Abstract. We prove results about completeness of the first-order theories of some classes of trees with respect to the subclasses of finitely branching trees.

TREES AND FINITE BRANCHING

VALENTIN GORANKO

1. Introduction

Here we study the following type of a question: *given a class of trees $\mathcal{K}$, is the first-order theory of $\mathcal{K}$ complete with respect to the finitely branching trees in $\mathcal{K}$*, i.e., is it true that every first-order sentence satisfiable in $\mathcal{K}$ is satisfiable in a finitely branching member of $\mathcal{K}$? The answer to that question has important consequences, both logical and computational for the first-order theory of $\mathcal{K}$, and the current study was originally motivated from a problem arising in the study of computation tree logics.

In this paper we develop some techniques and prove some general positive "finite branching completeness" results for a number of classes of trees.

2. Preliminaries

2.1. **Terminology and notation.** Here we summarize some basic definitions and results which will be used further.

For reference on the model-theoretic background see e.g. [Ebbinghaus et al, 94], or [Hodges, 93], and for reference on related results about trees and linear orderings —[Doets, 87], [Backofen et al, 95], [Rosenstein, 82].

Definition 2.1. *By a* ***tree*** *we mean any (strictly) partially ordered set with a least element, in which every element has a linearly ordered set of predecessors. The elements of a tree will be called* ***nodes***; *the least node is the* ***root***; *a* ***leaf*** *(or a* ***terminal node****) is any node without successors.*

A ***child*** *(or an* ***immediate successor****) of a node x is a successor of x which is not a successor of any successor of x. If a node has an immediate predecessor, it is called its* ***parent.***

Given a node a in a tree, the set of nodes $\{b|b < a\}$ will be called the ***stem of*** *a. A* ***sibling*** *of a node t in a tree T is any node in T which has the same stem as t. (Thus, every node is a sibling of itself.)*

Given a tree T and a node $t \in T$, we denote by $T(t)$ the ***subtree of*** *T* ***rooted at*** *t, i.e. the tree consisting of t and all successors of t, inheriting the ordering of T. A* ***pruned subtree*** *of a tree T is the tree obtained from T by removing some nodes, in such a way that all siblings of a node are removed if and only if some of their predecessors is removed too.*

A ***path*** *in a tree is any maximal linearly ordered subset. If $\mathbf{p}$ is a path and a is a node on $\mathbf{p}$, the* ***branch of*** *$\mathbf{p}$* ***stemming from*** *a is the set $\{b \in \mathbf{p}|a \leq b\}$.*

If τ is any order type, a ***τ-path*** *in a tree is a path of order type τ.*

Definition 2.2. *A tree is:*

- ***backward discrete*** *if every node which is not the root has an immediate predecessor.*
- ***forward discrete*** *if every node on any path, which is not a leaf, has an immediate successor on that path (i.e. every successor of a node is either a child or a successor of a child of that node).*
- ***discrete*** *if it is both backward and forward discrete.*
- ***well-founded*** *if every path in it has an ordinal order type. Clearly, every well-founded tree is forward discrete. (Note that by a tree some authors mean a well-founded tree.)*
- *a* τ***-tree*** *if every path in it is a* τ*-path, where* τ *is any order type.*
- *an* ***ordinal tree*** *if it is a* τ*-tree where* τ *is an ordinal order type.*

Definition 2.3. *For any order type* τ*, a* τ***-level*** *in a tree* T *is the set of all nodes in* T *whose stems have an order type* τ*.*

Note that for any ordinal α*, the nodes on level* α *are incomparable.*

The ***finite levels*** *in a tree are all* α*-levels for finite ordinals* α*. (Thus, the 0-level consists of the root of the tree.)*

Definition 2.4. *A* ***node in a tree is finitely branched*** *if it has finitely many siblings. A tree is* ***finitely branched on a level*** τ *if all nodes on that level are finitely branched; it is* m***-boundedly branched on a level*** τ*, where* $m \in \mathbf{N}$*, if every node on that level has no more than* m *siblings;* ***boundedly branched,*** *if it is boundedly branched on every level;* ***uniformly boundedly branched on a set of levels*** Λ *if there is an* $m \in \mathbf{N}$ *such that it is* m*-boundedly branched on every level from* Λ*; when* Λ *is the set of all (non-empty) levels of the tree, it is* ***uniformly boundedly branched****.*

Definition 2.5. *A tree is* ***finitely branching*** *if every node is finitely branched. Likewise,* ***(uniformly) boundedly branching*** *trees are defined.*

In this paper we shall consider trees enriched with finitely many additional unary predicates, which will be called **colours**, and the resulting structures **coloured trees**. (This terminology may be a little confusing though, since a node can have more than one colours or none at all.)

Some notation:

- If $\mathcal{K}$ is a class of first-order structures, $TH(\mathcal{K})$ denotes the first-order theory of $\mathcal{K}$.
- If Γ is a first-order theory then $MOD(\Gamma)$ is the class of models of Γ.
- The quantifier rank of a formula ϕ will be denoted by $qr(\phi)$.
- Elementary equivalence will be denoted by $\equiv$, n-equivalence — by $\equiv_n$.

