *to any non-terminal history in* $\mathcal{H}_i$.
- $\mathbf{g} : \mathcal{H} \to \mathbf{G_N}$ *is a function associating to each finite history the* **currently accepted NFG**, *defined below.*
- $\mathsf{out} : \prod_{i\in N} \Sigma_i \to \mathcal{H}^t$ *is an* **outcome play function**, *assigning to each strategy profile* σ *the terminal history* $\mathsf{out}(\sigma)$ *generated by* σ.

  Respectively, the **outcome NFG** *of* σ *is* $\mathbf{g}(\mathsf{out}(\sigma))$.

- $\mathbf{u} : \mathrm{N} \to (\mathcal{H}^t \to \mathbb{R})$ *is the* **utility function of the PNG**, *associating to each player the payoff function* $\mathbf{u}_i$ *such that* $\mathbf{u}_i(z) = \mathbf{v}^{\mathfrak{S}}{}_i(\mathbf{g}(z))$ *for every finite* $z \in \mathcal{H}^t$. *Further, for* $z, z' \in \mathcal{H}^t$, *with* z *finite and* z' *infinite, we require that* $\mathbf{u}_i(z) \geq \mathbf{u}_i(z')$ *for all players* i, *and* $\mathbf{u}_j(z) > \mathbf{u}_j(z')$ *for some* j, *i.e., no disagreement is better for all players than any agreement.*

Now we define the function $\mathbf{g}$. *Its intended meaning is that* $\mathbf{g}(h)$ *would be the outcome of the PNG if the game ended at* h. *Its precise definition depends on the repertoire of moves that are allowed in the PNG, as follows:*

- $\mathbf{g}(\epsilon)$ *is the starting normal form game* $\mathcal{G}$.
- *If* $h = h'; o$, *where the last move* o *is an unconditional offer, then* $\mathbf{g}(h) = \mathbf{g}(h')(o)$, *i.e., the transformation of* $\mathbf{g}(h')$ *by the offer* o.
- *If* $h = h'; a$, *where the last move* a *is an acceptance of a conditional offer* o, *then* $\mathbf{g}(h) = \mathbf{g}(h')(o)$.
- *In all other cases of actions* a, $\mathbf{g}(h; a) = \mathbf{g}(h)$.

Solution of PNG. By solution of a PNG we mean *the set of all transformed normal form games* $\mathbf{g}(h)$ *for all outcomes* h *of plays effected by subgame perfect equilibrium (SPE) strategy profiles in the PNG.*

4.5 Disagreements

Clearly, players would only be interested in making preplay offers inducing payoffs that are "optimal" for them. Therefore, rational players are expected to "negotiate" in the preplay phase the play of Pareto optimal outcomes. In particular, if the game has a unique strictly Pareto dominant outcome then the players can negotiate a transformation of the game to make it the (unique) dominant strategy equilibrium. Yet, players that are getting lesser shares of the total payoff may still want to negotiate a redistribution, so even in this case the outcome of the preplay negotiations is not a priori obvious. In particular, there is no guarantee that the PNG will ever terminate, i.e. that its solution is non-empty.

The PNG may terminate if all players pass at some stage, in which case we say that the players have reached agreement, or may go on forever, in which case the players have failed to reach agreement; we call such situation a *(passive) disagreement* and we denote any such infinite history with D. We will not discuss disagreements and their consequences here, but will make the explicit assumption that *any agreement is better for every player than disagreement* in terms of the payoffs, by assigning payoffs of $-\infty$ in the entire game for each player if the PNG evolves as a disagreement. However, we also outline a more flexible and possibly more realistic alternative, whereby players can explicitly express tentative agreements with the status quo before every move they make, essentially by saying "*So far so good, but let me try to improve the game further by offering ...*", or express disagreements, by essentially saying "*No, I am not happy with the way the negotiations have developed since the last time I agreed, so I'd like to improve the game by offering instead ...*". This type of negotiations involves, besides the other moves listed above, also formal statements of acceptance or non-acceptance of the current NFG, where the input NFG is automatically accepted by all players and at every stage of the negotiations, the current NFG is the one on which they are currently negotiating by making offers, whereas the currently accepted NFG is the last current one for which all players have explicitly stated acceptance. Then if at any stage of the PNG any player is currently unhappy and realises that he cannot improve further because of the other players not willing to accept his best conditional offers, then he can terminate the negotiations by explicitly **opting out**, which would leave as an outcome game the currently accepted NFG.

4.5.1 Efficient negotiation strategies

Definition 3 (Efficient negotiation strategies) *A strategy in the PNG is an* efficient negotiation strategy *if it only involves making (minimal) feasible offers and it passes once they are accepted. It is* strongly efficient *if the vector of payoffs of the outcome it attains is a redistribution of the vector of payoffs of a maximal outcome.*

A number of important relevant questions arise:

- Is it the case that every subgame perfect equilibrium (SPE) strategy of a PNG is an efficient negotiation strategy and vice versa?
- If not, can the inefficient ones be replaced by efficient ones generating the same, or at least as good solution?
- Under what conditions can a given (maximal) Pareto optimal outcome in the starting NFG become the unique outcome of the final NFG?

To answer these questions we need an analysis of the solutions of the PNG game. Further we provide such partial analysis for the case of two players.

5 Two-players preplay negotiation games with conditional offers

In this section we allow the possibility of players to make conditional offers to each other and obtain results about the efficiency of the resulting negotiation process and its possible outcomes, under several optional assumptions.

Before analysing some cases with additional optional assumptions, let us state a useful general result, also valid in the case of many players PNG. An extensive form game is said to have the **One Deviation Property** (ODP) [Osborne and Rubinstein [1994], Lemma 98.2] if, in order to check that a strategy profile is a Nash equilibrium in (some subgame of) that game, it suffices to consider the possible profitable deviations of each player not amongst *all* of its strategies (in that subgame), but only amongst the ones differing from the considered profile in the first subsequent move.

