Tableau Systems for Logics of Subinterval Structures over Dense Orderings

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Abstract. We construct a sound, complete, and terminating tableau system for the interval temporal logic $D_{\sqsubset}$ interpreted in interval structures over dense linear orderings endowed with *strict* subinterval relation (where both endpoints of the sub-interval are strictly inside the interval). In order to prove the soundness and completeness of our tableau construction, we introduce a kind of finite pseudo-models for our logic, called $D_{\sqsubset}$-structures, and show that every formula satisfiable in $D_{\sqsubset}$ is satisfiable in such pseudo-models, thereby proving small-model property and decidability in PSPACE of $D_{\sqsubset}$, a result established earlier by Shapirovsky and Shehtman by means of filtration. We also show how to extend our results to the interval logic $D_{\subset}$ interpreted over dense interval structures with *proper* (irreflexive) subinterval relation, which differs substantially from $D_{\sqsubset}$ and is generally more difficult to analyze. Up to our knowledge, no complete deductive systems and decidability results for $D_{\subset}$ have been proposed in the literature so far.

1 Introduction

Interval-based temporal logics provide a natural framework for temporal representation and reasoning. However, while many tableau systems have been developed for point-based temporal logics, few tableau systems have been constructed for interval temporal logics [3,6,9], as these are generally more complex. Even fewer tableau systems for interval logics provide decision procedures – a reflection of the general phenomenon of undecidability of interval-based temporal logics. Notable recent exceptions are [4,5,6,7,8].

In this paper we consider *interval temporal logics interpreted in interval structures over dense linear orderings endowed with subinterval relations.* These structures arise quite naturally and appear deceivingly simple, while actually they are not. Perhaps for that reason they have been studied very little yet, and we are aware of very few publications containing any representation results, complete

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deductive systems, or decidability results for subinterval structures and logics. The only known simple case is the logic $\mathrm{D}_{\sqsubseteq}$, where the reflexive subinterval relation is considered, which has been proved to be equivalent to the modal logic S4 of reflexive and transitive frames when interpreted over dense orderings in [1]. Neither of the two (irreflexive) cases we take into consideration in this work reduces to K4. Besides the purely mathematical attraction arising from the combination of conceptual simplicity with technical challenge, the study of subinterval structures and logics turns out to be important because they provide, together with the neighborhood interval logics, the currently most intriguing and under-explored fragments of Halpern-Shoham's interval logic HS [10]. They occupy a region on the very borderline between decidability and undecidability of propositional interval logics, and since decidability results in that area are preciously scarce, complete and terminating tableau systems like those constructed in the paper are of particular interest. (It should be noted that the decidability results obtained here do not follow from the decidability of the MSO over the rational order, because the semantics of the considered interval logics is essentially dyadic second-order).

Here we focus our attention on the logic $\mathrm{D}_{\sqsubset\!\!\!\cdot\,}$, corresponding to the case of strict subinterval relation (where both endpoints of the subinterval are strictly inside the interval) over the class of dense linear orderings. These subinterval structures turn out to be intimately related (essentially, interdefinable) with Minkowski space-time structures. The relations between the logic $\mathrm{D}_{\sqsubset\!\!\!\cdot\,}$ and the logic of Minkowski space-time were studied by Shapirovsky and Shehtman in [12]. They established a sound and complete axiomatic system for $\mathrm{D}_{\sqsubset\!\!\!\cdot\,}$ and proved its decidability and PSPACE-completeness by means of a non-trivial filtration technique [11,12]. In this paper, we construct a sound, complete, and terminating tableau system for $\mathrm{D}_{\sqsubset\!\!\!\cdot\,}$. In order to prove the soundness and completeness of our tableau construction, we introduce a kind of finite pseudo-models for $\mathrm{D}_{\sqsubset\!\!\!\cdot\,}$, called $\mathrm{D}_{\sqsubset\!\!\!\cdot\,}$-structures, and show that every formula satisfiable in $\mathrm{D}_{\sqsubset\!\!\!\cdot\,}$ is satisfiable in such pseudo-models, thereby proving small-model property and decidability in PSPACE of $\mathrm{D}_{\sqsubset\!\!\!\cdot\,}$. Moreover, we extend our results to the case of the interval logic $\mathrm{D}_{\sqsubset}$ interpreted in interval structures over dense linear orderings with *proper* (irreflexive) subinterval relation, which differs substantially from $\mathrm{D}_{\sqsubset\!\!\!\cdot\,}$ and is generally more difficult to analyze. Up to our knowledge, no decidability or completeness results for deductive systems for that logic have been proposed yet.

2 Syntax and Semantics of $\mathrm{D}_{\sqsubset\!\!\!\cdot\,}$

Let $\mathbb{D} = \langle D, < \rangle$ be a dense linear order. An *interval* over $\mathbb{D}$ is an ordered pair $[b, e]$, where $b < e$. We denote the set of all intervals over $\mathbb{D}$ by $\mathbb{I}(\mathbb{D})^-$ (we use the superscript $^-$ to indicate that point-intervals $[b, b]$ are excluded).

We consider three *subinterval relations*: the *reflexive subinterval relation* (denoted by $\sqsubseteq$), defined by $[d_k, d_l] \sqsubseteq [d_i, d_j]$ iff $d_i \leq d_k$ and $d_l \leq d_j$, the *proper (or irreflexive) subinterval relation* (denoted by $\sqsubset$), defined by $[d_k, d_l] \sqsubset [d_i, d_j]$ iff $[d_k, d_l] \sqsubseteq [d_i, d_j]$ and $[d_k, d_l] \neq [d_i, d_j]$, and the *strict subinterval relation* (denoted by $\sqsubseteq$), defined by $[d_k, d_l] \sqsubseteq [d_i, d_j]$ iff $d_i < d_k$ and $d_l < d_j$. In this paper we will only deal with the latter two cases, beginning with $\sqsubseteq$.

The language of the modal logic $\mathrm{D}_{\sqsubseteq}$ of interval structures with strict subinterval relation consists of a set $\mathcal{AP}$ of propositional letters, the propositional connectives $\neg$ and $\vee$, and the modal operator $\langle D \rangle$. The other propositional connectives, as well as the logical constants $\top$ (*true*) and $\bot$ (*false*) and the dual modal operator $[D]$, are defined as usual. The formulae of $\mathrm{D}_{\sqsubseteq}$ are defined as usual: $\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle D \rangle \varphi$.

The semantics of $\mathrm{D}_{\sqsubseteq}$ is given in *interval models* $\mathbf{M}^- = \langle \mathbb{I}(\mathbb{D})^-, \sqsubseteq, \mathcal{V} \rangle$. The *valuation function* $\mathcal{V} : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})^-}$ assigns to every propositional variable p the set of intervals $\mathcal{V}(p)$ over which p holds[1]. The semantics of $\mathrm{D}_{\sqsubseteq}$ is recursively defined by the satisfiability relation $\Vdash$ as follows:

- for every propositional variable $p \in \mathcal{AP}$, $\mathbf{M}^-, [d_i, d_j] \Vdash p$ iff $[d_i, d_j] \in \mathcal{V}(p)$;
- $\mathbf{M}^-, [d_i, d_j] \Vdash \neg\psi$ iff $\mathbf{M}^-, [d_i, d_j] \not\Vdash \psi$;
- $\mathbf{M}^-, [d_i, d_j] \Vdash \psi_1 \vee \psi_2$ iff $\mathbf{M}^-, [d_i, d_j] \Vdash \psi_1$ or $\mathbf{M}^-, [d_i, d_j] \Vdash \psi_2$;
- $\mathbf{M}^-, [d_i, d_j] \Vdash \langle D \rangle \psi$ iff $\exists [d_k, d_l] \in \mathbb{I}(D)^-$ such that $[d_k, d_l] \sqsubseteq [d_i, d_j]$ and $\mathbf{M}^-, [d_k, d_l] \Vdash \psi$.

A $\mathrm{D}_{\sqsubseteq}$-formula is $\mathrm{D}_{\sqsubseteq}$-*satisfiable* if it is true in some interval in some interval model and it is $\mathrm{D}_{\sqsubseteq}$-*valid* if it is true in every interval in every interval model. The logic $\mathrm{D}_{\sqsubset}$ has the same language as $\mathrm{D}_{\sqsubseteq}$, but it is interpreted in *irreflexive interval models* $\mathbf{M}^- = \langle \mathbb{I}(\mathbb{D})^-, \sqsubset, \mathcal{V} \rangle$.