Finally, we introduce some abbreviations in the fist-order language of trees as follows:

- "*y is a successor of x*" will be denoted by $x < y$;
- "*x is the root*": $r(x) := \forall y(x < y \vee x = y)$;
- "*x is a leaf*": $d(x) := \forall y(\neg x < y)$;
- "*y is a child of x*": $c(y, x) := x < y \wedge \neg\exists z(x < z \wedge z < y)$;
- "*y is a sibling of x*": $s(y, x) := \forall z(z < y \leftrightarrow z < x)$;
- "*the level of x is not less than k*", where $k \in \mathbf{N}$: Inductively on k : $l(x, 0) = (x = x)$; $l(x, k+1) = \exists y(c(x, y) \wedge l(y, k))$.
- "*x has no more than k children*": $c_k(x) := \forall y_1 ... \forall y_{k+1}(\bigwedge_{1 \leq i \leq k+1} c(y_i, x) \rightarrow \bigvee_{1 \leq i < j \leq k+1} y_i = y_j)$.

- "*x has no more than k siblings*":
  $s_k(x) := \forall y_1 ... \forall y_{k+1}(\bigwedge_{1\leq i\leq k+1} s(y_i, x) \rightarrow \bigvee_{1\leq i<j\leq k+1} y_i = y_j)$.

2.2. **Some preliminary results.** Besides well-known model theoretic results, incl. compactness, Ehrenfeucht's theorem, and ommiting types theorems, in this paper we shall use the following results.

2.2.1. *Relative completeness.*

Lemma 2.1. *Let $\mathcal{M}$ be a class of structures in a language L of finite signature and $\mathcal{K} \subseteq \mathcal{M}$. Then the following are equivalent:*

1. *$TH(\mathcal{M})$ is complete with respect to $\mathcal{K}$.*
2. *$TH(\mathcal{K}) = TH(\mathcal{M})$.*
3. *Every satisfiable in $\mathcal{M}$ sentence is satisfiable in $\mathcal{K}$.*
4. *For every $n \in \mathbf{N}$, every structure from $\mathcal{M}$ is n-equivalent to a structure from $\mathcal{K}$.*

Proof. We show that (1)⇒(2)⇒(3)⇒(4)⇒(1). All these implications except for (3)⇒(4) are trivial. As for (3)⇒(4), it suffices to note that (see[Ebbinghaus et al, 94]) for every $n \in \mathbf{N}$ and an L-structure A of finite signature there is a sentence $\Phi_{A,n}$ such that for any L-structure B, $B \models \Phi_{A,n}$ iff $A \equiv_n B$. □

Definition 2.6. *A class of first-order structures is* ***co-elementary*** *if its complement is elementary.*

Theorem 2.2. *Let Γ be a theory in a first-order language L, $\{\mathcal{M}_k\}_{k\in\mathbf{N}}$ be a family of co-elementary classes of L-structures and $\mathcal{M}^f = \bigcap_{k\in\mathbf{N}} \mathcal{M}_k$. If Γ is complete with respect to the class $\mathcal{M}_k$ for each $k \in \mathbf{N}$, then Γ is complete with respect to $\mathcal{M}^f$.*

Proof. For each $k \in \mathbf{N}$ we consider the 0-type $\mathbf{t}_k = \left\{\varphi | \overline{\mathcal{M}_k} \models \varphi\right\}$ where $\overline{\mathcal{M}_k}$ is the complement of $\mathcal{M}_k$.

Now, let ψ be a sentence such that $\Gamma \cup \{\psi\}$ is satisfiable. Then each type $\mathbf{t}_k$ is locally omitted (i.e. non-principal) in $\Gamma \cup \{\psi\}$. For, suppose $\Gamma \cup \{\psi\} \models \theta \rightarrow \bigwedge \mathbf{t}_k$ for some θ such that $\Gamma \cup \{\psi, \theta\}$ is satisfiable. But then this set is satisfiable in some structure $A_k \in \mathcal{M}_k$, hence $A_k \models \mathbf{t}_k$, so $A_k \in \overline{\mathcal{M}_k}$ – a contradiction.

Thus, $\Gamma \cup \{\psi\}$ has a model A which omits every type $\mathbf{t}_k$, hence $A \in \mathcal{M}_k$ for each $k \in \mathbf{N}$. □

Remark 2.1. *The same result can be proved (in a more general situation) as follows. Every class $\mathcal{M}_k$ is open and dense in $\mathcal{M}$ with respect to the elementary topology $\mathbf{E}$ on the class $\mathcal{M}$, and $\mathbf{E}$ has Baire's property, being compact and Hausdorff, hence $\mathcal{M}^f$ is dense in $\mathbf{E}$, which is equivalent to the statement of the theorem. For detailed proof see* [Goranko, 99].