Lemma 4 *Every PNG has the One Deviation Property.*

Proof. Let $\mathcal{E} = (\mathrm{N}, \mathcal{A}, \mathcal{H}, turn, \{\Sigma_A, \Sigma_B\}, o, \mathcal{G}, \mathfrak{S}, \mathbf{g}, \mathbf{u})$ be a PNG, and $\mathcal{H}_f \subseteq \mathcal{H}$ the set of finite histories in $\mathcal{H}$. Let moreover $\mathcal{E}_f$ be the restriction of $\mathcal{E}$ to $\mathcal{H}_f$, where the individual components are defined in the expected way. But $\mathcal{E}_f$ is a game of finite horizon, and by [Osborne and Rubinstein [1994], Lemma 98.2] it has the One Deviation Property. But by the fact that no disagreement is better for any player than any agreement, (Definition 2) then $\mathcal{E}$ has that property, too. ■

Furthermore, to analyse equilibrium strategies of PNG we consider so called **stationary acceptance strategies** where players have a minimal acceptance threshold d and a minimal passing threshold $d' \geq d$ (both of which may vary among the players).

5.1 Conditional offers with non-valuable time

The value for a player of a history in a PNG is the value for the player of the current NFG associated with that history. When time is not valuable players assign the same value to the NFG associated with the current moment and the same game associated with any other moment in the future, which means that players can afford a delay.

Proposition 5 *Every SPE strategy profile of stationary acceptance strategies of a two-player PNG with non-valuable time is strongly efficient.*

Proof. Suppose not. Let d^- be a vector of expected values that is not the redistribution of a maximal outcome of the starting game, associated to some SPE strategy profile. Such strategy profile yields a history h that ends with: 1) proposal of d^-; 2) acceptance of that proposal; 3) pass; 4) pass. Consider now some redistribution d^* of a maximal outcome where both players get more than in d^- and the history h with the the last four steps substituted by: 1) proposal of d^*; 2) acceptance of that proposal; 3) pass; 4) pass. By stationarity of strategies and the ODP, the player moving at step 1) is better off deviating from d^- and instead proposing d^*: a contradiction. ■

The condition of stationarity of acceptance strategies is needed if we want to prevent SPE that lead to inefficiency. Indeed, if players were not adhering to stationary acceptance strategies there could be a *suboptimal* outcome, guaranteeing for both players expected values respectively of d_A and d_B. To enforce that outcome it then suffices to design a strategy profile whereby off the equilibrium path player A threatens player B with a *stubborn* but maximal stationary acceptance strategy giving him less than d_B, while player B threatens A with an expected payoff of strictly less than d_A. So, if players are not obliged to be consistent in their acceptance policies, d_A and d_B can be the result of a subgame perfect equilibrium strategy.

The example below provides a detailed instance of such games.

Example 6 (Attaining inefficiency) *In what follows we say that a player 'proposes a given outcome with a given payoff distribution' to mean that the player makes a conditional offer which, when accepted, would make that specific outcome, with that specific distribution of the payoffs, the unique (dominant strategy equilibrium) outcome in the solution of the transformed game. More generally, we say that a player "proposes a payoff distribution" to mean that the player makes a conditional offer which, when accepted, would make that specific payoff distribution the vector of expected utilities of the players.*

Consider the starting NFG on Figure 7. As there are no dominant strategy equilibria, there are acceptable solution concepts assigning 2 *to each player.*

	L	R
U	$2,2$	$4,3$
D	$3,3$	$2,2$

Figure 7: Attaining inefficient divisions

We now will construct a strategy profile of the PNG starting from that game, that is a SPE strategy profile and attains an inefficient outcome:

1. *At the root node player* A *proposes outcome* (D, L) *with payoffs* $(3, 3)$.

2. *After such proposal player* B *accepts. However, if* A *had made a different offer (so, off the equilibrium path)* B *would reject and keep proposing outcome* (U, R) *with distribution of* 5 *for him and* 2 *for* A *and accepting (and passing on) maximal outcomes guaranteeing him at least* 5*.* A*, on the other hand, would not have better option than proposing the same distribution (*5 *for* B *and* 2 *for her) and accepting only maximal outcomes guaranteeing her at least* 2*. Notice that once they enter this subgame neither* A *nor* B *can profitably deviate from such distribution.*

3. *If, however,* B *did not accept the* $(3, 3)$ *deal then* A *would keep proposing outcome* (U, R) *with a redistribution of* $(5, 2)$ *(*5 *for her,* 2 *for him) and accepting at least that much. Respectively,* B *would also stick to the same distribution, accepting at least* 2*. Again, no player can profitably deviate from this stationary strategy profile starting from* B*'s rejection.*

4. *After player* B *has accepted the deal* $(3, 3)$*, then* A *passes. If* A *did not pass, player* B *would go back to his* $(2, 5)$ *redistribution threat.*

   Likewise with the next round. That eventually leads to the inefficient outcome $(3, 3)$.

It is easy to check that the strategy profile described above is a SPE. No player can at any point deviate profitably by proposing the outcome (U, L) *with dominating payoff distribution, e.g.,* $(3.5, 3.5)$.

We first focus on PNG where the opt out option is not available, and introduce it as an additional feature later on.

Negotiations without 'opt out' moves. In PNG with non-valuable time and without the possibility of opting out every redistribution of a maximal outcome can be attained as a solution.