In [12] a logic $\mathbf{L}_1$ was considered and completely axiomatized as follows:

$$\mathbf{L}_1 = \mathbf{K}_4 + \Diamond\top + \Diamond p_1 \wedge \Diamond p_2 \rightarrow \Diamond(\Diamond p_1 \wedge \Diamond p_2).$$

That logic was shown in [12] to be essentially the logic of the strict subinterval structure over the rational ordering $(\mathbb{Q}, <)$. In the next section we show that the latter coincides with the logic $\mathrm{D}_{\sqsubseteq}$, and therefore $\mathrm{D}_{\sqsubseteq}$ and $\mathbf{L}_1$ turn out to be the same.

3 Structures for $\mathrm{D}_{\sqsubseteq}$

To devise a decision procedure for $\mathrm{D}_{\sqsubseteq}$, we first interpret it over a special class of graphs, that we call $\mathrm{D}_{\sqsubseteq}$-graphs. We will prove that a $\mathrm{D}_{\sqsubseteq}$-formula is satisfiable in a dense interval structure if and only if it is satisfiable in a model over a $\mathrm{D}_{\sqsubseteq}$-graph. Furthermore, it will turn out that this is equivalent to satisfiability in an interval structure over the interval $[0, 1]$ of the rational line.

We begin by introducing the key notion of φ-*atom* and the relation D_φ connecting φ-atoms. Given a $\mathrm{D}_{\sqsubseteq}$-formula φ, let $\mathrm{CL}(\varphi)$ be the *closure of* φ, defined as the set of all sub-formulae of φ and their negations. A φ-atom is defined as follows.

[1] We emphasize that formulae are evaluated only *relative to intervals* and not to points. Thus, intervals are regarded as primitive entities and the formulae only express properties of intervals and their subintervals. In particular, no assumptions are made relating truth of an atomic formula at an interval to its truth at subintervals of that interval.

Definition 1. *Given a* $\mathrm{D}_{\sqsubset}$*-formula* φ*, a* φ-atom A *is a subset of* $\mathrm{CL}(\varphi)$ *such that (i) for every* $\psi \in \mathrm{CL}(\varphi)$*,* $\psi \in A$ *if and only if* $\neg\psi \notin A$ *and (ii) for every* $\psi_1 \vee \psi_2 \in \mathrm{CL}(\varphi)$*,* $\psi_1 \vee \psi_2 \in A$ *if and only if* $\psi_1 \in A$ *or* $\psi_2 \in A$*.*

We denote the set of all φ-atoms by $\mathcal{A}_\varphi$.

Definition 2. *Let* D_φ *be a binary relation over* $\mathcal{A}_\varphi$ *such that, for every pair of* φ*-atoms* $A, A' \in \mathcal{A}_\varphi$*,* $A\, D_\varphi\, A'$ *holds if and only if* $\psi \in A'$ *and* $[D]\psi \in A'$ *for every formula* $[D]\psi \in A$*.*

A φ*-atom* A *is* reflexive *if* $A\, D_\varphi\, A$ *holds, otherwise it is* irreflexive*.*

When φ is clear from the context, φ-atoms will be called simply 'atoms'.

Given a directed graph $\mathbb{G} = \langle V, E\rangle$, a vertex $v \in V$ is *reflexive* if the edge (v, v) belongs to E, otherwise v is an *irreflexive vertex.*

Definition 3. *A finite directed graph* $\mathbb{G} = \langle V, E\rangle$ *is a* $\mathrm{D}_{\sqsubset}$-graph *if (and only if) the following conditions hold:*

1. *there exists an irreflexive vertex* $v_0 \in V$*, called the* root *of* $\mathbb{G}$*, such that any other vertex* $v \in V$ *is reachable from it;*
2. *every irreflexive vertex* $v \in V$ *has a unique successor* v_D*, which is reflexive;*
3. *every successor of a reflexive vertex* v*, different from* v*, is irreflexive.*

A $\mathrm{D}_{\sqsubset}$-graph is depicted in Figure 1. $\mathrm{D}_{\sqsubset}$-graphs are finite by definition, but they may include loops involving irreflexive vertices.

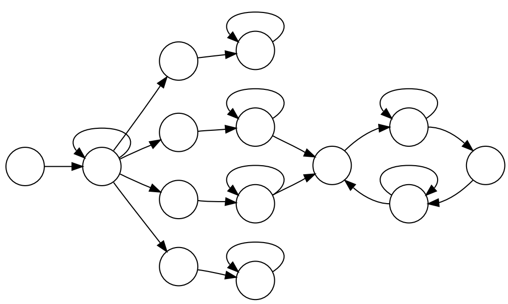


Fig. 1. An example of $\mathrm{D}_{\sqsubset}$-graph

A $\mathrm{D}_{\sqsubset}$-structure is a $\mathrm{D}_{\sqsubset}$-graph paired with a labeling function that assigns an atom to every vertex in the graph. It is formally defined as follows.

Definition 4. *A* $\mathrm{D}_{\sqsubset}$-structure *is a pair* $\mathbf{S} = \langle\langle V, E\rangle, \mathcal{L}\rangle$ *where* $\langle V, E\rangle$ *is a* $\mathrm{D}_{\sqsubset}$*-graph and* $\mathcal{L} : V \rightarrow \mathcal{A}_\varphi$ *is a* labeling function *that assigns to every vertex* $v \in V$ *an atom* $\mathcal{L}(v)$ *such that* $\mathcal{L}(v)\, D_\varphi\, \mathcal{L}(v')$ *for every edge* $(v, v') \in E$*. Let* v_0 *be the root of* $\langle V, E\rangle$*. If* $\varphi \in \mathcal{L}(v_0)$*, we say that* $\mathbf{S}$ *is a* $\mathrm{D}_{\sqsubset}$-structure for φ*.*

$\mathrm{D}_{\sqsubset}$-structures can be viewed as 'pseudo-models' for $\mathrm{D}_{\sqsubset}$. Formulae devoid of temporal operators are satisfied by the definition of φ-atom; moreover, $[D]$-formulae are satisfied by the definition of D_φ. To guarantee the satisfiability of $\langle D\rangle$-formulae, we introduce the notion of *fulfilling* $\mathrm{D}_{\sqsubset}$*-structures.*

Definition 5. *A* $\mathrm{D}_{\sqsubset}$*-structure* $\mathbf{S} = \langle\langle V, E\rangle, \mathcal{L}\rangle$ *is* fulfilling *if and only if for every vertex* $v \in V$ *and every formula* $\langle D\rangle\psi \in \mathcal{L}(v)$*, there exists a descendant (i.e., vertex reachable by a path of successors)* v' *of* v *such that* $\psi \in \mathcal{L}(v')$*.*

Theorem 1. *Let* φ *be a* $\mathrm{D}_{\sqsubset}$*-formula which is satisfied in a dense interval model. Then, there exists a fulfilling* $\mathrm{D}_{\sqsubset}$*-structure* $\mathbf{S} = \langle\langle V, E\rangle, \mathcal{L}\rangle$ *such that* $\varphi \in \mathcal{L}(v_0)$*, where* v_0 *is the root of* $\langle V, E\rangle$*.*

Proof. Let $\mathbf{M} = \langle \mathbb{I}(\mathbb{D})^-, \sqsubset, \mathcal{V}\rangle$ be a dense interval model and let $[b_0, e_0] \in \mathbb{I}(\mathbb{D})^-$ be an interval such that $\mathbf{M}, [b_0, e_0] \Vdash \varphi$. We recursively build a fulfilling $\mathrm{D}_{\sqsubset}$-structure $\mathbf{S} = \langle\langle V, E\rangle, \mathcal{L}\rangle$ for φ as follows.

We start with the one-node graph $\langle\{v_0\}, \emptyset\rangle$ and the labeling function $\mathcal{L}$ such that $\mathcal{L}(v_0) = \{\psi \in \mathrm{CL}(\varphi) : \mathbf{M}, [b_0, e_0] \Vdash \psi\}$.

Next, for every formula $\langle D\rangle\psi \in \mathcal{L}(v_0)$ we pick up an interval $[b_\psi, e_\psi]$ such that $[b_\psi, e_\psi]\sqsubset[b_0, e_0]$ and $\mathbf{M}, [b_\psi, e_\psi] \Vdash \psi$; it exists by definition of $\mathcal{L}(v_0)$.