From the previous two results we immediately obtain the following.

Corollary 2.3. *Let $\mathcal{M}$ be an elementary class of structures in a language of finite signature, $\{\mathcal{M}_k\}_{k\in\mathbf{N}}$ be a family of co-elementary subclasses of $\mathcal{M}$, and let $\mathcal{M}^f = \bigcap_{k\in\mathbf{N}} \mathcal{M}_k$. If for every $n \in \mathbf{N}$, every structure from $\mathcal{M}$ is n-equivalent to a structure from $\mathcal{M}_k$ for each $k \in \mathbf{N}$, then for every $n \in \mathbf{N}$ every structure from $\mathcal{M}$ is n-equivalent to a structure from $\mathcal{M}^f$.*

2.2.2. *Relativized formulae.* Let θ be a formula with no bound occurrences of the variable w. Then by $\theta^{<w}$ we denote the formula obtained from θ by relativizing all quantifiers in θ as follows: $\forall x(...)$ to $\forall x(x < w \rightarrow ...)$ and likewise $\exists x(...)$ to $\exists x(x < w \wedge ...)$. We shall call $\theta^{<w}$ the **downward w-relativization of** θ. Likewise we define the **upward w-relativization** $\theta^{>w}$.

Given a partial ordering A and $a \in A$, we denote by $A \downarrow a$ the substructure of A over $\{b \in A | b < a\}$, and likewise $A \uparrow a = \{b \in A | b > a\}$.

We shall use the following proposition, which is a particular case of a general fact relating truth of a formula in a definable substructure to the truth of a corresponding relativization of the formula in the whole structure (see e.g. [Rosenstein, 82], ch.13).

Proposition 2.4. *For any partial ordering A, sentence θ and $a \in A$,*

$$A \downarrow a \models \theta \text{ iff } A \models \theta^{<w}(a/w), \text{ and likewise, } A \uparrow a \models \theta \text{ iff } A \models \theta^{>w}(a/w).$$

Corollary 2.5. *Let T be a tree, $t \in T$ and $\mathbf{p}$ be a path containing t. Then for any sentence θ, $T \models \theta^{<y}(t/y)$ iff $\mathbf{p} \models \theta^{<y}(t/y)$.*

Proof. From proposition 2.4, since $T \downarrow t = \mathbf{p} \downarrow t$. □

Lemma 2.6. *For any sentence θ, and a partial ordering A, $A \models y \leq w \rightarrow ((\theta^{<y})^{<w} \leftrightarrow \theta^{<y})$.*

Proof. Straigthforward induction on θ. □

2.2.3. *Definable ordinals.* It is known (see e.g. [Rosenstein, 82], ch 13) that for every ordinal $\alpha < \omega^\omega$ there is a first-order sentence φ_α which completely axiomatizes α, i.e. for any structure $(A, <)$, $A \models \varphi_\alpha$ iff $A \equiv \alpha$. Moreover, φ_α uniquely determines α in the class of all ordinals $\mathcal{ORD}$ in the sense that for any ordinal β, $\beta \vDash \varphi_\alpha$ iff $\beta = \alpha$.

Proposition 2.7. *For any ordinal $\alpha < \omega^\omega$ and a sentence θ the following are equivalent:*

1. $\alpha \vDash \theta$.
2. $\varphi_\alpha \wedge \theta$ *is satisfiable.*
3. $\vDash \varphi_\alpha \rightarrow \theta$.
4. *For any linear ordering A, $A \vDash \forall w(\varphi_\alpha^{<w}(w) \rightarrow \theta^{<w})$.*

Proof. The equivalence of (1)–(3) is trivial.

(3)⇒(4): Suppose $A \nvDash \forall w(\varphi_\alpha^{<w}(w) \rightarrow \theta^{<w})$, i.e.

$A \vDash (\varphi_\alpha^{<w}(w) \wedge \neg\theta^{<w})(t/w)$ for some $t \in A$. Then, by proposition 2.4, $A \downarrow t \vDash \varphi_\alpha \wedge \neg\theta$, so $\nvDash \varphi_\alpha \rightarrow \theta$.

(4)⇒(3): Suppose $A \vDash \varphi_\alpha \wedge \neg\theta$ for some structure $(A, <)$. Then $<$ is a linear ordering in A. Let $A + \mathbf{1}$ be the end extension of A with one new element t. Then $A = (A + \mathbf{1}) \downarrow t$, so by proposition 2.4: $A \vDash (\varphi_\alpha^{<w}(w) \wedge \neg\theta^{<w})(t/w)$. □

Corollary 2.8. *For any ordinals $\alpha, \beta < \omega^\omega$:*

1. $\alpha < \beta$ *iff* $\beta \vDash \exists y \varphi_\alpha^{<y}$.
2. $\vDash \forall y(\varphi_\alpha^{<y} \rightarrow \forall z(z < y \rightarrow \neg\varphi_\alpha^{<z}))$.