Proposition 7 *Let* $\mathcal{E}$ *be PNG with non-valuable time starting from a NFG* $\mathcal{G}$ *and let* $d = (x_A, x_B)$ *be any redistribution of a maximal outcome of the starting NFG. The following strategy profile* $\sigma = (\sigma_A, \sigma_B)$ *is a SPE:*

For each player $i \in \{A, B\}$*:*

- *if i is the first player to move, he proposes a transformation of $\mathcal{G}$ where the vector of expected values in the transformed game is d;*
- *when i can make an offer and the previously made offer has not been accepted, he proposes a transformation of the current NFG where the vector of expected values in the transformed game is d;*
- *when i can make an offer and the previously made offer has been accepted, he passes;*
- *when i has a pending offer of a suggested transformation where the vector of expected values in the transformed game is d', he accepts it if and only if $x'_i \geq x_i$, and rejects it otherwise;*
- *if i can pass and the other player has just passed, he passes;*
- *if i can pass and the opponent has not just passed, i proposes d;*
- *if i has just accepted a proposal he passes;*

Proof. We have to show that there is no subgame where a player i can profitably deviate from this strategy at its root. By Lemma 4 it suffices to consider only first move deviations to the above described strategy.

Suppose the player has a pending offer that induces a transformation of the current NFG where the vector of expected values is d^*. If she accepts it then the outcome will be d^*, due to the definition of the strategy profile; if she rejects it, it will be the starting offer d. And she will accept if and only if she will get more from d^* than from d. So the acceptance component is optimal. For the remaining cases, if player i deviates from the prescribed strategy, due to the construction of the strategy and Lemma 4, the vector of payoffs associated to the outcome of $\mathcal{E}$ will be d anyway. ■

Corollary 8 *The game associated to the outcome of a subgame perfect equilibrium strategy profile consisting of stationary acceptance strategies in a two-player PNG with non-valuable time is optimally solvable.*

In summary, our analysis of two-player PNG with non-valuable time shows that efficiency can be attained when conditional offers are allowed and stationary acceptance strategies are followed. Indeed, any redistribution of the vector of payoffs of a maximal outcome can be made the unique solution of the final NFG by such SPE strategies. However, non-stationary acceptance strategies may lead to inefficient equilibria, as Example 6 clearly shows: there exist SPE strategy profiles of a two-player PNG with conditional offers and non-valuable time where (i) offers are made that are not feasible, (ii) the vector of payoffs of the outcome it attains is not a redistribution of the vector of payoffs of a maximal outcome, i.e., it is not strongly efficient.

Negotiations with 'opt out' moves. To address the issues related to possible inefficiency we consider the possibility for players to make an *opt out* move and unilaterally put an end to the negotiations, by making the currently accepted NFG the outcome of the whole PNG.

Proposition 9 *Let σ be a subgame perfect equilibrium strategy profile of a PNG with* opt out *move and let h be the resulting history. Then σ guarantees to all players at least as much as they had in the currently accepted NFG; in particular, at least as much as in the original game.*

Proof. Starting with the original, automatically accepted game, each currently accepted NFG must make each player better off than in the previous one; otherwise opting out would be a profitable deviation. ■

By introducing the possibility of opting out, the set of subgame perfect equilibria reduces further. Strategies, such as the one described in Example 6 demanding an unreasonably high reward or an unreasonably low one for the proponent, will not be equilibria anymore. However, this option does not solve the problem of attaining inefficiency, as the comment to Proposition 5 still applies. It has, however, several advantages: first, the equilibrium strategies of the PNG will guarantee for both players at least the expected payoff of the starting NFG; and second, the threat of opting out gives the players the possibility of making a more effective use of unconditional offers.

To sum up, while SPE strategies in a two-player PNG can attain efficiency, some important issues are still remaining:

- some SPE strategies, e.g., non-stationary acceptance strategies, are not strongly efficient.
- players can keep making unfeasible moves as a part of a SPE strategy, i.e., there are forms of equilibria where some players strictly decrease their expected payoff with respect to the original game;
- even strongly efficient strategies do not always yield perfectly solved games, as there is no notion of *most fair* redistribution of the payoff vectors in the solution of the original game.

Thus, when time is of no value, even the possibility of making conditional offers does not guarantee that fair and efficient outcomes are ever reached.

5.2 Conditional offers with valuable time

We will show here that when time is of value the problems mentioned above can be at least partially solved. To impose value on time we introduce for each player i a *payoff discounting factor* $\delta_i \in (0, 1)$ applied at every round of the PNG *associated to offers that are made* to his payoffs. These factors measure the players' impatience, i.e., how much they value time, and reduce the payoffs accordingly as time goes by. Thus, the players have no interest in delaying the negotiations by making redundant moves and sub-optimal offers. The intuition now, which we will justify further, is that for the sake of time efficiency, in a SPE strategy profile:

1. If a player intends to make an offer, she has never made any earlier offer that, if accepted, would give her a lesser value of the resulting game.

2. If any player is ever going to accept a given offer (or any other offer which is at least as good for her) she should do it the first time when she receives such offer.

In analysing PNG with valuable time we consider several cases, depending on whether opting out is allowed.

5.2.1 No opting out

For technical reasons we impose some additional constraints:

- every game associated with a history of a PNG does not have *in its solution* outcomes assigning negative utility to players. NB: we do allow payoff vectors consisting of negative reals to be present in the game matrix, only we do not allow such vectors to be associated to outcomes in the solution. This constraint has several practical consequences:
  - players' expected payoffs *decrease* in time, i.e., the discounting factor δ has always a negative effect on the expected payoff.
  - players can make offers that redistribute the payoff vectors associated with outcomes in the solution, leaving some nonnegative amount to each player and some strictly positive amount to some.
- each player's expected payoff at a disagreement history is assumed 0.

We will use the following notational conventions:

- (x, t) denotes the payoff vector x at time t, where each component x_i is discounted by δ_i^t; $(x, t)_i$ is the payoff of player i in the vector x at t.
- $\mathcal{G}_X$ will denote the set of all possible redistributions of payoffs of outcomes in a NFG $\mathcal{G}$ that assign nonnegative payoffs to all players. This set is compact, but generally not connected, as in the bargaining games of [4]. However, it is a finite union of compact and connected sets, and that will suffice to generalise the results from [4] that we need.