Then, since $\mathbb{D}$ is a dense ordering and $\mathrm{CL}(\varphi)$ is a finite set of formulae, we can find two intervals $[b_1, e_1]$ and $[b_2, e_2]$ such that:

- $[b_2, e_2]\sqsubset[b_1, e_1]\sqsubset[b_0, e_0]$;
- for every $\langle D\rangle\psi \in \mathcal{L}(v_0)$, $[b_\psi, e_\psi]\sqsubset[b_2, e_2]$;
- $[b_1, e_1]$ and $[b_2, e_2]$ satisfy the same formulae of $\mathrm{CL}(\varphi)$.

Since $\mathbf{M}$ is a model and $[b_\psi, e_\psi]\sqsubset[b_2, e_2]$ for every interval $[b_\psi, e_\psi]$, $[b_2, e_2]$ satisfies $\langle D\rangle\psi$ for every $\langle D\rangle\psi \in \mathcal{L}(v_0)$. Moreover, since $[b_1, e_1]$ and $[b_2, e_2]$ satisfy the same formulae of $\mathrm{CL}(\varphi)$ and $[b_2, e_2]\sqsubset[b_1, e_1]$, for every $[D]\psi \in \mathrm{CL}(\varphi)$, if $[b_2, e_2]$ satisfies $[D]\psi$, then it satisfies ψ as well.

Accordingly, we add a new (reflexive) vertex v_D and the edges (v_0, v_D) and (v_D, v_D) to the graph and we label v_D by $\mathcal{L}(v_D) = \{\xi \in \mathrm{CL}(\varphi) : \mathbf{M}, [b_2, e_2] \Vdash \xi\}$. Furthermore, for every interval $[b_\psi, e_\psi]$, we add a new (irreflexive) vertex v_ψ, together with the edge $(v_D, v_\psi) \in E$, and we label it by $\mathcal{L}(v_\psi) = \{\xi \in \mathrm{CL}(\varphi) : \mathbf{M}, [b_\psi, e_\psi] \Vdash \xi\}$. Finally, to obtain a $\mathrm{D}_{\sqsubset}$-structure for φ, we recursively apply the above construction to the vertices $v_{\psi_1}, \ldots, v_{\psi_k}$.

To keep the construction finite, whenever the above procedure requests us to introduce a successor v' of a reflexive (resp., irreflexive) node $v \in V$, but there exists an irreflexive (resp., reflexive) node $w \in V$ such that $\mathcal{L}(w) = \mathcal{L}(v')$, we replace the addition of the node v' with the addition of an edge from v to w. Since the set of atoms is finite, this guarantees the termination of the construction process. □

Now, let $\mathbf{S}$ be a fulfilling $\mathrm{D}_{\sqsubset}$-structure for a formula φ. We will prove that φ is satisfiable in a dense interval structure. Moreover, we will show that such a structure can be constructed on the interval $[0, 1]$ of the rational line. To begin with, we define a function $f_{\mathbf{S}}$ connecting intervals in $\mathbb{I}([0, 1])^-$ with vertices in $\mathbf{S}$. Such a function will allow us to define a model for φ.

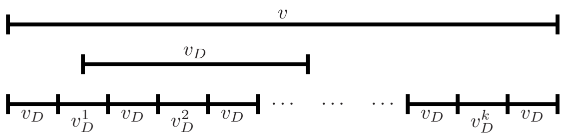
Definition 6. *Let* $\mathbf{S} = \langle\langle V, E\rangle, \mathcal{L}\rangle$ *be a* $\mathrm{D}_{\sqsubset}$*-structure. The function* $f_{\mathbf{S}} : \mathbb{I}([0, 1])^- \mapsto V$ *is recursively defined as follows:*

– $f_{\mathbf{S}}([0,1]) = v_0$;

– *let* $[b,e]$ *be an interval such that* $f_{\mathbf{S}}([b,e]) = v$ *and* $f_{\mathbf{S}}$ *has not yet been defined for any subinterval of* $[b,e]$. *Let* v_D *be the unique reflexive successor of* v *if* v *is irreflexive, and let* $v_D = v$ *otherwise. There are two alternatives:*

1. v_D has no successors other than itself. *In such a case, we put* $f_{\mathbf{S}}([b',e']) = v_D$ *for every proper subinterval* $[b',e']$ *of* $[b,e]$.
2. v_D has at least one successor different from itself. *Let* $v_D^1, \ldots, v_D^k$ *be the successors of* v_D *different from* v_D. *We consider the intervals defined by the points* $b, b+p, b+2p \ldots, b+2kp, b+(2k+1)p = e$, *with* $p = \frac{e-b}{2k+1}$. *The function* $f_{\mathbf{S}}$ *over such intervals is defined as follows:*
   - *for every* $i = 1, \ldots, k$, *we put* $f_{\mathbf{S}}([b+(2i-1)p, b+2ip]) = v_D^i$.
   - *for every* $i = 0, \ldots, k$, *we put* $f_{\mathbf{S}}([b+2ip, b+(2i+1)p]) = v_D$.

   We complete the construction by putting $f_{\mathbf{S}}([b',e']) = v_D$ *for every subinterval* $[b',e']$ *of* $[b,e]$ *which is not a subinterval of any of the intervals* $[b+ip, b+(i+1)p]$. *The resulting structure is depicted below.*



It is easy to show that $f_{\mathbf{S}}$ satisfies the following properties.

Lemma 1

1. *For every pair of intervals* $[b,e], [b',e'] \in \mathbb{I}([0,1])^-$ *such that* $[b',e'] \sqsubset\!\cdot\, [b,e]$, $f_{\mathbf{S}}([b',e'])$ *is reachable from* $f_{\mathbf{S}}([b,e])$.
2. *For every interval* $[b,e] \in \mathbb{I}([0,1])^-$, *if* $f_{\mathbf{S}}([b,e]) = v$ *and* v' *is reachable from* v, *then there exists* $[b',e'] \sqsubset\!\cdot\, [b,e]$ *such that* $f_{\mathbf{S}}([b',e']) = v'$.

Remark 1. The lemma above means that the mapping $f_{\mathbf{S}}$ defined above is a bounded morphism from $\mathbb{I}([0,1])^-$ to (in fact, onto) V, and the theorem below follows from a general result in modal logic stating truth-preservation under bounded morphisms, see e.g. [2]. However, since we do not regard $\mathrm{D}_{\sqsubset\!\cdot}$-structures a standard models for our logic, we will not make use of this reference, but will give a direct proof instead.

For any fulfilling $\mathrm{D}_{\sqsubset\!\cdot}$-structure $\mathbf{S}$ for φ, let $\mathbf{M_S}$ be the triplet $\langle \mathbb{I}([0,1])^-, \sqsubset\!\cdot, \mathcal{V} \rangle$, where $\mathcal{V}(p) = \{[b,e] : p \in \mathcal{L}(f_{\mathbf{S}}([b,e]))\}$ for every $p \in \mathcal{AP}$. It turns out that $\mathbf{M_S}$ is a model for φ.

Theorem 2. *Let* $\mathbf{S}$ *be a fulfilling* $\mathrm{D}_{\sqsubset\!\cdot}$*-structure for* φ. *Then* $\mathbf{M_S}, [0,1] \Vdash \varphi$.

Proof. We prove that for every interval $[b,e] \in \mathbb{I}([0,1])^-$ and every formula $\psi \in \mathrm{CL}(\varphi)$, $\mathbf{M_S}, [b,e] \Vdash \psi$ if and only if $\psi \in \mathcal{L}(f_{\mathbf{S}}([b,e])$. The proof is by induction on the structure of the formula:

– the case of propositional letters as well as those of Boolean connectives are straightforward and thus omitted;

– let $\psi = \langle D \rangle \xi$, and suppose that $\psi \in \mathcal{L}(f_{\mathbf{S}}([b,e]))$. Since $\mathbf{S}$ is fulfilling, there exists a vertex v', which is reachable from $f_{\mathbf{S}}([b,e])$, such that $\xi \in \mathcal{L}(v')$.

By Lemma 1, there exists $[b', e'] \sqsubset [b, e]$ such that $f_{\mathbf{S}}([b', e']) = v'$. By the inductive hypothesis, $\mathbf{M_S}, [b', e'] \Vdash \xi$ and thus $\mathbf{M_S}, [b, e] \Vdash \langle D \rangle \xi$.

