Undecidability and temporal logic: some landmarks from Turing to the present

(Extended abstract)

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I. Introduction

Temporal logic was born in the mid 1950s, right after the death of Alan Turing, with the pioneering work of Arthur Prior. Yet, one can argue that, in hindsight, the first undecidability result in temporal logic was essentially Alan Turing's celebrated Undecidability of the Halting problem of (what subsequently became known as) Turing machines, established 20 years earlier, in 1936 [1]. Indeed, the Halting problem can be interpreted both as a model checking problem and as a satisfiability problem for suitably defined temporal logics.

Numerous systems of temporal logics started proliferating since the early 1970s and the question of decidability of logical validity or satisfiability of formulas in a given temporal logic became one of the main technical problems of the research agenda in the area. Later on, in the early 1980s, the importance in computer science of model checking of temporal logics on transition systems was widely recognized and that sparked a new wave of research on decidability (and complexity) of model checking for a variety of systems of temporal logics. It gradually became evident that, while standard modal and temporal logics were generally known and praised for their 'robust decidability' [2], they could easily be extended in various ways to undecidable ones, too. This 'swaying' behaviour of temporal logics and their conspicuous presence on both sides of the decidability border has made the quests for ever stronger decidable, and for ever weaker undecidable, temporal logics one of the most intense and fruitful directions of research in the area.

Here I offer a brief journey into the land of undecidable temporal logics, focusing mostly on the validity/satisfiability decision problem. It is hardly possible to cover all relevant results in a single study, and to even mention them all in one lecture, so I will only present a (inevitably biased) selection of some of the more interesting undecidability phenomena in the realm of temporal logics.

II. Some landmarks of undecidability in temporal logics

Undecidability results in temporal logics can be classified along at least two interleaved dimensions:

- patterns and techniques for proving undecidability;
- families of typically undecidable logics.

A. Patterns and techniques for proving undecidability in temporal logics

The usual approach to proving undecidability is by reduction from other, already established undecidable problems, most notably the halting (or, non-halting) problem of Turing machines. This has been one of the favorite methods for proving undecidability in modal and temporal logics since their birth up to date. A sample of such undecidability results based on reductions from variations of the halting problem include [3], [4], [5], [6]. The method has been refined and honed over the years to a variety of simpler or more involved other reductions and encodings of other classical undecidable problems, such as:

- reduction from the satisfiability/validity in first-order logic (FOL), as e.g., in [7];
- encoding of various tiling problems on the integer grid or a suitable part of it [8], [9], [10], [11], [5], [12].
- encoding of the Post correspondence problem, [5].
- encoding of counter machines, Minsky machines, etc.

B. Families of undecidable temporal logics

Various families of temporal logics where undecidability is common have emerged over the years, including:

- *first-order temporal logics*. As a rule, these automatically inherit the undecidability of FOL. Even decidable fragments of FOL, such as the monadic and two-variable fragments, can become undecidable when temporalized [7]. Furthermore, unlike plain FOL, various first-order extensions of (even, decidable) temporal logics turn out to be *highly undecidable* and thus, not recursively axiomatizable.

- *temporal logics with propositional quantification*, [13].
- *many-dimensional temporal (or, temporalized) logics*, such as e.g., the compass logic [10], products of temporal and modal logics [5], etc.
- propositional *interval logics* [14], [15], [3], [16], [12], [17]. [6] and *duration calculi* [18], [19].
- *real time logics* with clocks, on both dense and discrete time flows [20].
- *metric temporal logics* [21].
- *temporal epistemic logics* [22].
- *hybrid temporal logics* with binders [9].
- *temporal description logics* [23].

III. Concluding remarks

Undecidability is an undesirable but inevitable price payable for the rich expressiveness of any logical system. Undecidability results are a priori discouraging, at least in theory. But, how much should they matter in practice? And, how much better is a decidable logic with non-elementary complexity, or even ExpTime complete one? These questions must be addressed in earnest in order to put the undecidability phenomenon in the right perspective, in general and specifically in the field of temporal logics.

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