Undecidability of Interval Temporal Logics with the *Overlap* Modality *

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Abstract

We investigate fragments of Halpern-Shoham's interval logic HS *involving the modal operators for the relations of left or right overlap of intervals. We prove that most of these fragments are undecidable, by employing a non-trivial reduction from the octant tiling problem.*

1. Introduction

Interval temporal logics are based on temporal structures over linearly (or partially) ordered domains, where time intervals, rather than time instants, are the primitive ontological entities. A systematic analysis of the variety of relations between intervals on linear orders was first accomplished by Allen [1], who explored the use of interval reasoning in systems for time management and planning. The problem of representing and reasoning about time intervals arises naturally in various other fields of computer science, artificial intelligence, and temporal databases, such as theories of action and change, natural language processing, and constraint satisfaction problems. Temporal logics with interval-based semantics have also been proposed as a useful formalism for the specification and verification of hardware [16] and of real-time systems [8].

Interval temporal logics feature modal operators that correspond to various relations between intervals, in particular the thirteen different binary interval relations on linear orders, known as Allen's relations [1]. In [11], Halpern and Shoham introduce a modal logic for reasoning about interval structures, nowadays known as HS, with modal operators corresponding to Allen's interval relations. This logic turns out to be undecidable under very weak assumptions on the class of interval structures [11]. In particular, undecidability holds for any class of interval structures over linear orders that contains at least one linear order with an infinite ascending (or descending) chain, thus including the natural time flows $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$. The complex and generally bad computational behavior of interval temporal logics is essentially due to the fact that formulas are evaluated over pairs of points and translate into binary relations. In a few cases, decidability has been recovered by imposing severe restrictions on the set of modalities and/or on the interval-based semantics, which essentially reduce the logic to a point-based one. For a long time, the sweeping undecidability results of Halpern and Shoham have discouraged attempts for practical applications and further research on interval logics. A renewed interest in the area has been recently stimulated by the discovery of some interesting decidable fragments of HS [4, 5, 6, 7]. The classification of decidable and undecidable fragments of HS has thus become one of the major topics of the current research agenda in interval temporal logics. The current state of affairs in that area has recently been been summarized in [3], where the main techniques so far exploited for proving decidability and undecidability have been presented. Such results have so far been obtained for fragments of HS involving all, but one, pairs of modal operators, with respect to various classes of linear orders. The only so far unexplored case is the one of fragments involving the operators $\langle O\rangle$ and/or its transpose $\langle\overline{O}\rangle$, which respectively capture the *Overlap* relation and its inverse. Such logics have received almost no attention in the literature (the only exception we are aware of is a number of representation theorems for interval structures containing such a relation, which have recently been obtained in [9]).

In this paper, we show that most extensions of the interval logic of *Overlap* are undecidable, thus making a further step toward the complete classification of the (un)decidability of all HS fragments. The proofs employ a non-trivial reduction from the octant tiling problem. The idea of using tiling problems to prove undecidability of interval logics and many-dimensional logics goes back to [15] and it has been subsequently applied to the compass logic [14], to various product logics [10], to modal spatial logics

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of topological relations [13], and to other fragments of HS [3, 6]. Most of these results, however, apply to (relatively) more expressive logical languages than those considered in the present paper. Moreover, the techniques for the encoding of tiling problems used so far do not transfer to our cases in any obvious way.

The paper is structured as follows. In Section 2 we introduce syntax and semantics of the logics of the *Overlap* relation. In Section 3, we provide a detailed account of the undecidability proofs for the fragments AO, $\overline{\mathsf{A}}\mathsf{O}$, $\mathsf{A}\overline{\mathsf{O}}$, and $\overline{\mathsf{A}}\,\overline{\mathsf{O}}$. In Section 4, we summarize the undecidability results for other extensions. In the conclusion, we provide an assessment of the work and outline future research directions.

2. The logics of the *Overlap* relation

Let $\mathbb{D} = \langle D, < \rangle$ be a linearly ordered set. An *interval* over $\mathbb{D}$ can be defined as an ordered pair $[a, b]$, where $a, b \in D$ and $a < b$, thus excluding intervals with coincident endpoints (strict semantics). As an alternative, one can define an interval over $\mathbb{D}$ as a pair $[a, b]$, with $a, b \in D$ and $a \leq b$ (non-strict semantics). Hereafter, we confine our attention to *strict* semantics; as we will show later, the non-strict case can be easily reduced to the strict one.

In general, the language of a propositional interval logic with unary modalities consists of a set $\mathcal{AP}$ of propositional letters, any complete set of boolean operators (such as $\vee$ and $\neg$), and a set of unary modal operators $\langle X_1 \rangle, \ldots, \langle X_k \rangle$, each of them associated with a specific binary relation over intervals (decidability issues for *binary* modal operators have been addressed in [12]). Formulas are defined by the grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X_1 \rangle\varphi \mid \ldots \mid \langle X_k \rangle\varphi,$$

The semantics is given in terms of *interval models* $\mathbf{M} = \langle \mathbb{I}(\mathbb{D}), V \rangle$, where $\mathbb{I}(\mathbb{D})$ is the set of all intervals over $\mathbb{D}$ and the valuation function $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})}$ assigns to every $p \in \mathcal{AP}$ the set of intervals $V(p)$ over which it holds. The *truth* of a formula over a given interval $[a, b]$ in a model $\mathbf{M}$ is defined by structural induction on formulas:

- $\mathbf{M}, [a, b] \Vdash p$ iff $[a, b] \in V(p)$, for all $p \in \mathcal{AP}$;
- $\mathbf{M}, [a, b] \Vdash \neg\psi$ iff it is not the case that $\mathbf{M}, [a, b] \Vdash \psi$;
- $\mathbf{M}, [a, b] \Vdash \varphi \vee \psi$ iff $\mathbf{M}, [a, b] \Vdash \varphi$ or $\mathbf{M}, [a, b] \Vdash \psi$;
- $\mathbf{M}, [a, b] \Vdash \langle X_i \rangle\psi$ iff there exists an interval $[c, d]$ such that $[a, b]\ R_{X_i}\ [c, d]$ and $\mathbf{M}, [c, d] \Vdash \psi$,

where R_{X_i} is the binary interval relation corresponding to the unary modal operator $\langle X_i \rangle$.

In the following, we will focus our attention on a specific family of HS fragments and, for that purpose, we will assume all operators listed in Fig. 1 and their transposes to be primitive in the language, where by a transpose of a unary modal operator $\langle X \rangle$ we mean the modal operator $\langle \overline{X} \rangle$ for the inverse of X. It is worth pointing out that the semantics of the operators slightly differs from the original one [11], which do not perfectly match the semantics of Allen's relations. As a general notation rule, we will denote any fragment of HS with the set of its operators; for example, the HS fragment featuring the operators $\langle A \rangle$ and $\langle O \rangle$ will be denoted by AO. Also, we will denote by $\mathsf{X}_1^*\mathsf{X}_2^*$ the set consisting of the fragments $\mathsf{X}_1\mathsf{X}_2, \mathsf{X}_1\overline{\mathsf{X}}_2, \overline{\mathsf{X}}_1\mathsf{X}_2$, and $\overline{\mathsf{X}}_1\overline{\mathsf{X}}_2$.