To prove the converse implication, suppose for reductio ad absurdum that $\mathbf{M_S}, [b, e] \Vdash \langle D \rangle \xi$ but $\langle D \rangle \xi \notin \mathcal{L}(f_{\mathbf{S}}([b, e]))$. By the definition of φ-atom, this implies that $[D]\neg\xi \in \mathcal{L}(f_{\mathbf{S}}([b, e]))$. Thus, by Lemma 1, we have that, for every $[b', e'] \sqsubset [b, e]$, $\neg\psi \in \mathcal{L}(f_{\mathbf{S}}([b', e'])$. By the inductive hypothesis, this implies that $\mathbf{M_S}, [b', e'] \Vdash \xi$ for every $[b', e'] \sqsubset [b, e]$, which is a contradiction.

Let v_0 be the root of $\mathbf{S}$. Since $\varphi \in \mathcal{L}(v_0)$ and $f_{\mathbf{S}}([0, 1]) = v_0$, it immediately follows that $\mathbf{M_S}, [0, 1] \Vdash \varphi$. □

4 A Small-Model Theorem for $\mathrm{D}_{\sqsubset}$-Structures

In this section we prove a small-model theorem for $\mathrm{D}_{\sqsubset}$, that is, we will show that a $\mathrm{D}_{\sqsubset}$-formula is satisfiable if and only if there exists a fulfilling $\mathrm{D}_{\sqsubset}$-structure of *bounded size*. To this end, we introduce the following two measurements for $\mathrm{D}_{\sqsubset}$-graph: (i) the *breadth* of a $\mathrm{D}_{\sqsubset}$-graph, which is the maximum number of outgoing edges of a vertex, and (ii) the *depth* of a $\mathrm{D}_{\sqsubset}$-graph, which the maximum length of a *simple path* of vertices (i.e., a path with no repetition of vertices).

We will show that a $\mathrm{D}_{\sqsubset}$-formula is satisfiable if and only if there exists a fulfilling $\mathrm{D}_{\sqsubset}$-structure whose breadth and the depth are linear in the size of the formula. In order to make the proofs easier to understand we provide the following definition.

Definition 7. *Given a* $\mathrm{D}_{\sqsubset}$*-formula* φ *and a* φ*-atom* $A \in \mathcal{A}_\varphi$*, the* set of temporal requests of A *is* $\mathrm{REQ}(A) = \{\langle D \rangle \psi \in \mathrm{CL}(\varphi) : \langle D \rangle \psi \in A\}$.

Notice that $\mathrm{REQ}(A)$ identifies all temporal formulae in A: if $\langle D \rangle \psi \notin \mathrm{REQ}(A)$, then $[D]\neg\psi \in A$ (by definition of φ-atom). We denote by REQ_φ the set of all $\langle D \rangle$-formulae in $\mathrm{CL}(\varphi)$.

Theorem 3. *For every satisfiable* $\mathrm{D}_{\sqsubset}$*-formula* φ*, there exists a fulfilling* $\mathrm{D}_{\sqsubset}$*-structure whose breadth and depth are bounded by* $2 \cdot |\varphi|$.

Proof. Let $\mathbf{S} = \langle \langle V, E \rangle, \mathcal{L} \rangle$ be a fulfilling $\mathrm{D}_{\sqsubset}$-structure for φ. The following algorithm builds a fulfilling $\mathrm{D}_{\sqsubset}$-structure $\mathbf{S}' = \langle \langle V', E' \rangle, \mathcal{L}' \rangle$ for φ with the required property.

Initialization. Initialize $\mathbf{S}'$ as the one-vertex $\mathrm{D}_{\sqsubset}$-structure $\langle \langle \{v_0\}, \emptyset \rangle, \mathcal{L}' \rangle$, where v_0 is the root of $\mathbf{S}$ and $\mathcal{L}'(v_0) = \mathcal{L}(v_0)$. Call the procedure *Expansion*(v_0).

Expansion(v). If v is irreflexive, execute *Step 1*; otherwise, execute *Step 2*.

Step 1. Let v' be unique reflexive successor of v in $\mathbf{S}$. Add v' to V' and $(v, v'), (v', v')$ to E'; moreover, put $\mathcal{L}'(v') = \mathcal{L}(v')$. Call *Expansion*$(v')$.

Step 2. Let $\mathrm{REQ}(\mathcal{L}'(v)) = \{\langle D \rangle \psi_1, \ldots, \langle D \rangle \psi_k\}$. Since $\mathbf{S}$ is fulfilling, for every formula $\langle D \rangle \psi_i \in \mathrm{REQ}(\mathcal{L}'(v))$, there exists a descendant v_i of v in $\mathbf{S}$ such that $\psi_i \in \mathcal{L}(v_i)$. For $i = 1 \ldots, k$, add v_i to V' and (v, v_i) to E'; moreover, put $\mathcal{L}'(v_i) = \mathcal{L}(v_i)$. Next, for every v_i such that $\mathrm{REQ}(\mathcal{L}'(v_i)) =$

REQ($\mathcal{L}'(v)$), add an edge (v_i, v) to E'. For the remaining vertices v_i, it holds that REQ($\mathcal{L}'(v_i)$) $\subset$ REQ($\mathcal{L}'(v)$), because every $[D]$-formula in $\mathcal{L}'(v)$ also belongs to $\mathcal{L}'(v_i)$ and there exists at least one ψ_j such that $\langle D\rangle\psi_j \in$ REQ($\mathcal{L}'(v)$) and $[D]\neg\psi_j \in$ REQ($\mathcal{L}'(v_i)$). For every vertex v_i in this latter set, call *Expansion*(v_i).

This algorithm terminates because it calls the *Expansion* procedure on every vertex in V at most once. Moreover, it follows immediately from the construction that it produces a fulfilling $D_{\sqsubset}$-structure $\mathbf{S}'$ for φ. To prove that both the breadth and the depth of $\mathbf{S}'$ are less than or equal to $2 \cdot |\varphi|$, it suffices to observe that:

– every irreflexive vertex has exactly one outgoing edge;
– the number of outgoing edges of reflexive vertices is bounded by the number of $\langle D\rangle$-formulae in CL(φ), not exceeding the size of the formula;
– in *Step 2*, the procedure *Expansion* is called only on those vertices v_i such that REQ($\mathcal{L}'(v_i)$) $\subset$ REQ($\mathcal{L}'(v)$). It follows that at every step the number of $\langle D\rangle$-formulae strictly decreases. As a consequence, we have that every simple path in $\mathbf{S}'$ contains at most $|\varphi|$ different irreflexive vertices. Since in every simple path reflexive and irreflexive vertices alternate, the depth of the $D_{\sqsubset}$-structure is bounded by $2 \cdot |\varphi|$. □

Given a formula φ, let n be the number of $\langle D\rangle$-formulae in CL(φ). It follows from Theorem 3 that there is a fulfilling $D_{\sqsubset}$-structure for φ whose simple vertex paths $v_0, ..., v_k$, starting from the root, have length at most $2n$. Taking advantage of Theorem 1 and Theorem 2, a PSPACE non-deterministic algorithm $D_{\sqsubset}$-sat for checking the satisfiability of a $D_{\sqsubset}$-formula φ can be easily obtained as follows. It non-deterministically generates a φ-atom A containing φ and calls a procedure $D_{\sqsubset}$-step on it. Such a procedure non-deterministically generates a reflexive φ-atom A' such that $A\ D_\varphi\ A'$ and REQ(A') = REQ(A), if there is such atom; otherwise it returns 'NO' and halts. Next, for all $\langle D\rangle\psi \in$ REQ(A'), it non-deterministically generates a φ-atom A'' such that $A'\ D_\varphi\ A''$ and $\psi \in A''$, if there is such atom; otherwise it returns 'NO' and halts. Finally, if REQ(A'') $\neq$ REQ(A'), then it executes a recursive call on A''. $D_{\sqsubset}$-sat fails and returns 'NO' whenever an atom with the requested properties cannot be generated, and it returns 'YES' if and only if there exists a fulfilling $D_{\sqsubset}$-structure for φ. It does not exceed polynomial space because the number of nested calls of $D_{\sqsubset}$-step is bounded by $\mathcal{O}(n)$ (the maximum length of a simple path) and every call needs $\mathcal{O}(n)$ memory space for local operations. In [11] Shapirovsky proved the PSPACE-hardness of $D_{\sqsubset}$ by providing a reduction of the validity problem for prenex quantified boolean formulae, that is known to be PSPACE-complete, to the satisfiability problem for $D_{\sqsubset}$.