The following properties of every 2-person PNG with valuable time starting from a given NFG $\mathcal{G}$ are the four fundamental assumptions of the bargaining model in [Rubinstein [1982]] and [Osborne and Rubinstein [1994], p.122].

1. For each $x, y \in \mathcal{G}_X$ such that $x \neq y$, if $(x, 0)_i = (y, 0)_i$ then $(x, 0)_{-i} \neq (y, 0)_{-i}$. This holds because the set $\mathcal{G}_X$ is made by payoff vectors and subtracting some payoff to a player means adding it to the other.
2. $(b^i, 1)_{-i} = (b^i, 0)_{-i} = (D)_{-i}$, where b^i is the highest payoff that i obtains in $\mathcal{G}_X$ and $(D)_{-i}$ the payoff for $-i$ in any disagreement history. As b^i is the best agreement for player i it is also the worst one for player $-i$.

3. If x is Pareto optimal amongst the payoff vectors in $\mathcal{G}_X$ then, by definition of $\mathcal{G}_X$, there is no y with $(x,0)_i \geq (y,0)_i$ for each $i \in N$. Moreover, x is a redistribution of a maximal outcome in $\mathcal{G}$.

4. There is a unique pair (x^*, y^*) with $x^*, y^* \in \mathcal{G}_X$ such that $(x^*, 1)_A = (y^*, 0)_A$ and $(y^*, 1)_B = (x^*, 0)_B$ and both x^*, y^* are Pareto optimal amongst the payoff vectors in $\mathcal{G}_X$.

The first 3 statements above are quite straightforward. To see the last one, let $x^* = (x^*_A, x^*_B)$ and $y^* = (y^*_A, y^*_B)$ and let the sum of the payoffs in any maximal outcome in $\mathcal{G}$ be d. Then $(x^*_A, x^*_B, y^*_A, y^*_B)$ is the unique solution of the following, clearly consistent and determined system of equations:
$y_A = \delta_A x_A$, $x_B = \delta_B y_B$, $x_A + x_B = d$, $y_A + y_B = d$.

The solution (see also [Osborne and Rubinstein [1994]]) is:

$$x_A = d\frac{1-\delta_B}{1-\delta_A\delta_B};\ y_A = \delta_A d\frac{1-\delta_B}{1-\delta_A\delta_B}$$

$$x_B = \delta_B d\frac{1-\delta_A}{1-\delta_A\delta_B};\ y_B = d\frac{1-\delta_A}{1-\delta_A\delta_B}.$$

Relation with bargaining games In the rest of the section we will explicitly view pre-play negotiation as a bargaining process on how to play the starting normal form game. Using our observations and assumptions, we can adapt the results from [Osborne and Rubinstein [1994]] to show that when time is valuable not only all equilibria consisting of stationary acceptance strategies attain efficiency but they also do so by redistributing the payoff vector in relation to players' impatience. Stationary acceptance strategies will be needed to focus only on the maximal connected subspace of the set $\mathcal{G}_X$. We extend the efficiency and fairness results obtained in [Osborne and Rubinstein [1994]] for bargaining games of the type of 'division of a cake' to somewhat more general bargaining games of the type where players have to choose a cake from a set of cakes, of possibly different sizes and divide it. Our claim, in a nutshell, is that, when players employ stationary acceptance strategies, they immediately choose the largest cake and then bargain on how to divide it.

First, recall that in our framework time passes as new offers are made. So, from a technical point if the PNG start with a game that is already perfectly solved, the player moving first will not be punished by passing immediately.

Then, without restriction of the generality of our analysis, we can assume a *unique* discounting factor for both players. Indeed, the discount factor of e.g., player A can be made equal to that of B while preserving the relative preferences of A on the set of outcomes by suitably re-scaling the payoffs of A in the input NFG, and therefore the expected value for A of that game; for technical details see [Osborne and Rubinstein [1994], p.119] following an idea of Fishburn and Rubinstein quoted there.

Now we are ready to state the main result for this case:

Theorem 1 *Let (x^*, y^*) be the unique pair of payoff vectors defined above. Then, in a PNG with valuable time starting from a NFG $\mathcal{G}$ with a unique discounting factor δ for both players, the strategy of player A in every subgame perfect equilibrium consisting of* *stationary acceptance strategies satisfies the following (to obtain the strategy for B simply swap x^* and y^*):*

- *if A is the first player to move, then she 'proposes' outcome x^*, i.e., makes a conditional offer that, if accepted, would update the game into one with a dominant strategy equilibrium yielding the Pareto maximal outcome x^* as payoff vector;*
- *when A has a pending offer y', she accepts it if and only if the payoff she gets in y' is at least as much as in y^*;*
- *if A can pass, she passes if and only if the expected value associated to the proposed game y'_A is at least y^*_A; otherwise she proposes x^*.*

Proof. It is easy to check, using the ODP, that no player can improve at any history of the game by deviating from this strategy. Consider for instance the case when player A at time t can choose whether to pass or not on the proposal of a distribution z on which player B has already passed. If A passes then the payoff vector will be (z, t); if not, it will be $(x^*, t+1)$. Obviously $(z, t)_A \geq (x^*, t+1)_A$ if and only if $(z, 0)_A \geq (x^*, 1)_A = (y^*, 0)_A$, so the acceptance rule is optimal. The reasoning for the other cases is similar.

To prove the claim we are going to use a variant of the argument provided by [Osborne and Rubinstein [1994]] for bargaining games, summarised as follows. We first show [Step 1] that the best SPE payoff for player A in any subgame $\mathcal{G}'_A$ starting with her proposal and where $\mathcal{G}'$ is the currently accepted game — let us denote it by $M_A(G'_A)$ — yields the same utility as the worst one — $m_A(\mathcal{G}'_A)$ — which, in turn, is the payoff of A at x^*. The argument for B is symmetric. Then we show [Step 2] that in every SPE the initial proposal is x^*, which is immediately accepted by the other player, followed by each player passing. Finally, we show [Step 3] that the acceptance and the passing conditions given are shared by every SPE strategy profile.