The family of fragments of HS we consider here includes $\mathsf{A}^*\mathsf{O}^*$, $\mathsf{B}^*\mathsf{O}^*$, $\mathsf{E}^*\mathsf{O}^*$, and $\mathsf{D}^*\mathsf{O}^*$. We will prove that all these logics are undecidable. We will give the details of the undecidability proof for the logics $\mathsf{A}^*\mathsf{O}^*$; the proofs for the other logics in the class are quite similar. Besides, we have been able to prove the undecidability of the logic $\mathsf{O}\overline{\mathsf{O}}$ with respect to any class of discrete linear orders. For lack of space, proof details for this case are omitted, but they will be included in an extended forthcoming version. All results are readily transferable to the case of non-strict semantics, using the formula $\langle O \rangle\top$ to impose the requirement that every interval we deal with has an internal point.

3. Undecidability of the logics $\mathsf{A}^*\mathsf{O}^*$

In this section, we show that the logics AO, $\mathsf{A}\overline{\mathsf{O}}$, $\overline{\mathsf{A}}\mathsf{O}$, and $\overline{\mathsf{A}}\,\overline{\mathsf{O}}$ are undecidable. In fact, we will prove the undecidability of the satisfiability problem for AO and then we will show that the proof can be tailored to deal with the other cases. For these proofs we will use a reduction from the tiling problem for the second octant $\mathcal{O}$ of the integer plane $\mathbb{Z} \times \mathbb{Z}$.

3.1. The tiling problem for $\mathcal{O}$

The tiling problem for $\mathcal{O}$ is the problem of establishing whether a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ can tile $\mathcal{O} = \{(i, j) : i, j \in \mathbb{N} \wedge 0 \leq i \leq j\}$. For every tile type $t_i \in \mathcal{T}$, let $right(t_i)$, $left(t_i)$, $up(t_i)$, and $down(t_i)$ be the colors of the corresponding sides of t_i. To solve the problem, one must find a function $f : \mathcal{O} \to \mathcal{T}$ such that

$$right(f(n, m)) = left(f(n + 1, m)), \text{ with } n < m,$$
$$\text{and } up(f(n, m)) = down(f(n, m + 1)).$$

Using König's lemma one can prove that a tiling system tiles an octant if and only if it tiles arbitrarily large squares if and only if it tiles $\mathbb{N} \times \mathbb{N}$ if and only if it tiles $\mathbb{Z} \times \mathbb{Z}$. The undecidability of the former thus immediately follows from that of the latter [2].

3.2. Generic reduction of the tiling problem for $\mathcal{O}$ to satisfiability in interval logics

Hereafter, we assume that $\mathcal{AP}$ contains some special propositional letters: $\{\mathsf{u}, \mathsf{ld}, \mathsf{tile}, *, \mathsf{b}, \mathsf{f}\}$ and others that will

Op.	Semantics
$\langle A\rangle$	$\mathbf{M},[a,b]\Vdash\langle A\rangle\phi\Leftrightarrow\exists c(b<c.\mathbf{M},[b,c]\Vdash\phi)$
$\langle L\rangle$	$\mathbf{M},[a,b]\Vdash\langle L\rangle\phi\Leftrightarrow\exists c,d(b<c<d.\mathbf{M},[c,d]\Vdash\phi)$
$\langle B\rangle$	$\mathbf{M},[a,b]\Vdash\langle B\rangle\phi\Leftrightarrow\exists c(a<c<b.\mathbf{M},[a,c]\Vdash\phi)$
$\langle E\rangle$	$\mathbf{M},[a,b]\Vdash\langle E\rangle\phi\Leftrightarrow\exists c(a<c<b.\mathbf{M},[c,b]\Vdash\phi)$
$\langle D\rangle$	$\mathbf{M},[a,b]\Vdash\langle D\rangle\phi\Leftrightarrow\exists c,d(a<c<d<b.\mathbf{M},[c,d]\Vdash\phi)$
$\langle O\rangle$	$\mathbf{M},[a,b]\Vdash\langle O\rangle\phi\Leftrightarrow\exists c,d(a<c<b<d.\mathbf{M},[c,d]\Vdash\phi)$

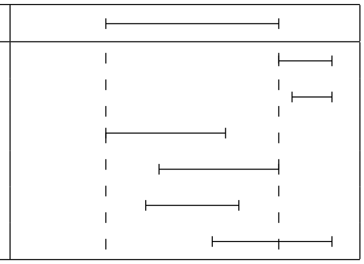

Figure 1. The semantics of basic interval modalities.

be introduced in due course. For every propositional letter q, we denote by the expression q-*interval* an interval satisfying q. We will provide a reduction of the tiling problem for $\mathcal{O}$ to the satisfiability problem for any fragment F of HS considered here in any class of interval models containing at least one model $\mathbf{M}$ with an unbounded-to-the-right (for the logics AO and $\mathsf{A\overline{O}}$) or unbounded-to-the-left (for the logics $\mathsf{\overline{A}O}$ and $\mathsf{\overline{A}\,\overline{O}}$) sequence of points. The reduction is based on the following main steps. First, we set our framework by forcing the existence of a unique infinite chain of u-intervals (u-*chain*, for short) on the linear order, which covers an initial segment of the domain. Such u-intervals will be used as cells ('unit-intervals') to arrange the tiling. Next, we define a chain of Id-intervals (Id-*chain*, for short), each of them representing a row of the octant. An Id-interval is composed by a sequence of u-intervals; each u-interval is used either to represent a part of the plane or to separate two Id-intervals. In the former case it is labelled with the propositional letter tile, in the latter case it is labelled with the propositional letter $*$. Then, we define two relations that connect each tile with its above neighbor and right neighbor in the octant, respectively. By using these relations, we force the j-th Id-interval to contain exactly j tile-intervals. Finally, we introduce a set of propositional letters $\mathtt{T}=\{\mathtt{t}_1,\mathtt{t}_2,\ldots,\mathtt{t}_\mathtt{k}\}$ corresponding to the set of tile types $\mathcal{T}=\{t_1,t_2,\ldots,t_k\}$ and we construct a formula $\Phi_\mathcal{T}$ belonging to the fragment F which is satisfiable if and only if there exists a proper tiling of the octant $\mathcal{O}$ by $\mathcal{T}$, i.e., one that satisfies the color constraints on vertically- and horizontally-adjacent tiles.

In the rest of this section we will illustrate this technique by constructing the formula $\Phi_\mathcal{T}$ for the fragment AO.

3.3. Definition of u-chain and Id-chain

Given an interval $[a,b]$, we define $\mathcal{G}_{[a,b]}$ as the set of intervals that contains the interval $[a,b]$ and all the intervals starting after a and ending after b. Moreover, we define the operator $[G]$ (*always in the future*) as follows:

$$[G]p\equiv p\wedge[O]p\wedge[A]p\wedge[A][A]p.$$

$[G]p$ holds over the interval $[a,b]$ iff p holds over each interval in $\mathcal{G}_{[a,b]}$.