5 A Tableau Method for $D_{\sqsubset}$

In the previous section we proved that a $D_{\sqsubset}$-formula is satisfiable iff it is fulfilled by an effectively constructed $D_{\sqsubset}$-structure of a suitably bounded size. We will use that construction here to design a sound, complete, and terminating tableau method for $D_{\sqsubset}$.

A tableau for a $D_{\sqsubset}$-formula φ is a finite, tree-like graph, in which every node is a subset of $CL(\varphi)$. Nodes are grouped into *macronodes*, that is, finite subtrees of the tableau, dealt with by the expansion rules. Branching inside a macronode corresponds to disjunctions. Macronodes and edges connecting them represent vertices and edges in the $D_{\sqsubset}$-structure for φ. We distinguish two types of rules: *local rules*, that generate new nodes belonging to the same macronode, and *global rules*, that generate new nodes belonging to new macronodes.

Given two nodes n and n' such that n' is a descendant of n, we say that n' is a *local descendant* of n (or, equivalently, that n is a *local ancestor* of n') if n and n' belong to the same macronode and that n' is a *global descendant* of n (n is a *global ancestor* of n') if n and n' belong to different macronodes.

Local Rules:

(NOT) $\dfrac{\neg\neg\psi, F}{\psi, F}$ (OR) $\dfrac{\psi_1 \vee \psi_2, F}{\psi_1, F \mid \psi_2, F}$ (AND) $\dfrac{\neg(\psi_1 \vee \psi_2), F}{\neg\psi_1, \neg\psi_2, F}$

(REFL) $\dfrac{[D]\psi, F}{\psi, [D]\psi, F}$ where ψ does not occur in any local ancestor of the node

Global Rules:

(2-DENS) $\dfrac{[D]\psi_1, \ldots, [D]\psi_m, \langle D\rangle\varphi_1, \ldots, \langle D\rangle\varphi_n, F}{\psi_1, \ldots, \psi_m, [D]\psi_1, \ldots, [D]\psi_m, \langle D\rangle\varphi_1, \ldots, \langle D\rangle\varphi_n}$
where $m, n \geq 0$ and F contains no temporal formulae;

(STEP) $\dfrac{[D]\psi_1, \ldots, [D]\psi_m, \langle D\rangle\varphi_1, \ldots, \langle D\rangle\varphi_n, F}{G_1 \mid \ldots \mid G_n}$
where $m \geq 0$, $n > 0$, F contains no temporal formulae and, for every $i = 1, \ldots, n$, $G_i = \{\varphi_i, \psi_1, \ldots, \psi_m, [D]\psi_1, \ldots, [D]\psi_m\}$

Reflexive and irreflexive macronodes. As the vertices of a $D_{\sqsubset}$-graph, macronodes can be either reflexive or irreflexive. A macronode is *irreflexive* if (i) it contains the initial node of the tableau, or (ii) it has been created by an application of the STEP rule. In the other cases, viz., when the macronode has been created by an application of the 2-DENS rule, it is *reflexive*. A node of the tableau is reflexive (resp., irreflexive) if it belongs to a reflexive (resp., irreflexive) macronode.

Expansion strategy. Given a formula φ, the tableau for φ is obtained from the one-node initial tableau $\{\varphi\}$ by recursively applying the following expansion strategy, until it cannot be applied anymore:

1. do not apply the expansion rules to nodes of the tableau containing both ψ and $\neg\psi$, for some $\psi \in CL(\varphi)$;
2. apply the NOT, OR, and AND rules to both reflexive and irreflexive nodes;

3. apply the 2-DENS rule to irreflexive nodes;
4. apply the REFL and STEP rules to reflexive nodes;

To keep the construction finite, when the application of the 2-DENS rule to a node n would generate a new reflexive node such that there exists another reflexive ancestor node n' in the tableau with the same set of temporal formulae, add an edge from n to n' instead of generating such a new node.

Open and closed tableau. A node n in a tableau $\mathcal{T}$ is *closed* if one of the following conditions holds: (i) there exists a formula $\psi \in \mathrm{CL}(\varphi)$ such that $\psi, \neg\psi \in n$; (ii) in the tableau construction, the NOT, OR, AND, 2-DENS, or REFL rules has been applied to n and *all* the immediate successors of n are closed; (iii) in the tableau construction, the STEP rule has been applied to n and *at least one* of the immediate successors of n is closed.

A tableau $\mathcal{T}$ is *closed* if its initial node is closed; otherwise, it is *open*. Figure 2 depicts a closed tableau for the unsatisfiable formula $\langle D\rangle\langle D\rangle p \wedge \langle D\rangle q \wedge [D]([D]\neg p \vee [D]\neg q)$, with boxes representing macronodes (we underlined formulae to which local rules are applied).

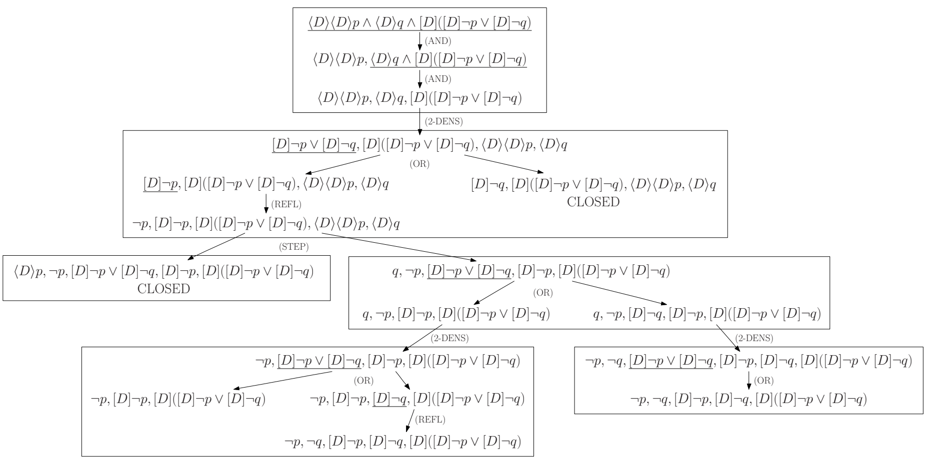


Fig. 2. The tableau for $\langle D\rangle\langle D\rangle p \wedge \langle D\rangle q \wedge [D]([D]\neg p \vee [D]\neg q)$

The proposed tableau method terminates because:

– the local rules can be applied only finitely many times to the nodes of a macronode (the application of the NOT, AND, and OR rules to a node generates one or two nodes where one of the formulae in the original node has been replaced by one or two formulae of strictly lower size; moreover, for any branch in the subtree associated with a macronode and any $[D]$-formula in $\mathrm{CL}(\varphi)$, the REFL rule can be applied at most one time);

- the 2-DENS rule generates a new node only when the set of temporal formulae of such a node is different from that of the other reflexive nodes in the tableau, and thus it can be applied only finitely many times;
- along any branch in the tableau, the applications of the 2-DENS and of STEP rules alternate.

As for the complexity, we have shown that a formula φ is satisfiable if and only if there exists a $\mathrm{D}_{\sqsubset}$-structure for it whose breadth and depth are linear in $|\varphi|$. Such a property holds for any tableau $\mathcal{T}$ for φ as well. Indeed, let n be the number of $\langle D\rangle$-formulae in $\mathrm{CL}(\varphi)$. The number of outgoing edges of a node is bounded by n. Moreover, as in $\mathrm{D}_{\sqsubset}$-structures, every simple path of macronodes starting from the root is of length at most $2n$. Hence, both the breadth and the depth of $\mathcal{T}$ are linear in $|\varphi|$ and a tableau $\mathcal{T}$ for φ can be non-deterministically generated and explored by using a polynomially bounded amount of space. Thus, we obtain the following theorem.