[Step 1] WLOG let A be the player moving first and call $\mathcal{G}'_A$ each subgame of the PNG beginning with a proposal by player A and where $\mathcal{G}'$ is the currently accepted NFG at its root ($\mathcal{G}_A$ is the game itself). Analogously let us call $\mathcal{G}'_B$ each subgame of the PNG beginning with a proposal by player B. For each player i let $M_i(\mathcal{G}'_i)$ be the best SPE outcome that player i can get from $\mathcal{G}'_i$, i.e., $M_i(\mathcal{G}'_i) = sup\{\delta^t x_i \mid \text{there is a SPE of } \mathcal{G}'_i \text{ consisting of stationary acceptance strategies with value } (x,t)_i\}$. Let $m_i(\mathcal{G}'_i)$ be the corresponding infimum. Recall that b^i is the highest payoff that i obtains in $\mathcal{G}_X$. Hereafter we write b^i_j instead of $(b^i, 0)_j$ for $i, j \in N$. By our assumptions the observations above, $b^A_B = b^B_A = 0$.

We can now show that for each $\mathcal{G}'$, $M_A(\mathcal{G}'_A) = m_A(\mathcal{G}'_A) = x^*_A$ and $M_B(\mathcal{G}'_B) = m_B(\mathcal{G}'_B) = y^*_B$. We first show that $m_B(\mathcal{G}'_B) \geq b^B_B - \delta M_A(\mathcal{G}'_A)$. Therefore, if player A rejects a proposal of player B in the first period of $\mathcal{G}'_B$ then she cannot get more than $\delta M_A(\mathcal{G}'_A)$. This means that in any SPE of $\mathcal{G}'_B$ she must accept any proposal giving her more than $\delta M_A(\mathcal{G}'_A)$ (otherwise she could be at least as well off by rejecting it). Thus what is left for player B is no less than $b^B_B - \delta M_A(\mathcal{G}'_A)$ in any SPE of $\mathcal{G}'_B$.

It is easy to see that $M_A(\mathcal{G}'_A) \leq b^A_A - \delta m_B(\mathcal{G}'_B)$, because player A cannot get more than her best agreement minus what player B could guarantee with a rejection. That is, player A needs to pay B with the difference between her ideal (appropriately discounted) payoff and what B could guarantee alone. We can show now that $M_A(\mathcal{G'}_A) = x^*_A$. That $M_A(\mathcal{G'}_A) \geq x^*_A$ is easily observed from the properties satisfied by every SPE and the fact that each $\mathcal{G}'$ is a transformation of $\mathcal{G}$ by conditional offers. To show that $M_A(\mathcal{G'}_A) \leq x^*_A$ we argue the following. We know that $\delta b^A_B = 0$. We also know that $\delta(b^B_B - \delta b^A_A) > 0 = b^A_B = b^B_B - b^A_A$. In turn we have that $b^A_A > b^A_A - (\delta(b^B_B - \delta b^A_A))$. By the previous observations we can conclude that $M_A(\mathcal{G'}_A) \leq b^A_A - (\delta(b^B_B - \delta M_A(\mathcal{G'}_A)))$. But, by a similar argument to that in the proof of Proposition 5, M_A is obtained from a strongly efficient SPE. So, as the set of maximal outcomes in $\mathcal{G}_X$ is compact and connected, it also follows that there exists $U_A \in [M_A(\mathcal{G'}_A), b^A_A)$ such that $U_A = b^A_A - (\delta(b^B_B - \delta U_A))$. Now if $M_A(\mathcal{G'}_A) > x^*_A$ then $U_A \neq x^*_A$. Then, taking any pair of efficient agreements (a^*, b^*) such that $a^*_A = U_A$ and $b^*_A = \delta U_A$ we have obtained a pair of efficient agreements contradicting Property 5.2.1 (4). Similar reasoning shows that $m_A(\mathcal{G'}_A) = x^*_A$, $M_B(\mathcal{G'}_B) = y^*_B$ and finally $m_B(\mathcal{G'}_B) = y^*_B$.

[Step 2] Step 1 implies that if A is the first player to move, she starts by proposing x^* which is immediately accepted. Likewise for player B.

[Step 3] Step 1 and 2 imply that every SPE shares the same acceptance and passing condition. Consider first the acceptance condition. If B rejects an offer in $\mathcal{G'}_A$ we go to $\mathcal{G'}_B$ where, by what was observed before, he gets y^*_B. But $y^*_B = \delta x^*_B$ so every proposal giving him in $\mathcal{G'}_A$ at least x^*_B should be accepted, otherwise rejected. Putting everything together we have that player B must accept any proposal giving him exactly x^*_B. Similar reasoning applies for the passing condition and for player A. ■

One important consequence of Theorem 1 is that every SPE strategy profile, consisting of stationary acceptance strategies, of a two-player PNG with valuable time and with $N = \{A, B\}$ starting from $\mathcal{G}$ and with A (resp. B) first player to move induces a play h of length 4 and of value for player A of x^*_A while for player B of δy^*_B (resp. $(y^*_A, \delta x^*_B)$ if B moves first).

To summarise, when time is valuable and players' value of time (impatience) is measured by a vector of discount factors δ and no opting out allowed, the SPEs following stationary acceptance strategies are essentially unique, efficient and redistribute a maximal payoff vector in a *fair* way, depending on players' impatience, viz. in each SPE play, players agree as soon as possible and divide (almost) evenly any of the maximal outcomes in the game. Thus, introducing value of time solves both problems of efficiency and fairness at once.