Let $[a,b]$ be the interval over which we evaluate formulas (we may think of it as the interval to the right of which the u-chain starts). From now on, when we talk about an interval or a set of intervals, we will implicitly refer to intervals belonging to $\mathcal{G}_{[a,b]}$.

We start the encoding by constructing a u-chain of unit intervals:

$$\neg\mathsf{u}\wedge\langle A\rangle\mathsf{u}\wedge\langle O\rangle\top\wedge\neg\langle O\rangle\mathsf{u} \tag{1}$$

$$[G](\mathsf{u}\rightarrow\langle A\rangle\mathsf{u})\wedge[G](\mathsf{u}\rightarrow\langle O\rangle\top) \tag{2}$$

$$[G](\langle A\rangle\mathsf{u}\rightarrow\neg\langle O\rangle\mathsf{u}) \tag{3}$$

$$[G]((\mathsf{u}\leftrightarrow *\vee\mathsf{tile})\wedge(\mathsf{tile}\rightarrow\neg *)) \tag{4}$$

and a Id-chain that encodes the levels of the octant:

$$[G](\langle A\rangle *\leftrightarrow\langle A\rangle\mathsf{Id}) \tag{5}$$

$$[G](\mathsf{Id}\rightarrow\langle A\rangle *) \tag{6}$$

$$[G](\langle A\rangle *\rightarrow\neg\langle O\rangle\mathsf{Id}) \tag{7}$$

$$\begin{aligned}&\neg\mathsf{Id}\wedge\neg\langle O\rangle\mathsf{Id}\wedge\langle A\rangle(*\wedge\langle A\rangle(\mathsf{tile}\wedge\\&\langle A\rangle *\wedge[G](*\rightarrow\langle A\rangle(\mathsf{tile}\wedge\langle A\rangle\mathsf{tile}))))\end{aligned} \tag{8}$$

$$(1)\wedge\ldots\wedge(8) \tag{9}$$

Lemma 1. *If $\mathbf{M},[a,b]\Vdash(9)$, then there exists a sequence of points $b=b_1^0<b_1^1<\ldots<b_1^{k_1}=b_2^0<b_2^1<\ldots<b_2^{k_2}=b_3^0<\ldots$ such that for each $j>0$ we have:*

a) $\mathbf{M},[b_j^i,b_j^{i+1}]\Vdash\mathsf{u}$ *for each* $0\le i<k_j$ *and no other interval* $[c,d]\in\mathcal{G}_{[a,b]}$ *satisfies* u, *unless* $c>b_j^i$ *for each* $i,j>0$*;*

b) $\mathbf{M},[b_j^0,b_j^{k_j}]\Vdash\mathsf{Id}$ *and no other interval* $[c,d]\in\mathcal{G}_{[a,b]}$ *satisfies* Id, *unless* $c>b_j^i$ *for each* $i,j>0$*;*

c) $\mathbf{M},[b_j^0,b_j^1]\Vdash *$, *and no other interval* $[c,d]\in\mathcal{G}_{[a,b]}$ *satisfies* $*$, *unless* $c>b_j^i$ *for each* $i,j>0$*;*

d) $\mathbf{M},[b_j^i,b_j^{i+1}]\Vdash\mathsf{tile}$ *for each* $1\le i<k_j$, *and no other interval* $[c,d]\in\mathcal{G}_{[a,b]}$ *satisfies* tile, *unless* $c>b_j^i$ *for each* $i,j>0$*;*

e) $k_1=2$ *and* $k_l>2$ *for each* $l>1$.

Proof. a) The existence of such a sequence is guaranteed by (1) and the left conjunct of (2). Now, suppose, for contradiction, that there exists an interval $[c,d]\in\mathcal{G}_{[a,b]}$

satisfying u such that $c \leq b_j^i$ for some $i, j > 0$, and $[c, d] \neq [b_j^i, b_j^{i+1}]$ for each $i \geq 0$ and for each $j > 0$. We distinguish the following cases:

- if $[c, d] = [a, b]$ or $a < c < b < d$, then we have a contradiction with (1);
- if $c = b_j^i$ for some $i \geq 0$, $j > 0$, then we have $d \neq b_j^{i+1}$. By (1), there exists a point a' such that $a < a' < b$. If $d < b_j^{i+1}$, $[a', d]$ meets the u-interval starting at d, by the first conjunct of (2), and overlaps the u-interval $[b_j^i, b_j^{i+1}]$, contradicting (3). Otherwise, if $d > b_j^{i+1}$, then $[a', b_j^{i+1}]$ meets the u-interval starting at b_j^{i+1} and overlaps the u-interval $[c, d]$, contradicting (3);
- if $c \neq b_j^i$ for each $i \geq 0$, $j > 0$, then we have $b_j^i < c < b_j^{i+1}$ for some $i \geq 0$ and $j > 0$. In this case, the interval $[a', c]$ meets the u-interval starting at c and overlaps the u-interval $[b_j^i, b_j^{i+1}]$, contradicting (3).

b) The existence of a Id-chain is guaranteed by (5), (6), and (8). Now suppose, for contradiction, that there exists an interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfying Id such that $c \leq b_j^i$ for some $i, j > 0$, and $[c, d] \neq [b_j^0, b_j^{k_j}]$ for each $j > 0$. Notice that c and d start a $*$-interval by (5) and (6), respectively. We must consider the following cases:

- if $[c, d] = [a, b]$ or $a < c < b < d$, then we have a contradiction with (8);
- if $c = b_j^0$ for some $j > 0$, then we have $d \neq b_{j+1}^0$. If $d < b_{j+1}^0$, $[a', d]$ meets the $*$-interval starting at d and overlaps the Id-interval $[b_j^0, b_{j+1}^0]$, contradicting (7). Otherwise, if $d > b_{j+1}^0$, then $[a', b_{j+1}^0]$ meets the $*$-interval starting at b_{j+1}^0 and overlaps the Id-interval $[c, d]$, contradicting (7);
- if $c \neq b_j^0$ for each $j > 0$, then we have $b_j^i < c < b_j^{i+1}$ for some $i \geq 0$ and $j > 0$. In this case, the interval $[a', c]$ meets the $*$-interval starting at c and overlaps the Id-interval $[b_j^0, b_{j+1}^0]$, contradicting (7).

c) The first u-interval of each Id-interval ($[b_j^0, b_j^1]$) is a $*$-interval by (5). Now suppose, for contradiction, that there exists an interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfying $*$ such that $c \leq b_j^i$ for some $i, j > 0$, and $[c, d] \neq [b_j^0, b_j^1]$ for each $j > 0$. By point a) of this lemma and by (4), we have that $[c, d] = [b_j^i, b_j^{i+1}]$ for some $i, j > 0$. Since the interval $[a', c]$ meets the $*$-interval $[b_j^i, b_j^{i+1}]$ and overlaps the Id-interval $[b_j^0, b_j^{k_j}]$, we have a contradiction with (7).

d) By point c) of this lemma and (4), we can conclude that for each $i, j > 0$, $[b_j^i, b_j^{i+1}]$ satisfies tile. Moreover, by point c), (4), and point a), we have that no other interval $[c, d] \in \mathcal{G}_{a,b}$ is a tile-interval, unless $c > b_j^i$ for each $i, j > 0$.

e) $k_1 = 2$ and $k_l > 2$ for each $l > 1$ immediately follows from (8). □

3.4. The above-neighbor relation

We focus now on the 'above-neighbor' relation, whose encoding is shown in Fig. 2. Intuitively, the above-neighbor relation connects each tile-interval with its vertical neighbor in the octant (e.g., $t_{2,2}$ with $t_{2,3}$ in Fig. 2).