Proof. (1): $\beta \vDash \exists y \varphi_\alpha^{<y}$ iff $\beta \vDash \varphi_\alpha^{<y}(\gamma/y)$ for some $\gamma < \beta$ iff $\gamma \vDash \varphi_\alpha$ iff $\gamma = \alpha$.

(2): Suppose $A \nvDash \forall y(\varphi_\alpha^{<y} \rightarrow \forall z(z < y \rightarrow \neg\varphi_\alpha^{<z}))$, i.e. for some $a \in A$, $A \models (\varphi_\alpha^{<y} \wedge \exists z(z < y \wedge \varphi_\alpha^{<z}))(a/y)$ hence $A \downarrow a \vDash \varphi_\alpha \wedge \exists z \varphi_\alpha^{<z}$, so $A \downarrow a \equiv \alpha$. Therefore $\alpha \vDash \exists z \varphi_\alpha^{<z}$, hence $\alpha < \alpha$ — a contradiction. □

3. Some general results about trees.

We now fix a first-order language (with equality) $\mathcal{L}_m$ for m-**coloured trees,** containing one binary predicate $<$ and m unary predicates $P_1, ..., P_m$. The adjective "m-coloured" will be tacitly assumed and usually omitted henceforth.

Let $\mathcal{T}$ be the class of all trees for $\mathcal{L}_m$ and $\mathbf{T}$ be the first-order theory of trees. Note that the class of (backward, forward) discrete trees is first-order definable.

Definition 3.1. 1. *A class of trees $\mathcal{K}$ is **finite-branching reducible** if $TH(\mathcal{K})$ is complete with respect to the class of finitely branching members of $\mathcal{K}$, i.e. if every first-order sentence satisfiable in $\mathcal{K}$ is satisfiable in a finitely branching member of $\mathcal{K}$.*

2. *A first-order theory of trees Γ is **finite-branching complete** if $MOD(\Gamma)$ is finite-branching reducible.*

*Likewise we define **(uniformly) bounded-branching reducible** classes and **(uniformly) bounded-branching complete** theories of trees.*

Note that if a class $\mathcal{K}$ is finite-branching reducible then $TH(\mathcal{K})$ is finite-brancihng complete.

From lemma 2.1 we obtain the following.

Lemma 3.1. *A class of trees $\mathcal{K}$ is finite-branching reducible iff for each $n \in \mathbf{N}$, every tree from $\mathcal{K}$ is n-equivalent to a finitely-branching tree from $\mathcal{K}$.*

Likewise for bounded-branching reducibility.

Since $\mathcal{L}_m$ has a finite signature, for every $n \in \mathbf{N}$ there are finitely many non-equivalent sentences of rank n, hence finitely many classes of n-equivalent trees. Let a complete set of their representatives be $T_1, \ldots, T_{j_n}$ and let $b(n) = nj_n$.

Lemma 3.2. *Let T be a tree and X be an anti-chain (a set of pairwise incomparable nodes) in T. Then there is a prunned subtree T' of T which is n-equivalent to T and each $t \in X \cap T'$ has at most $b(n)$ siblings in T'.*

Proof. We can assume that there are no siblings in X, otherwise we ignore all but one from each family. For every node $t \in X$ we simultaneously do the following pruning. For each $i = 1, \ldots, j_n$ we consider those siblings s of t such that $T(s) \equiv_n T_i$. If there are no more than n of these we leave them intact, otherwise we select (arbitrarily) n of them and remove all others together with the subtrees rooted at them. Then we enumerate the remaining siblings of $t : s^i_1, s^i_2, \ldots$. Let the resulting tree be T'.

We prove that $T' \equiv_n T$ by demonstrating a winning strategy for Player II in the n-round Ehrenfeucht game for (T, T'). Let $k \leq n$ and the node chosen in the k-th round by Player I be $v \in T$. If v is not a successor of a pruned node $t \in X$, then Player II selects the same node from T'. Otherwise, suppose $v \in T(s)$ where s is a sibling of $t \in X$. If no other nodes from $T(s)$ have been chosen in the game yet, then Player II selects the subtree $T'(s')$ where s' is the first preserved sibling of t in T' which has not been used so far in the game and such that $T(s) \equiv_n T'(s')$ (there will be at least one available), and then responds by choosing an element from $T'(s')$ according to the winning strategy for the n-round game on $(T(s), T'(s'))$. Suppose now that other nodes from $T(s)$ have already been chosen in the game, viz. $v_1, \ldots v_r$, with counterparts $v'_1, \ldots v'_r$ from $T'(s')$. Then Player II responds by choosing an element from $T'(s')$ according to the winning strategy for the corresponding $(r+1)$-st round of the n-round game on $(T(s), T'(s'))$ and the existing configuration.

If the node chosen by Player I in the k-th round is from T' then the response of Player II is symmetric to the one described above.