6 Related work and comparisons

The present study has a rich pre-history and we do not purport to provide a comprehensive citation of all related previous work and literature here, but will only mention various links with earlier studies and then will discuss in more detail and compare with the most relevant recent work.

6.1 Related topics and relevant early references

Here is a selection of related topics and relevant earlier references:

▷ To begin with, preplay offers technically fall broadly in the scope of *externalities*. There is abundant literature on these, of which we only mention some of the early works: [Meade[1952]], [Maskin[1994]], [Varian [1994]], More specifically, preplay offers can be regarded as a special type of so called in cooperative game theory *side payments*.

▷ In his theorem, [Coase[1960]] describes how efficiency of an allocation of goods or simply an outcome can be obtained in presence of externalities, i.e. when actors' possible decisions affect positively or negatively the payoffs of the other actors involved. The claim, which is usually provided in a rather informal fashion, states that if there are no transaction costs and it is possible to bargain on the effect of the externalites, the process will lead to an efficient outcome regardless of the initial allocation of property rights, i.e. regardless of who is endowed with the capacity of performing the action in question.

▷ [Rosenthal[1975]] proposes one of the earliest models of preplay negotiations, where 'players successively commit themselves irrevocably, according to a specified exogenous ordering, to coalitional strategies conditionally on the rest of the players in the coalition agreeing to play their parts of the coalitional strategy'. He defines a special solution concept, the induced outcome, and provides some sufficient conditions for its existence and uniqueness.

▷ Several two-stage games with preplay communication have been studied in the literature. They seem to go back to [Guttman [1978]] and [Guttman [1987]]. [Kalai[1981]] studies preplay negotiation procedures as sequences of pre-defined length of "preplays", each being a joint strategy of all players. Furthermore, [Matthews and Postlewaite[1989]] consider preplay communication in the context of two-person sealed-bid double auctions. [Danziger and Schnytzer[1991]] consider a 2-stage game for implementing Lindahl's voluntary-exchange mechanism. In a series of papers, incl. [Farrel[1998]], Farrell considers two-stage games, with preplay 'cheap talk' followed by actual play, and discusses the role of preplay communication in ensuring Nash equilibrium profile in the actual play. Also, [Watson [1991]] studies two-stage 2-person normal form games with preplay communication and [d'Aspremont and Gérard-Varet[1980]] study Stackelberg solvable games with preplay communication.

▷ Our preplay negotiation games are closely related to *bargaining* games, see [Rubinstein [1982]], [Osborne and Rubinstein [1990]], [Osborne and Rubinstein [1994]], and [Myerson[1997]].

▷ Another related early work is [Varian [1994]] where he studies variations of 'compensatory mechanisms' where, instead of making offers, players declare compensations for which they are prepared to play one or another strategy (in favour of another player who is willing to pay such compensation and makes a binding offer for it). Although the flavour of such variation is somewhat different, technically it reduces to a type of games with preplay offers that we have considered here.

▷ [Fershtman, *et al.*[1991]], and more recently [Monderer and Tennenholtz [2009]], consider the use of 'agents' or 'mediators' playing on behalf of the players, and show how such mechanisms can be used to achieve more efficient outcomes in non-cooperative games.

▷ The idea of combining competition and cooperation in non-cooperative games has been considered often since the early times of game theory, and has later evolved in theories of *co-opetition* by [Brandeburger and Nalebuffs [1997]] and more recently [Carfì and Schilirò [2011]]. Related in spirit are some theories of coalitional rationality, see [Ambrus[2009]].

6.2 Detailed comparison with most relevant recent work

To our knowledge, Jackson and Wilkie have been the first to explicitly study arbitrary transfer functions from one to another player in a normal form game. That work was preceded by earlier relevant literature mentioned above, such as [Guttman [1978]], [Danziger and Schnytzer[1991]], [Varian [1994]], [Qin[2002]], where only limited forms of payments were considered, such as payments proportional to the actions taken by the other players or only contingent on own actions. Jackson and Wilkie's framework bears substantial similarities with ours, as it studies a two-stage transformations on a normal form game where players announce transfers functions which update the initial normal form game and then play the updated game. Jackson and Wilkie study the subgame perfect equilibria of the two stage game and show under what conditions equilibria of the original game survive in the update game. They focus on the 2-player case, but they also extend their results to the N-player case. However, there are some essential conceptual and technical differences between this framework and our, which we describe and discuss below. In [Jackson and Wilkie [2005]]:

▷ Transfers from a player A to a player B are of the form (in our notation) $A \xrightarrow{\delta/\sigma} B$ where $\sigma \in \prod_{i \in N} \Sigma_i$, $\delta \in \mathbb{R}^+$ and $\delta = 0$ whenever $A = B$, i.e. players are allowed to make positive side payments to other players *that are conditional on the entire strategy profile played*, and not only on the recipient's individual strategy, as in our framework. Technically, every unconditional offer from player A to player B can be simulated by a set of such transfers from A to B. This is not the case for conditional offers, which would instead require a *set* of transfers from B to A as well, or the possibility for δ to be negative, i.e. the introduction of punishments. So, these two types of offers are generally incompatible.

▷ Players announce their transfer functions *simultaneously*. This is a reasonable choice in many situations, e.g., where players only have the possibility of once-off exchange before the actual play, and where the sequential aspects of decisions do not play any essential role, but it is not so in many others where they would rather bargain on their choice of actions, which a central feature of our framework. In this sense, the framework of [Jackson and Wilkie [2005]] and ours have essentially different scopes of applicability.