We distinguish between *backward* and *forward* Id-intervals, which alternate, by labeling each u-interval either with b, if it belongs to a backward Id-interval, or with f, if it belongs to a forward one (formulas from (10) to (12)). Intuitively, we have that the tile-intervals are placed in ascending order in forward Id-intervals and in descending order in backward Id-intervals. In particular, this means that the leftmost tile-interval of a backward Id-interval represents the last tile of that level (and not the first one) in the octant plane. Let $\alpha, \beta \in \{\mathsf{b}, \mathsf{f}\}$, with $\alpha \neq \beta$:

$$\langle A \rangle \mathsf{b} \wedge [G]((\mathsf{u} \leftrightarrow \mathsf{b} \vee \mathsf{f}) \wedge (\mathsf{b} \rightarrow \neg \mathsf{f})) \quad (10)$$

$$[G](\mathsf{u} \wedge \alpha \wedge \neg \langle A \rangle * \rightarrow \langle A \rangle \alpha) \quad (11)$$

$$[G](\mathsf{u} \wedge \alpha \wedge \langle A \rangle * \rightarrow \langle A \rangle \beta) \quad (12)$$

$$(10) \wedge \ldots \wedge (12) \quad (13)$$

Lemma 2. *If* $\mathbf{M}, [a, b] \Vdash (9) \wedge (13)$*, then there exists a sequence of points like that defined in Lemma 1 such that* $\mathbf{M}, [b_j^i, b_j^{i+1}] \Vdash \mathsf{b}$ *if and only* j *is an odd number and* $\mathbf{M}, [b_j^i, b_j^{i+1}] \Vdash \mathsf{f}$ *if and only* j *is an even number. Furthermore, we have that no other interval* $[c, d] \in \mathcal{G}_{[a,b]}$ *satisfies* b *or* f*, unless* $c > b_j^i$ *for each* $i, j > 0$.

Because of the alternation between forward and backward Id-intervals, the encoding of the above-neighbor relation is not simple. For example, in Fig. 2b, consider the 3rd and the 4th Id's: the 1st tile of the 3rd Id ($t_{3,3}$) is connected with the next-to-last tile of the 4th Id ($t_{3,4}$), the 2nd tile of the 3rd Id ($t_{2,3}$) is connected with the third from last tile of the 4th Id ($t_{2,4}$), and so on. Notice that, in the forward (resp., backward) Id-intervals, the last (resp., first) tile-interval has no tile-intervals connected above with it, to constrain each level of the octant to have exactly one tile more than the previous one (these tile-intervals are labeled with last).

We define the above-neighbor relation as follows. If $[b_j^i, b_j^{i+1}]$ is a tile-interval belonging to a forward (resp., backward) Id-interval, then we say that it is *above-connected* with the tile-interval $[b_{j+1}^{k_{(j+1)}-i}, b_{j+1}^{k_{(j+1)}-i+1}]$ (resp., $[b_{j+1}^{k_{(j+1)}-i-1}, b_{j+1}^{k_{(j+1)}-i}]$). We capture this situation by labelling with $\mathsf{up\_rel}$ the interval $[b_j^{i+1}, b_{j+1}^{k_{(j+1)}-i}]$ (resp., $[b_j^{i+1}, b_{j+1}^{k_{(j+1)}-i-1}]$). Moreover, we distinguish between $\mathsf{up\_rel}$-intervals starting from a forward and a backward Id-interval and, for each one of these cases, between those starting from an odd and an even tile-interval.

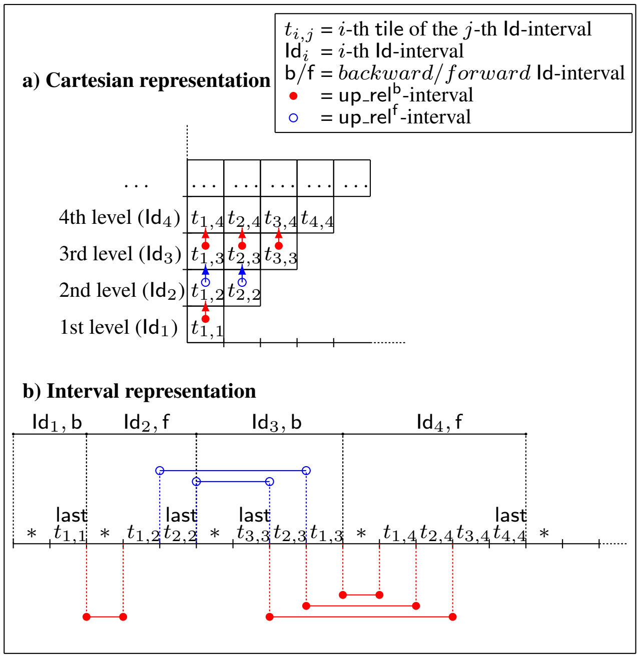

Figure 2. The *above-neighbor* **relation encoded in the fragment** AO**.**

To this end, we use a new propositional letter, namely, $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}}$ (resp., $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$, $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$, $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$) to label up_rel-intervals starting from an odd tile-interval of a backward Id-interval (resp., even tile-interval/backward Id-interval, odd/forward, even/forward). Moreover, to ease the reading of the formulas, we group $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}}$ and $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$ in $\mathsf{up\_rel}^{\mathsf{b}}$ ($\mathsf{up\_rel}^{\mathsf{b}} \leftrightarrow \mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}} \oplus \mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$), and similarly for $\mathsf{up\_rel}^{\mathsf{f}}$. Finally, up_rel is exactly one among $\mathsf{up\_rel}^{\mathsf{b}}$ and $\mathsf{up\_rel}^{\mathsf{f}}$ ($\mathsf{up\_rel} \leftrightarrow \mathsf{up\_rel}^{\mathsf{b}} \oplus \mathsf{up\_rel}^{\mathsf{f}}$). From the above conditions, it follows that the up_rel-intervals between any pair of consecutive Id-intervals are placed the one strictly contained in the other.