A straightforward induction on k shows that the resulting configurations after the k-th round are isomorphic. The lemma now follows from Ehrenfeucht's theorem. □

Corollary 3.3. *Let T be a tree and $\theta(x)$ be a formula of quantifier rank $\leq n$, satisfiable at a node t of T. Then $\theta(x)$ is satisfiable at a $b(n)$-boundedly branched node of an appropriately prunned subtree of T, n-equivalent to T.*

Corollary 3.4. *Let T be a tree and X be a set of nodes in T, such that the length of the chains in X is bounded by some natural number m. Then there is a prunned subtree T' of T which is n-equivalent to T and each $t \in X \cap T'$ has at most $b(n)$ siblings in T'.*

Proof. For any set of nodes Y in T let $M(Y)$ be the set of maximal in Y nodes. Now we apply repeatedly lemma 3.2 for the sets $M(Y_1), \ldots M(Y_m)$ defined recursively by $Y_1 = X$, $Y_{k+1} = Y_k \setminus M(Y_k)$. Note that each Y_k is an anti-chain and $Y_{m+1} = \emptyset$. □

Theorem 3.5. *Let $\mathcal{K}$ be any first-order definable class of trees or any elementary class of trees, closed under prunned subtrees. Then $\mathcal{K}$ is finite-branching reducible.*

Proof. Let ϕ be a sentence satisfiable in $\mathcal{K}$, i.e. let $TH(\mathcal{K}) \cup \{\phi\}$ be satisfiable. We need to show that it is satisfiable in a finitely branching tree from $\mathcal{K}$.

First take the case of $\mathcal{K}$ first-order definable. Consider the 1-type

$$\mathbf{t} = \left\{ \exists y_k, \ldots, \exists y_1 \left(\bigwedge_{1 \leq i < j \leq k} y_i \neq y_j \wedge \bigwedge_{1 \leq i \leq k} s(y_i, x) \right) \middle| \, k \in \mathbf{N} \right\}$$

Note that a node in a tree realizes $\mathbf{t}$ iff it has infinitely many siblings.

1. The type $\mathbf{t}$ is not principal in the theory $TH(\mathcal{K}) \cup \{\phi\}$. Indeed, suppose otherwise, i.e. for some $\theta(x)$, $TH(\mathcal{K}) \bigcup \{\phi, \exists x \theta(x)\}$ is satisfiable by some tree $T \in \mathcal{K}$ and $TH(\mathcal{K}) \cup \{\phi\} \models \theta(x) \rightarrow \psi(x)$ for every $\psi(x) \in \mathbf{t}$. Let $n = \max(qr(\gamma), qr(\phi), qr(\exists x \theta(x)))$ where γ is the sentence defining $\mathcal{K}$. Then, by corollary 3.3 $\theta(x)$ is satisfied by some finitely branched node a in a pruned subtree T' of T, n-equivalent to T. Then $T' \models TH(\mathcal{K}) \cup \{\phi\}$, hence a realizes $\mathbf{t}$, which is a contradiction.

   Thus, by the omitting types theorem, $\mathbf{t}$ is omitted in a (countable) model of $TH(\mathcal{K}) \cup \{\phi\}$ i.e. there is a (countable) finitely branching tree from $\mathcal{K}$ satisfying ϕ.

   If $\mathcal{K}$ is elementary and closed under prunned subtrees the proof is similar. We now take $n = \max(qr(\phi), qr(\exists x \theta(x)))$, and again $T' \models TH(\mathcal{K}) \cup \{\phi\}$, due to the closedness of $\mathcal{K}$ under prunned subtrees.

□

Corollary 3.6. *The class of (backward, forward) discrete trees is finite-branching reducible.*

4. Level-definable order types in trees.

Definition 4.1. *A formula $l(x)$ is* ***level defining in a class of trees*** $\mathcal{K}$ *if*

$$\mathcal{K} \models \forall x(l(x) \rightarrow \forall y(y < x \rightarrow \neg l(y))).$$

If $\mathcal{K} = \mathcal{T}$, we simply call $l(x)$ ***level defining****.*

(Here and henceforth, whenever we substitute a term for a variable in a formula, we assume that care has been taken to avoid capture, i.e. the term is free for the substitution).

Note that:

- For every formula $\theta(x)$, the formula $\theta_l(x) = \theta(x) \wedge \forall z(z < x \rightarrow \neg\theta(z))$ is level defining.
- If $\theta(x)$ is level defining then $\mathcal{T} \models \theta(x) \longleftrightarrow \theta_l(x)$.

Definition 4.2. *An order type τ is* ***level-definable (in a class of trees $\mathcal{K}$)*** *if there is a level-defining (in $\mathcal{K}$) formula $\delta_\tau(x)$ which is satisfied by every node on level τ in every tree (from $\mathcal{K}$).*

Example 4.1.