Theorem 4. *The proposed tableau method for* $\mathrm{D}_{\sqsubset}$ *has a PSPACE complexity.*

Theorem 5. *(*SOUNDNESS*) Let* φ *be a* $\mathrm{D}_{\sqsubset}$*-formula and* $\mathcal{T}$ *be a tableau for it. If* $\mathcal{T}$ *is open, then* φ *is satisfiable.*

Proof. Let $\mathcal{T}$ be an open tableau for φ. We build a fulfilling $\mathrm{D}_{\sqsubset}$-structure $\mathbf{S} = \langle\langle V, E\rangle, \mathcal{L}\rangle$ for φ step by step, starting from the root of the tableau. Let n_0 be the root of $\mathcal{T}$. Since $\mathcal{T}$ is open, then n_0 is not closed. We generate a one-node $\mathrm{D}_{\sqsubset}$-graph $\langle\{v_0\}, \emptyset\rangle$ and we put the formulae that belong to n_0 in $\mathcal{L}(v_0)$, thus associating n_0 with v_0. Now, let n be a non-closed node in $\mathcal{T}$ and let v be the associated vertex in the $\mathrm{D}_{\sqsubset}$-graph. We distinguish four possible cases, depending on the expansion rule R that has been applied to n in the tableau construction:

Case 1 *R is one of NOT, AND, and REFL.* Since n is not closed, its unique successor n' is not closed as well. We add the formulae contained in n' to $\mathcal{L}(v)$, thus associating n' with v, and then we proceed with the pair (n', v) (note that different nodes can be associated with the same vertex of the $\mathrm{D}_{\sqsubset}$-structure).

- *R is OR.* Since n is not closed, it has a successor n' that is not closed. We add the formulae contained in n' to $\mathcal{L}(v)$, thus associating n' with v, and then we proceed with the pair (n', v).
- *R is 2-DENS.* Since n is not closed, its unique successor n' is not closed as well. If n' has already been associated with a vertex v' during the construction of the $\mathrm{D}_{\sqsubset}$-structure, we simply add an edge (v, v') to E; otherwise, we add a new reflexive vertex v' to V, we add the edges (v, v') and (v', v') to E, and we put the formulae that belong to n' in $\mathcal{L}(v')$, thus associating n' with v'. Then we proceed with the pair (n', v').
- *R is STEP.* Since n is not closed, none of its successors $n_1, \ldots, n_k$ is closed either. Take any of them, n_i. If n_i has already been associated with a vertex v_i during the construction of the $\mathrm{D}_{\sqsubset}$-structure, we simply add an edge (v, v_i) to E; otherwise, we add a new irreflexive vertex v_i to V, we add the edge (v, v_i) to E, and we put the formulae that belong to n_i in $\mathcal{L}(v_i)$, thus associating n_i with v_i. Then we proceed with the pair (n_i, v_i).

Since any tableau for φ is finite, such a construction is terminating. However, the resulting pair $\langle\langle V, E\rangle, \mathcal{L}\rangle$ is not necessarily a $\mathrm{D}_{\sqsubset}$-structure: while $\langle V, E\rangle$ respects the definition of $\mathrm{D}_{\sqsubset}$-graph, the function $\mathcal{L}$ is not necessarily a labeling function. Since in the tableau construction we add new formulae only when necessary, there may exist a vertex $v \in V$ and a formula $\psi \in \mathrm{CL}(\varphi)$ such that neither ψ nor $\neg\psi$ belongs to $\mathcal{L}(v)$. Let $v \in V$ and $\psi \in \mathrm{CL}(\varphi)$ such that neither ψ nor $\neg\psi$ belongs to $\mathcal{L}(v)$. We can complete the definition of $\mathcal{L}$ as follows (by induction on the complexity of ψ):

- if $\psi = p$, with $p \in \mathcal{AP}$, we put $\neg p \in \mathcal{L}(v)$;
- If $\psi = \neg\xi$, we put $\psi \in \mathcal{L}(v)$ if and only if $\xi \notin \mathcal{L}(v)$;
- If $\psi = \psi_1 \vee \psi_2$, we put $\psi_1 \vee \psi_2 \in \mathcal{L}(v)$ if and only if $\psi_1 \in \mathcal{L}(v)$ or $\psi_2 \in \mathcal{L}(v)$;
- If $\psi = \langle D\rangle\xi$, we put $\psi \in \mathcal{L}(v)$ if and only if there exists a descendant v' of v such that $\xi \in \mathcal{L}(v')$.

It follows from the construction that the resulting $\mathrm{D}_{\sqsubset}$-structure $\langle\langle V, E\rangle, \mathcal{L}\rangle$ is a fulfilling $\mathrm{D}_{\sqsubset}$-structure for φ. Therefore, by Theorem 2, φ is satisfiable. □

Theorem 6. (COMPLETENESS) *Let φ be a $\mathrm{D}_{\sqsubset}$-formula and $\mathcal{T}$ be a tableau for it. If $\mathcal{T}$ is closed, then φ is unsatisfiable.*

Proof. Given a node n in a tableau, we say that (the set of formulae belonging to) n is *consistent* if there exists a fulfilling $\mathrm{D}_{\sqsubset}$-structure $\mathbf{S} = \langle\langle V, E\rangle, \mathcal{L}\rangle$ such that if n belongs to a reflexive (resp., irreflexive) macronode then there exists a reflexive (resp., irreflexive) vertex $v \in V$ such that $\mathcal{L}(v)$ contains all formulae in n; otherwise n is *inconsistent*.

We will prove that for any node n in a tableau $\mathcal{T}$, if n is closed, then n is inconsistent. If there exists a formula $\psi \in \mathrm{CL}(\varphi)$ such that n contains both ψ and $\neg\psi$, then n is obviously inconsistent. In the other cases, we proceed by induction, from the leaves to the root, on the expansion rule R that has been applied to the node n in the construction of the tableau. Since any tableau is finite, we eventually reach the initial node of $\mathcal{T}$, thus concluding that φ is an inconsistent formula.

- *R is NOT.* Then n is of the form $\neg\neg\psi, F$ and it has a unique successor $n' = \psi, F$ within the same macronode. If n' is closed then, by inductive hypothesis, ψ, F is an inconsistent set. Hence, $\neg\neg\psi, F$ is inconsistent.
- *R is OR.* Then n is of the form $\psi_1 \vee \psi_2, F$ and it has two immediate successors $n_1 = \psi_1, F$ and $n_2 = \psi_2, F$ within the same macronode. If both n_1 and n_2 are closed then, by inductive hypothesis, both ψ_1, F and ψ_2, F are inconsistent sets. Hence, $\psi_1 \vee \psi_2, F$ is inconsistent.
- *R is AND.* Then n is of the form $\neg(\psi_1 \vee \psi_2), F$ and it has a unique successor $n' = \neg\psi_1, \neg\psi_2, F$ within the same macronode. If n' is closed then, by inductive hypothesis, $\neg\psi_1, \neg\psi_2, F$ is an inconsistent set. Hence, $\psi_1 \wedge \psi_2, F$ is inconsistent.
- *R is REFL.* In this case, n belongs to a reflexive macronode and it is of the form $[D]\psi, F$. Suppose, for contradiction, that n is closed but consistent,

i.e., that there exists a fulfilling $D_{\sqsubseteq}$-structure $\mathbf{S} = \langle\langle V, E\rangle, \mathcal{L}\rangle$ and a reflexive vertex $v \in V$ such that $n \subseteq \mathcal{L}(v)$. Since v is reflexive, we have that $(v, v) \in E$ and thus $\mathcal{L}(v)\ D_{\varphi}\ \mathcal{L}(v)$. Hence, we have that $\psi \in \mathcal{L}(v)$ and thus the set of formulae $\psi, [D]\psi, F$ is consistent. By inductive hypothesis, this implies that the successor n' of n is not closed, which contradicts the hypothesis that n is closed.