Let $\alpha, \beta \in \{\mathsf{b}, \mathsf{f}\}$ and $\gamma, \delta \in \{\mathsf{o}, \mathsf{e}\}$, with $\alpha \neq \beta$ and $\gamma \neq \delta$:

$$[G](\mathsf{up\_rel} \leftrightarrow \mathsf{up\_rel}^{\mathsf{b}} \vee \mathsf{up\_rel}^{\mathsf{f}}) \tag{14}$$

$$[G](\mathsf{up\_rel}^{\alpha} \leftrightarrow \mathsf{up\_rel}^{\alpha}_{\mathsf{o}} \vee \mathsf{up\_rel}^{\alpha}_{\mathsf{e}}) \tag{15}$$

$$[G](\langle A\rangle \mathsf{up\_rel}^{\alpha}_{\gamma} \rightarrow \neg\langle A\rangle \mathsf{up\_rel}^{\alpha}_{\delta} \wedge \neg\langle A\rangle \mathsf{up\_rel}^{\beta}) \tag{16}$$

$$[G](\mathsf{up\_rel}^{\alpha} \rightarrow \neg\langle O\rangle \mathsf{up\_rel}^{\alpha}) \tag{17}$$

$$[G](\mathsf{up\_rel}^{\alpha}_{\gamma} \rightarrow \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle \mathsf{up\_rel}^{\beta}_{\gamma})) \tag{18}$$

$$(14) \wedge \ldots \wedge (18) \tag{19}$$

Lemma 3. *If* $\mathbf{M}, [a,b] \Vdash (9) \wedge (13) \wedge (19)$*, then there exists a sequence of points like that defined in Lemma 1 such that, for each* $i \geq 0$*,* $j > 0$*, the following properties hold:*

a) $[b^i_j, b^{i'}_{j'}]$ *satisfies* up_rel *if and only if it satisfies exactly one between* $\mathsf{up\_rel}^{\alpha}$ *and* $\mathsf{up\_rel}^{\beta}$ *and* $[b^i_j, b^{i'}_{j'}]$ *satisfies* $\mathsf{up\_rel}^{\mathsf{b}}$ *(resp.,* $\mathsf{up\_rel}^{\mathsf{f}}$*) if and only if it satisfies exactly one between* $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}}$ *and* $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$ *(resp., between* $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$ *and* $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$*);*

b) for each $\alpha, \beta \in \{\mathsf{b}, \mathsf{f}\}$ *and* $\gamma, \delta \in \{\mathsf{o}, \mathsf{e}\}$*, if* $[b^i_j, b^{i'}_{j'}]$ *satisfies* $\mathsf{up\_rel}^{\alpha}_{\gamma}$*, then there is no other interval starting at* b^i_j *satisfying* $\mathsf{up\_rel}^{\beta}_{\delta}$ *such that* $\mathsf{up\_rel}^{\alpha}_{\gamma} \neq \mathsf{up\_rel}^{\beta}_{\delta}$*;*

c) each $\mathsf{up\_rel}^{\mathsf{b}}$*-interval (resp.,* $\mathsf{up\_rel}^{\mathsf{f}}$*-interval) does not overlap any other* $\mathsf{up\_rel}^{\mathsf{b}}$*-interval (resp.,* $\mathsf{up\_rel}^{\mathsf{f}}$*-interval);*

d) if $[b^i_j, b^{i'}_{j'}]$ *satisfies* $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}}$ *(resp.,* $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$*,* $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$*,* $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$*), then* $[b^{i'}_{j'}, b^{i'+1}_{j'}]$ *satisfies* tile *and there exists a* $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$*-interval (resp.,* $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$*-interval,* $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}}$*-interval,* $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$*-interval) starting at* $b^{i'+1}_{j'}$*.*

Now we constrain each tile-interval, except for the one representing the last tile of some level, to have a tile-interval above-connected with it. To this end, we label the tile-intervals representing the last tile of some level in the octant with the new propositional letter last (formulas from (28) to (30)). Next, we force all and only tile-intervals not labelled with last to have a tile-interval above-connected with them (formulas from (31) to (34)):

$$\neg\mathsf{up\_rel} \wedge \neg\langle O\rangle \mathsf{up\_rel} \wedge \neg\langle A\rangle \mathsf{up\_rel} \tag{20}$$

$$[G](\mathsf{tile} \rightarrow \langle A\rangle \mathsf{up\_rel}) \tag{21}$$

$$[G](\langle A\rangle \mathsf{up\_rel} \rightarrow \langle A\rangle \mathsf{u}) \tag{22}$$

$$[G](\mathsf{u} \wedge \langle A\rangle \mathsf{up\_rel} \rightarrow \mathsf{tile}) \tag{23}$$

$$[G](\alpha \rightarrow [A](\mathsf{up\_rel} \rightarrow \mathsf{up\_rel}^{\alpha})) \tag{24}$$

$$[G](\mathsf{up\_rel}^{\alpha} \rightarrow \langle A\rangle \beta) \tag{25}$$

$$[G](\langle O\rangle * \rightarrow \neg(\langle O\rangle \mathsf{up\_rel}^{\mathsf{b}} \wedge \langle O\rangle \mathsf{up\_rel}^{\mathsf{f}})) \tag{26}$$

$$[G](\langle A\rangle \mathsf{up\_rel}^{\alpha}_{\gamma} \wedge \langle A\rangle \mathsf{tile} \rightarrow \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle \mathsf{up\_rel}^{\alpha}_{\delta})) \tag{27}$$

$$[G](\mathsf{last} \rightarrow \mathsf{tile}) \tag{28}$$

$$[G]((* \wedge \mathsf{b} \rightarrow \langle A\rangle \mathsf{last}) \wedge (\mathsf{f} \wedge \langle A\rangle * \rightarrow \mathsf{last})) \tag{29}$$

$$[G]((\mathsf{last} \wedge \mathsf{f} \rightarrow \langle A\rangle *) \wedge (\mathsf{b} \wedge \langle A\rangle \mathsf{last} \rightarrow *)) \tag{30}$$

$$[G](* \wedge \mathsf{f} \rightarrow \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle(\mathsf{up\_rel} \wedge \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle *)))) \tag{31}$$

$$[G](\mathsf{last} \wedge \mathsf{b} \rightarrow \langle A\rangle(\mathsf{up\_rel} \wedge \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle *)))) \tag{32}$$

$$[G](\langle O\rangle(\mathsf{u} \wedge \langle A\rangle \mathsf{up\_rel}^{\alpha}_{\gamma}) \rightarrow [A](\langle O\rangle \mathsf{up\_rel}^{\alpha}_{\gamma} \wedge \langle O\rangle (\mathsf{u} \wedge \langle A\rangle(\mathsf{u} \wedge \neg\mathsf{last} \wedge \langle A\rangle \mathsf{up\_rel}^{\beta}_{\delta})) \rightarrow \langle O\rangle \mathsf{up\_rel}^{\alpha}_{\delta})) \tag{33}$$

$$[G](\mathsf{up\_rel} \rightarrow \neg\langle A\rangle \mathsf{last}) \tag{34}$$

$$(20) \wedge \ldots \wedge (34) \tag{35}$$

Lemma 4. *If* $\mathbf{M}, [a,b] \Vdash (9) \wedge (13) \wedge (19) \wedge (35)$*, then there exists a sequence of points like that defined in Lemma 1 such* *that the following properties hold:*