- *Every finite order type is definable on the class of all trees.*
- *Let α be an ordinal less than ω^ω and φ_α be the sentence axiomatizing α. Then, by corollary 2.8 the formula $\delta_\alpha(x) = \varphi_\alpha^{<x}$ is level defining and defines the order type of α on the class of all trees. Moreover, by the same corollary, α is the only level defined by $\delta_\alpha(x)$ on the class of well-founded trees. The order types level-definable by $\delta_\alpha(x)$ will be called* ***α-like order types.***

Examples of order types which are not level definable are $\omega + \tau + \omega^*$ for any order type τ, $\omega + \zeta.\eta$, and $\omega + \zeta.\lambda$, since every node in a tree on such a level has a predecessor on the same level.

Proposition 4.1.

1. *If the type τ is level-definable then every order type elementarily equivalent to τ is level-definable by the same formula.*
2. *If τ_1, τ_2 are level definable then $\tau_1 + 1 + \tau_2$ is level-definable too.*

Proof. (1): Follows from proposition 2.4.

(2): If τ_1, τ_2 are level definable by $\delta_1(x)$ and $\delta_2(x)$ respectively, then $\tau_1 + 1 + \tau_2$ is level-definable by the formula $\delta(x) = \exists y(y < x \wedge \delta_1(y/x) \wedge \delta_2^{>y}(x))$. ☐

Proposition 4.2. *For any ordinal $\alpha < \omega^\omega$, a linear ordering τ is α-like iff $\tau \equiv \alpha$.*

Proof. Let $\tau + \mathbf{1}$ be the end extension of τ with a last element t. Then $\tau \equiv \alpha$ iff $\tau + \mathbf{1} \equiv \alpha + 1$ iff $(\tau + \mathbf{1}) \downarrow t \models \delta_\alpha(x)$ iff τ is α-like. ☐

Note that the nodes from any definable level in a tree are incomparable and therefore, applying 3.4 we obtain the following.

Corollary 4.3. *Let T be a tree, $l_1, ..., l_k$ be level defining formulae, X be the set of all nodes in T belonging to levels defined by $l_1, ..., l_k$, and $n \in \mathbf{N}$. Then there is a prunned subtree T' of T which is n-equivalent to T and each $t \in X \cap T'$ has at most $b(n)$ siblings in T'.*

Corollary 4.4. *For every $k, n \in \mathbf{N}$, every tree is n-equivalent to a pruned subtree, finitely branched on all finite levels up to k.*

Theorem 4.5. 1. *Let $\mathcal{K}$ be any elementary class of trees closed under prunned subtrees. For each $n \in \mathbf{N}$ every tree from $\mathcal{K}$ is n-equivalent to a countable tree from $\mathcal{K}$, $b(n)-$boundedly branched on all definable in $\mathcal{K}$ levels.*

2. *The same holds for any class of trees $\mathcal{K}$, first-order definable by a sentence of quantifier rank r, for each $n \geq r$.*

Proof. Let $\{l_1, l_2 ..., \}$ be an enumeration of all level defining formulae in $\mathcal{K}$ and for every $k, m \in \mathbf{N}$ let $l_k^m = \forall x(l(x,k) \rightarrow s_m(x))$.

Now let $T \in \mathcal{K}$ and $TH_n(T)$ be set of all sentences of quantifier rank at most n valid in T. Then the theory

$$\Delta = TH(\mathcal{K}) \cup TH_n(T) \cup \left\{ l_1^{b(n)}, l_2^{b(n)}, ... \right\}$$

is finitely satisfiable in either case, due to corollary 4.3, hence it is satisfiable in a countable model $T' \in \mathcal{K}$, n-equivalent to T, and $b(n)-$boundedly branched on all definable on $\mathcal{K}$ levels. □

Remark 4.1. *The result above can be strengthened somewhat by adding the universal theory of T to Δ.*

Corollary 4.6. *Let $\mathcal{K}$ be a definable class of trees or an elementary class closed under prunned subtrees. Then for every (large enough) $n \in \mathbf{N}$ every tree from $\mathcal{K}$ is n-equivalent to a tree from $\mathcal{K}$, $b(n)-$boundedly branched on all α-like levels for every $\alpha < \omega^\omega$.*

5. Well-founded trees and finite branching.

Definition 5.1. *A tree T is **definably well-founded** if every non-empty definable set in T has a minimal element, i.e. for every formula $\theta(x)$, $T \models \exists x \theta(x) \rightarrow \exists x(\theta(x) \wedge \forall y(y < x \rightarrow \neg\theta(y))$.*

Lemma 5.1. *Every finitely branching, definably well-founded coloured tree is n-equivalent to a finitely branching well-founded tree, for each $n \in$ $\mathbf{N}$.*

Proof. This is a strengthening of the following theorem proved in [Doets, 87], ch.4: *Every definably well-founded coloured tree is n-equivalent to a well-founded tree, for each $n \in$* $\mathbf{N}$. The proof of this result (which is long and elaborated and will not be repeated here) is based on a sequence of cut-and-paste surgeries, replacing parts of the tree by appropriate well-founded rooted subtrees, and the diligent reader can verify that if those surgeries are applied carefully to a *finitely branching* tree, then the result will again be *finitely branching.* □