- *R is 2-DENS.* In this case, n belongs to an irreflexive macronode and it is of the form $[D]\psi_1, \ldots, [D]\psi_m, \langle D\rangle\varphi_1, \ldots, \langle D\rangle\varphi_h, F$. Suppose, for contradiction, that n is closed but consistent. Hence, there exists a fulfilling $D_{\sqsubseteq}$-structure $\mathbf{S} = \langle\langle V, E\rangle, \mathcal{L}\rangle$ and an irreflexive vertex $v \in V$ such that $n \subseteq \mathcal{L}(v)$. By the definition of $D_{\sqsubseteq}$-structure, v has a reflexive successor v' such that $\mathcal{L}(v)\ D_{\varphi}\ \mathcal{L}(v')$. Hence, v' is such that $\{\psi_1, \ldots, \psi_m, [D]\psi_1, \ldots, [D]\psi_m, \langle D\rangle\varphi_1, \ldots, \langle D\rangle \varphi_h\} \subseteq \mathcal{L}(v')$. This proves that the successor n' of n in the tableau is consistent. By inductive hypothesis, n' is not closed – a contradiction.
- *R is STEP.* In this case, n belongs to a reflexive macronode and it is of the form $[D]\psi_1, \ldots, [D]\psi_m, \langle D\rangle\varphi_1, \ldots, \langle D\rangle\varphi_h, F$. Suppose, for contradiction, that n is closed but consistent. Hence, there exists a fulfilling $D_{\sqsubseteq}$-structure $\mathbf{S} = \langle\langle V, E\rangle, \mathcal{L}\rangle$ and a reflexive vertex $v \in V$ such that $n \subseteq \mathcal{L}(v)$. Since $\mathbf{S}$ is fulfilling, for every formula $\langle D\rangle\varphi_i$ there exists a descendant v_i of v such that $\varphi_i \in \mathcal{L}(v_i)$. This implies that, for every $i = 1, \ldots, h$, the set of formulae $G_i = \{\varphi_i, \psi_1, \ldots, \psi_m, [D]\psi_1, \ldots, [D]\psi_m\}$ is consistent. By inductive hypothesis, this implies that every immediate successor of n is not closed, which contradicts the assumption that n is closed.

Since the tableau $\mathcal{T}$ is closed if and only if its root is closed, and since the root of the tableau is an irreflexive node, from the above result we can conclude that the set $\{\varphi\}$ is inconsistent, namely, that there are no fulfilling $D_{\sqsubseteq}$-structures that features an irreflexive vertex v such that $\varphi \in \mathcal{L}(v)$. Since a formula is satisfiable if and only if it belongs to the labelling of the (irreflexive) root of some fulfilling $D_{\sqsubseteq}$-structure, this proves that φ is unsatisfiable. □

6 The Logic $D_{\sqsubset}$ of the Proper Subinterval Relation

For the case of *proper* subinterval relation, we replace $D_{\sqsubseteq}$-structures by $D_{\sqsubset}$-structures. Given an interval $[b, e]$ in a dense linear order $\mathbb{D} = \langle D, <\rangle$, a *beginning subinterval* of $[b, e]$ is any proper subinterval $[b, e']$, for $b < e' < e$; likewise, an *ending subinterval* of $[b, e]$ is any proper subinterval $[b', e]$, for $b < b' < e$.

The presence of the special families of beginning subintervals and ending subintervals of a given interval in a structure with proper subinterval relation causes an essential distinction from the case interval structures with strict subinterval relation studied in the previous sections. This distinction reflects essentially on the respective logic $D_{\sqsubset}$, and in particular on the definition of $D_{\sqsubset}$-structures and the construction of tableau for $D_{\sqsubset}$-formulae, where special care must be taken for these families.

Given a graph $\mathbb{G} = \langle V, E \rangle$, a *maximal cluster* in $\mathbb{G}$ is a maximal set $\mathcal{C}$ of reflexive vertices in V such that for every pair $v, v' \in \mathcal{C}$, v is reachable from v'.

Definition 8. *A finite directed graph* $\mathbb{G} = \langle V, E \rangle$ *is a* $\mathrm{D}_{\sqsubset}$-graph *if:*

1. *there exists an irreflexive vertex* $v_0 \in V$*, called the* root *of* $\mathbb{G}$*, such that any other vertex* $v \in V$ *is reachable from it;*
2. *every irreflexive vertex* $v \in V$ *has exactly two successors* v_b *and* v_e*, which are both reflexive; we denote by* $\mathcal{C}_b$ *the maximal cluster containing* v_b *and by* $\mathcal{C}_e$ *the maximal cluster containing* v_e*;*
3. v_b *and* v_e *have a unique common successor* v_c*, which is reflexive;*
4. v_b *and* v_e *may have at most one irreflexive successor and there are no other edges exiting from* $\mathcal{C}_b$ *and* $\mathcal{C}_e$*;*
5. *every successor of* v_c *different from itself is irreflexive.*

A portion of a $\mathrm{D}_{\sqsubset}$-graph is depicted in Figure 3.

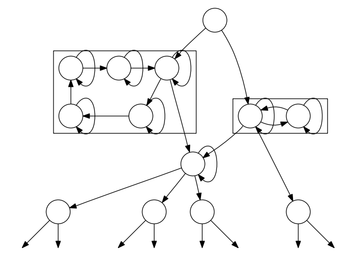


Fig. 3. An example of $\mathrm{D}_{\sqsubset}$-graph

Definition 9. *A* $\mathrm{D}_{\sqsubset}$-structure *is a quadruple* $\mathbf{S} = \langle \langle V, E \rangle, \mathcal{L}, \mathcal{B}, \mathcal{E} \rangle$*, where:*

1. $\langle V, E \rangle$ *is a* $\mathrm{D}_{\sqsubset}$*-graph.*
2. $\mathcal{L} : V \to \mathcal{A}_\varphi$ *is a* labeling function *that assigns to every vertex* $v \in V$ *an atom* $\mathcal{L}(v)$ *such that* $\mathcal{L}(v)\ D_\varphi\ \mathcal{L}(v')$ *for every edge* $(v, v') \in E$*.*
3. $\mathcal{B} : V \to 2^{\mathrm{REQ}_\varphi}$ *and* $\mathcal{E} : V \to 2^{\mathrm{REQ}_\varphi}$ *are mappings that assign to every node the set of the requests that must be satisfied respectively in the beginning subintervals and in the ending subintervals of the current interval.*
4. *For every irreflexive vertex* $v \in V$*, we have that:*
   - *the reflexive vertex* v_c *is such that* $\mathcal{E}(v_c) = \mathcal{B}(v_c) = \emptyset$ *and* $REQ(\mathcal{L}(v_c)) = REQ(\mathcal{L}(v)) - (\mathcal{B}(v) \cup \mathcal{E}(v))$*,*
   - *every reflexive vertex* $v' \in \mathcal{C}_b$ *is such that* $\mathcal{B}(v') = \mathcal{B}(v)$*,* $\mathcal{E}(v') = \emptyset$ *and* $REQ(\mathcal{L}(v')) = REQ(\mathcal{L}(v_c)) \cup \mathcal{B}(v)$*,*
   - *the unique irreflexive successor* v'' *from a vertex in* $\mathcal{C}_b$ *(if any) is such that* $\mathcal{B}(v'') \subset \mathcal{B}(v)$ *and* $\mathrm{REQ}(\mathcal{L}(v'')) \subset \mathrm{REQ}(\mathcal{L}(v))$*,*
   - *every reflexive vertex* $v' \in \mathcal{C}_e$ *is such that* $\mathcal{E}(v') = \mathcal{E}(v)$*,* $\mathcal{B}(v') = \emptyset$ *and* $REQ(\mathcal{L}(v')) = REQ(\mathcal{L}(v_c)) \cup \mathcal{E}(v)$*,*
   - *the unique irreflexive successor* v'' *from a vertex in* $\mathcal{C}_e$ *(if any) is such that* $\mathcal{E}(v'') \subset \mathcal{E}(v)$ *and* $\mathrm{REQ}(\mathcal{L}(v'')) \subset \mathrm{REQ}(\mathcal{L}(v))$*,*

   where $v_c, \mathcal{C}_b$*, and* $\mathcal{C}_e$ *are defined as in Definition 8.*

Let v_0 be the root of $\langle V, E\rangle$. If $\varphi \in \mathcal{L}(v_0)$, we say that $\mathbf{S}$ is a $\mathrm{D}_{\sqsubset}$*-structure for* φ. The notion of fulfilling $\mathrm{D}_{\sqsubset}$-structure is defined as in the case of $\mathrm{D}_{\sqsubseteq}$-structures. An analogous of Theorem 1 holds for $\mathrm{D}_{\sqsubset}$-structures as well.

Theorem 7. *Let* φ *be a* $\mathrm{D}_{\sqsubset}$*-formula which is satisfied in a dense interval model. Then, there exists a fulfilling* $\mathrm{D}_{\sqsubset}$*-structure* $\mathbf{S} = \langle\langle V, E\rangle, \mathcal{L}, \mathcal{B}, \mathcal{E}\rangle$ *such that* $\varphi \in \mathcal{L}(v_0)$*, where* v_0 *is the root of* $\langle V, E\rangle$*.*

Now, let $\mathbf{S}$ be a fulfilling $\mathrm{D}_{\sqsubset}$-structure for a formula φ. To prove that φ is satisfiable in a dense interval structure we consider the interval $[0, 1]$ of the rational line and define a function $f_{\mathbf{S}}$ mapping intervals in $\mathbb{I}([0, 1])^-$ to vertices in $\mathbf{S}$. Such a function will allow us to define a model for φ.