a) for each up_rel*-interval* $[c, d]$*, there exist* c', d' *such that* $[c', c]$ *and* $[d, d']$ *are* tile*-intervals and if* $[c, d]$ *satisfies* $\mathsf{up\_rel}^{\mathsf{b}}$ *(resp.,* $\mathsf{up\_rel}^{\mathsf{f}}$*), then* $[c', c]$ *satisfies* b *(resp.,* f*) and* $[d, d']$ *satisfies* f *(resp.,* b*);*

b) (strict alternation property) for each tile*-interval* $[b^i_j, b^{i+1}_j]$*, with* $i < k_j - 1$*, such that there exists a* $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}}$*-interval (resp.,* $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$*-interval,* $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$*-interval,* $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$*-interval) starting at* b^{i+1}_j*, there exists a* $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$*-interval (resp.,* $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}}$*-interval,* $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$*-interval,* $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$*-interval) starting at* b^{i+2}_j*;*

c) for every tile*-interval* $[b^i_j, b^{i+1}_j]$ *satisfying* last*, there is no* up_rel*-interval ending at* b^i_j*;*

d) for each up_rel*-interval* $[b^i_j, b^{i'}_{j'}]$*, with* $1 < i \le k_j$*, we have that* $j' = j + 1$*.*

Proof. a) Let $[c, d]$ be a up_rel-interval. By (18), we have that there exists d' such that $[d, d']$ is a tile-interval and by (22), (23), and Lemma 1, there exists c' such that $[c', c]$ is a tile-interval. Now, suppose that $[c, d]$ satisfies $\mathsf{up\_rel}^{\mathsf{b}}$ (the other case is symmetric) and that $[c', c]$ satisfies f. Then, (24) is contradicted. Similarly, if $[d, d']$ satisfies b, then (25) is contradicted.

b) Straightforward, by (27);

c) Straightforward, by (34);

d) Let $[b^i_j, b^{i'}_{j'}]$ be a up_rel-interval, with $1 < i \le k_j$, and suppose, for contradiction, that $j' \neq j + 1$. Suppose that $[b^i_j, b^{i'}_{j'}]$ is a $\mathsf{up\_rel}^{\mathsf{b}}$-interval (the other case is symmetric). By point a) of this lemma, we have that $[b^{i-1}_j, b^i_j]$ satisfies b and that $[b^{i'}_{j'}, b^{i'+1}_{j'}]$ satisfies f. Two cases are possible:

(i) if $j' = j$, then $[b^{i-1}_j, b^i_j]$ and $[b^{i'}_{j'}, b^{i'+1}_{j'}]$ belong to the same Id-interval. By Lemma 2, they must be both labelled with b or f, against the hypothesis;

(ii) if $j' > j + 1$, then consider a tile-interval $[b^h_{j+1}, b^{h+1}_{j+1}]$ belonging to the $(j + 1)$-th Id-interval. By Lemma 2, we have that $[b^h_{j+1}, b^{h+1}_{j+1}]$ satisfies f (since $[b^{i-1}_j, b^i_j]$ satisfies b) and, by (21) and (24), we have that there is a $\mathsf{up\_rel}^{\mathsf{f}}$-interval starting at b^{h+1}_{j+1} and ending at some point $b^{h'}_{j''}$ for some $j'' > j + 1$ (by point (i)). Consider the $*$-interval $[b^0_{j+2}, b^1_{j+2}]$. By the right conjunct of (2), there exists a point b' such that $b^0_{j+2} < b' < b^1_{j+2}$. Thus, the interval $[b, b']$ overlaps the $*$-interval $[b^0_{j+2}, b^1_{j+2}]$, the $\mathsf{up\_rel}^{\mathsf{f}}$-interval $[b^{h+1}_{j+1}, b^{h'}_{j''}]$ and the $\mathsf{up\_rel}^{\mathsf{b}}$-interval $[b^i_j, b^{i'}_{j'}]$, contradicting (26).

Hence, the only possibility is $j' = j + 1$. □

Lemma 5. *Each* tile*-interval* $[b^i_j, b^{i+1}_j]$ *is above-connected with exactly one* tile*-interval and if* $[b^i_j, b^{i+1}_j]$ *does not satisfy* last*, then there exists exactly one* tile*-interval which is above-connected with it.*

Proof. First of all, we observe that each tile-interval is above-connected with at least one tile, by (21) and by Lemma 4, item (a). Now suppose, for contradiction, that there exists a tile-interval $[b^i_j, b^{i+1}_j]$ not satisfying last and such that there is no tile-interval above-connected with it. If it is the rightmost interval of the j-th Id-interval not satisfying last (base case) and it satisfies f (resp., b), then we have that $i = k_j - 2$ (resp., $i = k_j - 1$) and (32) (resp., (31)) guarantees the existence of a up_rel-interval ending at b^i_j, leading to a contradiction. Now, suppose that there is a up_rel-interval ending at b^{i+1}_j and starting at some point $b^{i'}_{j-1}$ (inductive case). Without loss of generality, suppose that $[b^{i'}_{j-1}, b^{i+1}_j]$ satisfies $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$. Then, by Lemma 3, item (d), there exists a $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}}$-interval starting at b^{i+2}_j and, by the strict alternation property (Lemma 4, item (b)), there exists a $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$-interval starting at b^{i+1}_j. Let c be a point such that $b^{i'-1}_{j-1} < c < b^{i'}_{j-1}$ (the existence of such a point is guaranteed by the right conjunct of (2)). Similarly, let d be a point such that $b^{i-1}_j < d < b^i_j$. We show that, by applying (33) to any interval ending in c and starting before than $b^{i'-1}_{j-1}$, say $[c', c]$, we get a contradiction. Indeed, $[c', c]$ satisfies $\langle O\rangle(\mathsf{u} \wedge \langle A\rangle \mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}})$ and it meets $[c, d]$, which satisfies the following formulas:

- $\langle O\rangle \mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$: $[b^{i'}_{j-1}, b^{i+1}_j]$ satisfies $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$;
- $\langle O\rangle(\mathsf{u} \wedge \langle A\rangle(\mathsf{u} \wedge \neg\mathsf{last} \wedge \langle A\rangle \mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}))$: the interval $[b^{i-1}_j, b^i_j]$ satisfies u and meets the u-interval $[b^i_j, b^{i+1}_j]$, which does not satisfy last (by hypothesis) and whose right endpoint starts a $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$-interval.

We show that $[c, d]$ does not satisfy the formula $\langle O\rangle \mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$, getting a contradiction with (33). Suppose that there exists an interval $[e, f]$ satisfying $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$ and such that $c < e < d < f$. We distinguish the following cases:

- if $f > b^{i+1}_j$ and $e > b^{i'}_{j-1}$, then the $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$-interval $[b^{i'}_{j-1}, b^{i+1}_j]$ overlaps the $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$-interval $[e, f]$, contradicting Lemma 3, item (c);
- if $f > b^{i+1}_j$ and $e = b^{i'}_{j-1}$, then there are a $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$- and a $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$-interval starting at $b^{i'}_{j-1}$, contradicting Lemma 3, item (b);
- if $f = b^{i+1}_j$, then there are a $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$- and a $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$-interval ending at b^{i+1}_j and, by Lemma 3, item (d), there are a $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}}$- and a $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$-interval starting at b^{i+2}_j, contradicting Lemma 3, item (b);
- finally, if $f = b^i_j$, we have a contradiction with the hypothesis.