Theorem 5.2. *Every class $\mathcal{K}$ of well-founded trees, first-order definable in $\mathcal{WFT}$, is finite-branching reducible.*

Proof. Like in the proof of theorem 3.5, we can show that, given a sentence ϕ satisfiable in $\mathcal{K}$, the type $\mathbf{t} = \left\{ \exists y_k, ..., \exists y_1 \left(\bigwedge_{1 \leq i < j \leq k} y_i \neq y_j \wedge \bigwedge_{1 \leq i \leq k} s(y_i, x) \right) \middle| k \in \mathbf{N} \right\}$ is not principal in the theory $TH(\mathcal{K}) \cup \{\phi\}$. Suppose otherwise. Then for some generator $\theta(x)$ of $\mathbf{t}$ in $TH(\mathcal{K}) \cup \{\phi\}$, $\phi \wedge \exists x \theta(x)$ is satisfiable in some model of $TH(\mathcal{K})$, hence it is satisfiable by some tree $T \in \mathcal{K}$, and therefore $\theta(x)$ is satisfied by a finitely branched node a in a prunned subtree T' of T, n-equivalent to T and $T' \models \phi$ for a large enough n so that $T' \in \mathcal{K}$ (Note that a prunned subtree of a well-founded tree is well-founded). But then a realizes $\mathbf{t}$ which is a contradiction. Therefore, ϕ is satisfied in some (countable) model T'' of $TH(\mathcal{K})$ which omits $\mathbf{t}$. Then T'' is finitely branching and definably well-founded. Now, an application of lemma 5.1 completes the proof. □

Given an ordinal α we denote:

- by $\mathcal{T}_\alpha$ the class of all α-trees;
- by $\mathcal{T}_{<\alpha}$ the class of well-founded trees in which every path has an order type less than α. Likewise for $\mathcal{T}_{\leq\alpha}$, (which, of course, equals $\mathcal{T}_{<\alpha^+}$).
- by $\mathcal{T}_{\approx\alpha}$ the class of **almost** α**-trees**, consisting of all trees from $\mathcal{T}_{<\alpha^+}$ in which every node belongs to an α-path.

Thus, an almost α-tree is a well-founded tree in which every path is not longer than α and there are "plenty" α-paths.

Let $\Gamma_\alpha = TH(\mathcal{T}_\alpha)$, $\mathcal{M}_\alpha = MOD(\Gamma_\alpha)$, and likewise for $\mathcal{T}_{<\alpha}$ and $\mathcal{T}_{\approx\alpha}$.

Theorem 5.3. *For every ordinal $\alpha < \omega^\omega$, the classes $\mathcal{T}_{<\alpha}$ and $\mathcal{T}_{\approx\alpha}$ are finite-branching reducible.*

Proof. By theorem 5.2 it is sufficient to show that every class $\mathcal{T}_{<\alpha}$ and $\mathcal{T}_{\approx\alpha}$ is first-order definable in the class $\mathcal{WFT}$.

For $\mathcal{T}_{<\alpha}$ the defining sentence is $\forall x \neg\delta_\alpha(x)$.

As for $\mathcal{T}_{\approx\alpha}$, first note that, since α has a unique presentation in Cantor's normal form, there is a unique $n \in \mathbf{N}$ such that $\alpha = \beta + \omega^n$ for some ordinal β. Furthermore, since the sequence $\omega^{n-1}, \omega^{n-1}.2, ...\omega^{n-1}.m, ...$ is cofinal in ω^n, and $\omega^{n-1}.m + \omega^n = \omega^n$, the following are equivalent:

- $\alpha = \beta + \omega^n$, $n \geq 1$.
- there is a cofinal sequence $\beta_0, \beta_1, ...\beta_m, ...$ in α such that each of the intervals $[\beta_m, \alpha) = \{\gamma < \alpha | \beta_m \leq \gamma\}$ is isomorphic to ω^n.
- there is a cofinal sequence $\beta_0, \beta_1, ...\beta_m, ...$ in α such that each of the intervals $(\beta_m, \beta_{m+1}) = \{\gamma | \beta_m < \gamma < \beta_{m+1}\}$ is isomorphic to ω^{n-1}.

Now we shall define for every $\alpha < \omega^\omega$ a formula $\chi_\alpha(x)$ such that for every tree $T \in \mathcal{T}_{<\alpha^+}$ and a node $t \in T$,

$$T \models \chi_\alpha(t) \text{ iff } t \text{ and every successor of } t \text{ belong to paths of length } \alpha.$$

First, we define the formulae $\chi_{\omega^n}(x)$, for $n > 0$.

$$\chi_{\omega^n}(x) = \forall y(x \leq y \rightarrow \exists z(y < z \wedge (\delta_{\omega^{n-1}}(z))^{>y})) \wedge \neg\exists y \delta_{\omega^n}(y).$$

Now, we define for every $\alpha = \beta + \omega^n$:

$$\chi_\alpha(x) = \forall w(x \leq w \rightarrow \exists y(w < y \wedge \exists z(z \leq y \wedge \delta_\beta(z) \wedge (\chi_{\omega^n})^{>z}(z)))).$$

Indeed, $\chi_\alpha(x)$ says that x and each successor of x is comparable to a node z the stem of which has an order type β, and every successor of which belongs to a branch of type ω^n stemming from z.