▷ The authors study strategies that can be *supported*, i.e. that they are subgame perfect equilibria of the two-stage game and Nash-equilibria *of the original game* that also *survive* — i.e. remain equilibria — in the updated game. In particular, they focus on the (interesting) relation between the solo-payoff, i.e. the Nash equilibrium payoff that a player can guarantee by making offers, and the supportability of strategies. Jackson and Wilkie show two important results for the two-player case, the main bulk of their paper: (i) that every Nash equilibrium x of the starting game survives if and only if it yields for every player i a utility that is higher than the one given by i's solo-payoff; and (ii) that a transfer function together with an outcome are supportable if and only if they yield for every player i a utility that is higher than the one given by i's *minimal* solo-payoff, the solo-payoff obtained by making *minimal* offers. It is worth noticing that the definition of minimal offer they adopt is essentially the one we have adopted here: the minimal transfer function needed to change the game solution.

Ellingsen and Paltseva generalize Jackson and Wilkie's work as follows:

$\triangleright$ Transfers from a player A to a player B are again of the form $A \xrightarrow{\delta/\sigma} B$ where $\sigma \in \prod_{i \in N} \Sigma_i$, but now $\delta \in \mathbb{R}$ and $\delta = 0$ whenever $A = B$, i.e. players are allowed to propose both rewards and punishments contingent upon entire strategy profiles. This boils down to players not only making offers but also proposing contracts to the other players to sign or reject.

$\triangleright$ The game played is composed of three stages: (i) the one in which players propose contracts, (ii) the one in which players decide whether to sign a contract, (iii) the one in which players play the game updated by the signed contract.

$\triangleright$ Contracts are proposed on mix strategies, and non-deterministic contracts are considered, i.e. it is possible to make randomise offers.

While in [Jackson and Wilkie [2005]] each player A specifies the (nonnegative) transfer of payoff to the other players for each pure strategy profile σ, in [Elligsen and Paltseva[2011]] each player specifies a (possibly negative) transfer to the other players for each (possibly mixed) strategy profile σ and, at the same time, specifies a signing decision for each contract of the other players. Ellingsen and Paltseva show that their more general contracting game always has efficient equilibria. In particular they show that all the efficient outcomes guaranteeing to each player at least as much as the worst Nash-equilibrium payoff in the original game can be attained in some equilibrium.

[Yamada [2005]] considers variants of the games in [Jackson and Wilkie [2005]] where one player moves before the other and the move of the second ends the preplay phase, showing a clear advantage of the latter player in improving is own payoff. In particular, Yamada shows that:

- the second player can always increase his original payoff, i.e. the payoff he gets in the starting game, in every surviving Nash equilibrium
- every surviving Nash equilibrium that is also maximally Pareto optimal gives the second player at least his original payoff

Clearly Yamada's framework is a step closer to ours than Jackson and Wilkie's. However the games analysed there are a rather restricted sort of Stackelberg games, where the second player behaves like a dictator: not only can he best respond to the first player, but he can unilaterally decide that the game will end with him improving his original payoff.

All in all, the message conveyed by this stream of contributions is that efficiency can be reached if the structure of players' offers is complex enough. On the one hand Jackson and Wilkie show that promises are not enough to attain efficient outcomes, while Ellingsen and Paltseva show that contracting is. Possibly only Yamada's framework acknowledges that the structure of the game might influence the preplay phase. Our results lie on a rather different axis, as we restrict the type of offers to ones that only commit the proposer, not the recipient, and focus on the effects that additional factors in the preplay negotiation game, e.g. value of time and conditional offers, have on attaining outcomes with desirable properties, such as efficiency and fairness. We also discuss how equilibrium strategies themselves display desirable properties, i.e. being efficient negotiation strategies.

7 Further agenda and concluding remarks

The main purpose of the present paper is to initiate a systematic study of preplay negotiations in non-cooperative games, and to outline a broad and long-term research agenda for that study. We have indicated a number of conceptual and technical problems and have only sketched some results, but still much work needs to be done. In particular, we identify two natural and important directions of current and future extensions of our framework:

Coalitional offers. The analysis of N-player normal form games with preplay negotiations phase, for $N > 2$, is much more complicated than the 2-players case. To begin with, the benefit for a player A of player B playing a strategy induced by an offer from A to B crucially depend on the strategies that the remaining players choose to play, so an offer from a player to another player does not have the clear effect that it has in the 2-player case. Thus, a player may have to make a *collective offer* to several (possibly all) other players in order to orchestrate their plays in the best possible for him way. Furthermore, a player may be able to benefit in different ways by making offers for side payments to different players or groups of players, and the accumulated benefit from these different offers may or may not be worth the total price paid for it. Lastly, when all players make their offers pursuing their individual interests only, the total effect may be completely unpredictable, or even detrimental for all players. It is therefore natural that groups of players get to collaborate in coordinating their offers. Thus, a coalitional behaviour naturally emerges here, and the preplay negotiation phase incorporates playing a coalitional game to determine the partition of all players into coalitions that will coordinate their offers in the negotiation phase. However, we emphasise again that the transformed normal form game played after the preplay negotiation phase should remain a non-cooperative game where every player eventually plays for himself.

Inter-play offers in extensive form games. The problem of underperformance is not limited to normal form games, where players cannot observe the outcome of the opponents' actions during the play. It also arises in some extensive form games, such as the Centipede game, where the Backward Induction strategy profile can prescribe to players an utterly inefficient solution. The idea of preplay offers of payments to other players can be applied quite effectively in extensive form games by means of *inter-play offers*, where, before every move of a player, the other player(s) can make him individual or coalitional offers conditional on his forthcoming move. The players from both sides can consider these offers through some commonly accepted solution concept, e.g. Backward Induction, which would provide current values for each player of every subgame arising after the possible moves of A.

In conclusion, the focal problems of the study initiated here are to:

- analyse the game-theoretic effects of preplay/interplay offers for payments between individual players and coalitions in strategic and extensive form games, with complete and incomplete information;

- develop the theory of preplay negotiations and, in particular, the concept of efficient negotiations under various assumptions considered here;
- analyse the optimality and efficiency of the solutions that can be achieved in preplay negotiation games;
- expand the study into a systematic theory of cooperation through negotiations in non-cooperative games.
- apply the developed theory and the obtained results both descriptively and prescriptively to real-life scenarios where our framework applies.