Definition 10. *Let* $\mathbf{S} = \langle\langle V, E\rangle, \mathcal{L}, \mathcal{B}, \mathcal{E}\rangle$ *be a* $\mathrm{D}_{\sqsubset}$*-structure. The function* $f_{\mathbf{S}} : \mathbb{I}([0, 1])^- \mapsto V$ *is recursively defined as follows. First,* $f_{\mathbf{S}}([0, 1]) = v_0$*. Now, let* $[b, e]$ *be an interval such that* $f_{\mathbf{S}}([b, e]) = v$*, but* $f_{\mathbf{S}}$ *has not been defined for the subintervals of* $[b, e]$*.*

Case 1: v *is an irreflexive vertex. Let* v_b, v_e *be the reflexive successors of* v*,* $\mathcal{C}_b$ *and* $\mathcal{C}_e$ *be their clusters, and* v_c *be the reflexive vertex that they have as common successor. We consider the case when* v_b *has an irreflexive successor* v_b^{max}*,* v_e *has an irreflexive successor* v_e^{max}*, and* v_c *has* k *irreflexive successors* $v_1, \ldots, v_k$*. The other cases are simpler and can be treated in a similar way. Let* $p = \frac{e-b}{2k+3}$*. We define* $f_{\mathbf{S}}$ *in a way similar to the case of* $\mathrm{D}_{\sqsubseteq}$*-structures:*
1. *we put* $f_{\mathbf{S}}([b, b + p]) = v_b^{max}$ *and* $f_{\mathbf{S}}([e - p, e]) = v_e^{max}$*;*
2. *for every* $i = 1, \ldots, k$*, we put* $f_{\mathbf{S}}([b + 2ip, b + (2i + 1)p]) = v_i$*;*
3. *for every* $i = 1, \ldots, k + 1$*, we put* $f_{\mathbf{S}}([b + (2i - 1)p, b + 2ip]) = v_c$*;*
4. *for every* strict *subinterval* $[b', e']$ *of* $[b, e]$ *that is not a subinterval of any of the intervals* $[b + ip, b + (i + 1)p]$*, we put* $f_{\mathbf{S}}([b', e']) = v_c$*.*

To complete the construction we need to define $f_{\mathbf{S}}$ *for the beginning intervals* $[b, e']$ *such that* $b + p < e' < e$ *and for the ending intervals* $[b', e]$ *such that* $b < b' < e - p$*. We map such beginning intervals* $[b, e']$ *to vertices in* $\mathcal{C}_b$ *and ending intervals* $[b', e]$ *to vertices in* $\mathcal{C}_e$*, with the further constraint that for every interval* $[b, e']$ *and every node* $v' \in \mathcal{C}_b$ *there is an interval* $[b, e''] \sqsubset [b, e']$ *such that* $f_{\mathbf{S}}([b, e'']) = v''$ *and, conversely, that for every interval* $[b', e]$ *and every node* $v' \in \mathcal{C}_e$ *there is an interval* $[b'', e] \sqsubset [b'', e]$ *such that* $f_{\mathbf{S}}([b'', e]) = v''$*. The density of the rational line* $[0, 1]$ *allow us to define such a mapping.*

Case 2: v *is a reflexive vertex. We can assume, by construction, that* v *is one of the vertices* v_c *that have only irreflexive successors (except itself). In such a case we proceed as for* $\mathrm{D}_{\sqsubseteq}$*-structures in the definition of* $f_{\mathbf{S}}$*.*

Now, given a fulfilling $\mathrm{D}_{\sqsubset}$-structure $\mathbf{S}$ for φ, we define the corresponding interval model $\mathbf{M}_{\mathbf{S}} = \langle\mathbb{I}([0, 1])^-, \sqsubset, \mathcal{V}\rangle$, where, for every $p \in \mathcal{AP}$, $\mathcal{V}(p) = \{[b, e] : p \in \mathcal{L}(f_{\mathbf{S}}([b, e]))\}$. It turns out that $\mathbf{M}_{\mathbf{S}}$ is a model for φ. The following theorem can be proved by structural induction, similarly to Theorem 2.

Theorem 8. *Given a fulfilling* $\mathrm{D}_{\sqsubset}$*-structure* $\mathbf{S}$ *for* φ*, the corresponding interval model* $\mathbf{M}_{\mathbf{S}} = \langle\mathbb{I}([0, 1])^-, \sqsubset, \mathcal{V}\rangle$ *is such that* $\mathbf{M}_{\mathbf{S}}, [0, 1] \Vdash \varphi$*.*

The following observations imply a bound on the size of $\mathrm{D}_{\sqsubset}$-structures:

- given a cluster of reflexive vertices $\mathcal{C}$, we have that all the vertices in $\mathcal{C}$ share the same set of requests $\mathrm{REQ}_{\mathcal{C}}$. Furthermore, we can remove from $\mathcal{C}$ all the vertices that either does not satisfy any of the $\langle D \rangle$-formulae in $\mathrm{REQ}_{\mathcal{C}}$ or they satisfy only $\langle D \rangle$-formulae that are satisfied by some other vertex in $\mathcal{C}$. This implies that we can build a $\mathrm{D}_{\sqsubset}$-structure where the size of each cluster is bounded by the number of $\langle D \rangle$-formulae in $\mathrm{CL}(\varphi)$.
- we can bound the breadth and depth of the $\mathrm{D}_{\dot\sqsubset}$-structure by exploiting the same technique that we use for $\mathrm{D}_{\sqsubset}$-structures.

Theorem 9. *For every satisfiable* $\mathrm{D}_{\sqsubset}$*-formula* φ*, there exists a fulfilling* $\mathrm{D}_{\sqsubset}$*-structure with breadth and depth bounded by* $2 \cdot |\varphi|$*.*

As for the complexity, given a formula φ, let n be the number of $\langle D \rangle$-formulae in $\mathrm{CL}(\varphi)$. From the above theorem, every simple path $(v_0, ..., v_n)$ starting from the root of a $\mathrm{D}_{\sqsubset}$-structure for φ has length at most $2n$. A PSPACE non-deterministic algorithm $\mathrm{D}_{\sqsubset}$-sat for checking satisfiability of $\mathrm{D}_{\sqsubset}$ formulae can be obtained by revising the procedure $\mathrm{D}_{\dot\sqsubset}$-sat as follows. At each step, once it has nondeterministically generated an irreflexive atom, it nondeterministically generates one reflexive atom for v_b, one for v_e, and one for v_c; then, it generates the reflexive atoms for the vertices contained in the cluster of v_b and in the cluster of v_e, if any; finally, it generates the irreflexive successors of v_b and v_e, and one irreflexive successor for v_c, for every $\langle D \rangle$-formula contained in $\mathcal{L}(v_c)$. $\mathrm{D}_{\sqsubset}$-sat fails if an atom with the required properties cannot be generated; otherwise, it succeeds and it generates a fulfilling $\mathrm{D}_{\sqsubset}$-structure for φ. The very same reduction that has been used to prove the PSPACE-hardness of $\mathrm{D}_{\dot\sqsubset}$ can be applied to $\mathrm{D}_{\sqsubset}$, thus proving that this logic is PSPACE-complete as well.

The tableau method for $\mathrm{D}_{\dot\sqsubset}$ can be adapted to $\mathrm{D}_{\sqsubset}$, even though such an adaptation is not trivial at all (for lack of space, we omit its description).

7 Conclusions

In this paper we have started building tableau-based decision procedures for logics of subinterval structures. First, we have considered in detail the case of the logic $\mathrm{D}_{\dot\sqsubset}$; then, we have shown how to refine structures and methods for $\mathrm{D}_{\dot\sqsubset}$ to cope with the logic $\mathrm{D}_{\sqsubset}$. Our methodology can be suitably adapted to the case we admit the existence of point intervals $[b, b]$ in subinterval structures over dense orderings. The cases of discrete and general orderings present additional challenges which will be treated in subsequent publications.

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