Thus, there exists no such an interval, contradicting (33).

This proves that each tile-interval is above-connected with at least one tile-interval and if it does not satisfy last, then there exists at least one tile-interval above-connected with it. Now, we show that such connections are unique.

Suppose, for contradiction, that for some $[b^i_j, b^{i'}_{j+1}]$ and $[b^i_j, b^{i''}_{j+1}]$, with $b^{i'}_{j+1} < b^{i''}_{j+1}$ (the case $b^{i'}_{j+1} > b^{i''}_{j+1}$ is symmetric), we have that both $[b^i_j, b^{i'}_{j+1}]$ and $[b^i_j, b^{i''}_{j+1}]$ are $\mathsf{up\_rel}$-intervals. By Lemma 3 and Lemma 4, we have that they both satisfy the same propositional letter among $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$, $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$, $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}}$ and $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$, say $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$ (the other cases are symmetric). Then both $b^{i'+1}_{j+1}$ and $b^{i''+1}_{j+1}$ start a $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}}$-interval by Lemma 3, item (d). By the strict alternation property, a $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$-interval starts at the point $b^{i'+2}_{j+1}$. Since $[b^{i'+1}_{j+1}, b^{i'+2}_{j+1}]$ is not the rightmost tile of the $(j+1)$-th Id-interval, then, as we have already shown, there exists a point c such that $[c, b^{i'+1}_{j+1}]$ is a $\mathsf{up\_rel}$-interval. By Lemma 3, items (d) and (b), we have that $[c, b^{i'+1}_{j+1}]$ is a $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$-interval. We show that the existence of such an interval leads to a contradiction:

- if $c < b^i_j$, then the $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$-interval $[c, b^{i'+1}_{j+1}]$ overlaps the $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$-interval $[b^i_j, b^{i''}_{j+1}]$, contradicting Lemma 3, item (c);
- if $c = b^i_j$, then b^i_j starts both a $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$- and a $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$-interval, contradicting Lemma 3, item (b);
- if $c > b^i_j$, then the $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$-interval $[b^i_j, b^{i'}_{j+1}]$ overlaps the $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$-interval $[c, b^{i'+1}_{j+1}]$, contradicting Lemma 3, item (c).

In a similar way, we can prove that it cannot happen that two distinct $\mathsf{up\_rel}$-intervals end at the same point. □

3.5. The right-neighbor relation

Intuitively, the right-neighbor relation connects each tile-interval with its horizontal neighbor in the octant, if any (e.g., $t_{2,3}$ with $t_{3,3}$ in Fig. 2).

Again, we must distinguish between forward and backward Id-intervals: a tile-interval belonging to a forward (resp., backward) Id-interval is *right connected* with the tile-interval immediately on its right (resp., left), if any. For example, in Fig. 2b, the 2nd tile-interval of the 4th Id-interval ($t_{2,4}$) is right connected with the tile-interval immediately on its right ($t_{3,4}$), since the 4th Id-interval is a forward one, while the 2nd tile-interval of the 3rd Id-interval ($t_{2,3}$) is right connected with the tile-interval immediately on its left ($t_{3,3}$), since the 3rd Id-interval is a backward one.

As a consequence, we define the right-neighbor relation as follows. If $[b^i_j, b^{i+1}_j]$ is a tile-interval belonging to a forward (resp., backward) Id-interval, with $i \neq k_j - 1$ (resp., $i \neq 1$), then we say that it is *right connected* with the tile-interval $[b^{i+1}_j, b^{i+2}_j]$ (resp., $[b^{i-1}_j, b^i_j]$).

Lemma 6 (Commutativity property). *If* $\mathbf{M}, [a,b] \Vdash (9) \wedge (13) \wedge (19) \wedge (35)$*, then there exists a sequence of points like that defined in Lemma 1 such that the following commutativity property holds: given two* tile*-intervals* $[c,d]$ *and* $[e,f]$*, if there exists a* tile*-interval* $[d_1, e_1]$*, such that* $[c,d]$ *is right connected with* $[d_1,e_1]$ *and* $[d_1,e_1]$ *is above-connected with* $[e,f]$*, then there exists also a* tile*-interval* $[d_2,e_2]$ *such that* $[c,d]$ *is above-connected with* $[d_2,e_2]$ *and* $[d_2,e_2]$ *is right connected with* $[e,f]$.

3.6. Tiling the plane

The following formulas constrain each tile-interval (and no other interval) to be tiled by exactly one tile ((36) and (37)) and constrain the tiles that are right or above-connected to respect the color constraints (from (38) to (40)):

$$[G]((\bigvee_{i=1}^{k} \mathtt{t_i}) \leftrightarrow \mathsf{tile}) \tag{36}$$

$$[G] \bigwedge_{i,j=1, i\neq j}^{k} \neg(\mathtt{t_i} \wedge \mathtt{t_j}) \tag{37}$$

$$[G](\mathsf{tile} \rightarrow \bigvee_{up(t_i)=down(t_j)} (\mathtt{t_i} \wedge \langle A\rangle(\mathsf{up\_rel} \wedge \langle A\rangle \mathtt{t_j}))) \tag{38}$$

$$[G](\mathsf{tile} \rightarrow \bigvee_{right(t_i)=left(t_j)} (\mathtt{t_i} \wedge \mathsf{f} \wedge \langle A\rangle \mathsf{tile} \wedge \langle A\rangle \mathtt{t_j})) \tag{39}$$

$$[G](\mathsf{tile} \rightarrow \bigvee_{left(t_i)=right(t_j)} (\mathtt{t_i} \wedge \mathsf{b} \wedge \langle A\rangle \mathsf{tile} \wedge \langle A\rangle \mathtt{t_j})) \tag{40}$$

$$(36) \wedge \ldots \wedge (40) \tag{41}$$

Given the set of tiles $T = \{t_1, t_2, \ldots, t_k\}$, let $\Phi_{\mathcal{T}}$ be the formula:

$$(9) \wedge (13) \wedge (19) \wedge (35) \wedge (41)$$

Lemma 7. *Given any finite set of tile types* $\mathcal{T} = \{t_1, t_2, \ldots, t_k\}$*, the formula* $\Phi_{\mathcal{T}}$ *is satisfiable if and only if* $\mathcal{T}$ *can tile the second octant* $\mathcal{O}$.

Since the above construction can be carried out on any linear ordering containing an infinite ascending chain of points, such as, for instance, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, the following theorem holds.