Acknowledgements We thank anonymous reviewers who made constructive comments on earlier versions of this paper.

Paolo Turrini acknowledges the support of the Imperial College Junior Research Fellowship "Designing negotiation spaces for collective decision-making".

References

[Ambrus[2009]] Ambrus, A. [2009] Theories of coalitional rationality. *Journal of Economic Theory*, **144(2)**, 676–695.

[Brandeburger and Nalebuffs [1997]] Brandenburger, A. and Nalebuff, B. *Co-opetition: A Revolutionary Mindset That Combines Competition and Cooperation.* Doubleday, New York, 1997.

[Carfì and Schilirò [2011]] Carfì, D. and Schilirò, D. [2011] A model of coopetitive game and the Greek crisis. *http://arxiv.org/abs/1106.3543*

[Coase[1960]] Coase, R. [1960] The problem of social cost. *J. Law and Economics*, **3)**, 1–44.

[Danziger and Schnytzer[1991]] Danziger, L. and Schnytzer, A. [1991] Implementing the Lindahl voluntary-exchange mechanism. *European Journal of Political Economy*, **7)**, 55–64.

[d'Aspremont and Gérard-Varet[1980]] d'Aspremont, C. and Gérard-Varet, L. [1980] Stackelberg-solvable games and pre-play communication. *Journal of Economic Theory*, **23)**, 201–217.

[Elligsen and Paltseva[2011]] Elligsen, T. and Paltseva, E. [2011] Non-cooperative contracting *http://www.econ.ku.dk/okoep/Non-cooperativeContracting5.pdf*.

[Farrel[1998]] Farrel, J.[1998] Communication, coordination and nash equilibrium. *Economics Letters*, **27)**, 209–214.

[Fershtman, *et al.*[1991]] Fershtman,C., Judd,K., and Kalai,E. [1991] Cooperation through delegation. *International Economic Review*, **32)**,551–559.

[Goranko[2012]] Goranko, V., [2012] Transformations of normal form games by preplay offers for transfer of utility. Manuscript, *http://arxiv.org/abs/1208.1758*

[Goranko and Turrini [2013]] Goranko, V., and Turrini, P. [2013] An offer you cannot refuse: obtaining efficiency and fairness in preplay negotiation games with conditional offers. *In Huaxin Huang, Davide Grossi, and Olivier Roy, editors,* Proc. of the Fourth International Workshop on Logic, Rationality and Interaction (LORI'2013)*, volume 8196 of* LNCS, **8196**, 110–123 .

[Guttman [1978]] Guttman, J. [1978] Understanding collective action: Matching behavior *American Economic Review*, **68(2)**, 251–255.

[Guttman [1987]] Guttman, J. [1987] A non-Cournot model of voluntary collective action *Economica*, **54(213)**, 1–19.

[Hamilton and Slutsky [1990]] Hamilton, J., and Slutsky, S. [1990] Endogenous Timing in Duopoly Games: Stackelberg and Cournot Equilibria *Games and Economic Behaviour*, **2**, 29–46.

[Jackson and Wilkie [2005]] Jakson, M. and Wilkie, S. [2005] Endogenous games and mechanisms: Side payments among players. *Mathematical Social Sciences*, **2**, 543–566.

[Kalai[1981]] Kalai, E. [1981] Preplay negotiations and the prisoner's dilemma. *Review of Economic Studies*, **72(2)**, 375–379.

[Maskin[1994]] Maskin, E. [1994] The invisible hand and externalities. *American Economic Review*, **84(2)**, 333–337.

[Matthews and Postlewaite[1989]] Matthews, S., and Postlewaite, A., [1989] Pre-play communication in two- person sealed-bid double auctions. *Journal Of Economic Theory*, **48**, 238–263.

[Meade[1952]] Kalai, E. [1952] External economies and diseconomies in a competitive situation. *Economic Journal*, **62(245)**, 54–67.

[Monderer and Tennenholtz [2009]] Monderer, D., and Tennenholtz, M., [2009] Strong mediated equilibrium. *Artificial Intelligence*, **173(1)**, 180 – 195.

[Myerson[1997]] Myerson, R., [1997] Strong mediated equilibrium. *Game Theory: Analysis of Conflict.*, Harvard UP.

[Osborne and Rubinstein [1990]] Osborne, M., and Rubinstein, A., [1990] *Bargaining and Markets.*, Acad. Press.

[Osborne and Rubinstein [1994]] Osborne, M., and Rubinstein, A., [1994] *A Course in Game Theory.*, MIT Press.

[Qin[2002]] Qin, C.-Z., [2002] Penalties and rewards as inducements to cooperate. *Departmental Working Papers, Dept. of Economics, UC Santa Barbara,* .

[Rosenthal[1975]] Rosenthal, R.,[1975] Induced outcomes in cooperative normal-form games. *Review of Economic Studies* . Discussion Paper No. 178, Center for. Math. Studies, Northwestern University.

[Rubinstein [1982]] Rubinstein, A., [1982] Perfect equilibrium in a bargaining model. *Econometrica*, **50**, 97–110.

[Schelling [1960]] Schelling, T., [1960] *The strategy of conflict.* Harvard University Press.

[Segal [1999]] Segal, I., [1999] Contracting with externalities. *Quarterly Journal of Economics*, **114(2)**, 337–388.

[Varian [1994]] Varian, H., [1994] A solution to the problem of externalities when agents are well-informed. *American Economic Review*, **84(5)**, 1278–93.

[Watson [1991]] Watson, J., [1991] Communication and superior cooperation in two-player normal form games. *Economics Letters*, **35**, 267–271.

[Yamada [2005]] Yamada, A., [2005] Alternate contracts for side payments. *Economic Bulletin*, **3(46)**, 1–5..