Theorem 1. *The satisfiability problem for the logic* AO *is undecidable over any class of linear orders that contains at least one linear order with an infinite ascending sequence.*

3.7. Undecidability of $\mathsf{A\overline{O}}$, $\mathsf{\overline{A}O}$, and $\mathsf{\overline{A}\,\overline{O}}$

To prove the undecidability of the logic $\mathsf{A\overline{O}}$ we can exploit the same construction we use for AO, provided that we change the formulas containing the operators $\langle O\rangle$ or $[O]$ as follows. In formulas (1), (2), (17), and (26), we replace all occurrences of the operator $\langle O\rangle$ with $\langle \overline{O}\rangle$. Besides, we replace formulas (3), (7), and (33) by the following ones:

$$[G](\mathsf{u} \rightarrow \neg\langle\overline{O}\rangle\langle A\rangle\mathsf{u}) \quad (42)$$

$$[G](\mathsf{Id} \rightarrow \neg\langle\overline{O}\rangle\langle A\rangle *) \quad (43)$$

$$[G](\mathsf{tile} \wedge \neg\mathsf{last} \rightarrow [\overline{O}](\langle\overline{O}\rangle\mathsf{u} \wedge \neg\langle\overline{O}\rangle\mathsf{Id} \rightarrow \langle\overline{O}\rangle\mathsf{up\_rel})) \quad (44)$$

Theorem 2. *The satisfiability problem for the logic* $\mathsf{A}\overline{\mathsf{O}}$ *is undecidable over any class of linear orders that contains at least one linear order with an infinite ascending sequence.*

The previous reductions can easily be extended to the logics $\overline{\mathsf{A}}\mathsf{O}$ and $\overline{\mathsf{A}}\,\overline{\mathsf{O}}$, provided that the class of models contains at least one linear order with an infinite descending sequence of points, such as, for instance, $\mathbb{Z}^{\leq 0}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

Theorem 3. *The satisfiability problem for the logics* $\overline{\mathsf{A}}\mathsf{O}$ *and* $\overline{\mathsf{A}}\,\overline{\mathsf{O}}$ *is undecidable over any class of linear orders that contains at least one linear order with an infinite descending sequence.*

4. Other undecidable logics

Using analogous constructions we have proved the undecidability of the logics $\mathsf{B}^*\mathsf{O}^*$, $\mathsf{E}^*\mathsf{O}^*$, and $\mathsf{D}^*\mathsf{O}^*$. The obtained results are summarized by the following theorem.

Theorem 4. *The satisfiability problem for the logics* BO, $\overline{\mathsf{B}}\mathsf{O}$, EO, $\overline{\mathsf{E}}\mathsf{O}$, DO, *and* $\overline{\mathsf{D}}\mathsf{O}$ *(resp.,* $\mathsf{B}\overline{\mathsf{O}}$, $\overline{\mathsf{B}}\,\overline{\mathsf{O}}$, $\mathsf{E}\overline{\mathsf{O}}$, $\overline{\mathsf{E}}\,\overline{\mathsf{O}}$, $\mathsf{D}\overline{\mathsf{O}}$, *and* $\overline{\mathsf{D}}\,\overline{\mathsf{O}}$*) is undecidable over any class of linear orders that contains at least one linear order with an infinite ascending (resp., descending) sequence.*

We have also constructed a similar reduction for the logic $\mathsf{O}\overline{\mathsf{O}}$ interpreted on discrete linear orders, as formally stated by the following theorem.

Theorem 5. *The satisfiability problem for the logic* $\mathsf{O}\overline{\mathsf{O}}$ *is undecidable over any class of discrete linear orders that contains at least one linear order with an infinite ascending or descending sequence.*

The details of these constructions will be included in a forthcoming extended version.

5. Conclusions and future work

In this paper, we have shown that most extensions of the logics O and $\overline{\mathsf{O}}$ are undecidable. The undecidability proof for the various logics has essentially the same structure, based on a suitable reduction from the octant tiling problem. The only extensions for which the decision problem remains open are those of the form $\mathsf{L}^*\mathsf{O}^*$. The most interesting related open problem, however, is that concerning the decidability status of the logic O (and respectively $\overline{\mathsf{O}}$). They are the only one-modality fragments of HS for which we do not have yet any positive decidability result on important classes of linear orders.

References

[1] J. F. Allen. Maintaining knowledge about temporal intervals. *Communications of the ACM*, 26(11):832–843, 1983.

[2] E. Börger, E. Grädel, and Y. Gurevich. *The Classical Decision Problem*. Springer, 1997.

[3] D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco. Decidable and Undecidable Fragments of Halpern and Shoham's Interval Temporal Logic: Towards a Complete Classification. In *Proc. of 15th Int. Conf. on Logic for Programming, Artificial Intelligence, and Reasoning*, volume 5330 of *LNCS*, pages 590–604. Springer, 2008.

[4] D. Bresolin, V. Goranko, A. Montanari, and P. Sala. Tableau-based decision procedures for the logics of subinterval structures over dense orderings. *Journal of Logic and Computation*, doi:10.1093/logcom/exn063, 2008.

[5] D. Bresolin, V. Goranko, A. Montanari, and G. Sciavicco. On Decidability and Expressiveness of Propositional Interval Neighborhood Logics. In *Proc. of the International Symposium on Logical Foundations of Computer Science*, volume 4514 of *LNCS*, pages 84–99. Springer, 2007.

[6] D. Bresolin, V. Goranko, A. Montanari, and G. Sciavicco. Propositional Interval Neighborhood Logics: Expressiveness, Decidability, and Undecidable Extensions. Submitted, 2008.

[7] D. Bresolin, A. Montanari, P. Sala, and G. Sciavicco. Optimal Tableaux for Right Propositional Neighborhood Logic over Linear Orders. In *Proc. of the 11th European Conference on Logics in AI*, volume 5293 of *LNAI*, pages 62–75. Springer, 2008.

[8] Z. Chaochen, C. A. R. Hoare, and A. P. Ravn. A calculus of durations. *Inf. Proc. Letters*, 40(5):269–276, 1991.

[9] C. J. Coetzee. *Representation Theorems for Classes of Interval Structures, Master Thesis*. University of Johannesburg, South Africa, 2009.

[10] D. Gabbay, A. Kurusz, F. Wolter, and M. Zakharyaschev, editors. *Many-Dimensional Modal Logics: Theory and Applications*. Elsevier, 2003.

[11] J. Halpern and Y. Shoham. A propositional modal logic of time intervals. *Journal of the ACM*, 38(4):935–962, 1991.

[12] I. Hodkinson, A. Montanari, and G. Sciavicco. Non-finite axiomatizability and undecidbility of interval temporal logics with C, D, and T. In *Proc. of the 17th Conference of the European Association Computer Science Logic*, volume 5213 of *LNCS*, pages 308–322. Springer, 2008.

[13] C. Lutz and F. Wolter. Modal logics of topological relations. *Logical Methods in Computer Science*, 2(2), 2006.

[14] M. Marx and M. Reynolds. Undecidability of compass logic. *Journal of Logic and Computation*, 9(6):897–914, 1999.

[15] M. Marx and Y. Venema. *Multi-Dimensional Modal Logics*. Kluwer Academic Press, 1997.

[16] B. Moszkowski. *Reasoning about digital circuits*. Tech. rep. stan-cs-83-970, Dept. of Computer Science, Stanford University, Stanford, CA, 1983.