Now, $\mathcal{T}_{\approx\alpha}$ is defined in $\mathcal{T}_{<\alpha^+}$ by $\forall x \chi_\alpha(x)$. □

Combining the theorem for $\mathcal{T}_{<\omega}$ with König's lemma (which implies that every finitely branching tree with no infinite branches is finite) we obtain the following results, first proved in [Backofen et al, 95], where an explicit axiomatization of the theory of finite trees is given as well.

Corollary 5.4. 1. *Every tree with no infinite branches is n-equivalent to a finite tree, for each $n \in \mathbf{N}$.*

2. *The first-order theory of the class $\mathcal{T}_{<\omega}$ of trees with no infinite branches is complete with respect to the class of finite trees.*

For a finite ordinal α, the property of being an α-tree is first-order definable, hence the following is an immediate consequence of either of corollary 5.4.

Proposition 5.5. *The theory of α-trees for any finite ordinal α is complete with respect to all finite α-trees.*

Theorem 5.6. *The class $\mathcal{T}_\omega$ is finite-branching reducible.*

Proof. Follows from theorem 5.3, since $\mathcal{T}_\omega = \mathcal{T}_{\approx\omega}$. □

6. Level definable classes of trees and bounded branching.

Here we shall prove a more general and stronger than theorem 5.3 result about finite branching reducibility, without referring to theorem 5.2 or lemma 5.1 used in its proof.

Definition 6.1. *A class of trees $\mathcal{K}$ is **level-definable** iff there is a set of formulae Δ of a free variable x such that:*

1. *Every $\delta(x) \in \Delta$ is level defining on $\mathcal{K}$.*
2. *A tree T belongs to $\mathcal{K}$ iff $T \models TH(\mathcal{K})$ and every node T satisfies some $\delta(x) \in \Delta$ (i.e. $\mathcal{K}$ consists of the models of $TH(\mathcal{K})$ which omit the type $\{\neg\delta(x) | \delta(x) \in \Delta\}$).*

Theorem 6.1. *Every level definable class of trees $\mathcal{K}$ which is closed under pruned subtrees is bounded-branching reducible.*

Proof. Let $\{l_k(x)\}_{k\in\mathbf{N}}$ be an enumeration of all formulae level defining in $\mathcal{K}$ and Λ_k be the level defined by $l_k(x)$.

Let $\Gamma = TH(\mathcal{K})$, $\mathcal{M} = MOD(\Gamma)$ and let $\mathcal{M}^f$ consist of the trees T from $\mathcal{M}$ satisfying the following two conditions:

1. T is boundedly branched on all definable levels;
2. Every formula $\theta(x)$ satisfiable in T is satisfiable by a node on some definable level.

Lemma 6.2. *The theory Γ is complete with respect to the class $\mathcal{M}^f$.*

Proof of the lemma: We shall prove that $\mathcal{M}^f$ satisfies the conditions of theorem 2.2.

First, note that for every $k \in \mathbf{N}$ the set $\Phi_k(x)$ of formulae $\theta(x)$ of a free variable x and quantifier rank not greater than k is finite, up to logical equivalence.

Now, for every $k, m \in \mathbf{N}$ we denote

$$\theta_{k,m} = \bigwedge \{\exists x\theta(x) \to \exists x(\theta(x) \wedge (l_0(x) \vee ... \vee l_m(x)) | \theta(x) \in \Phi_k(x)\}.$$

The sentence $\theta_{k,m}$ says in a tree T that every formula $\theta(x)$ from $\Phi_k(x)$ satisfiable in T is satisfied by a node on a level Λ_i for some $i \leq m$.

Note that for every k, every tree from $\mathcal{K}$ satisfies all $\theta_{k,m}$ for a large enough m.

Denote by $\chi_{k,m}$ the sentence $\forall x(l_k(x) \to s_m(x))$.

For every $k \in \mathbf{N}$ let

$$\mathcal{M}_k = \bigcup_{m\in\mathbf{N}} (\mathcal{M} \cap MOD(\chi_{k,m} \wedge \theta_{k,m}))$$

$\mathcal{M}_k$ consists of all trees T from $\mathcal{M}$ which are finitely branched on each level Λ_i for $i \leq k$ and in which every formula $\theta(x)$ from $\Phi_k(x)$, if satisfiable in T at all, is satisfied there at a definable level.

Note that:

- $\mathcal{M}^f = \bigcap_{k \in \mathbf{N}} \mathcal{M}_k$.
- Every $\mathcal{M}_k$ has an elementary complement in $\mathcal{M}$, being a union of classes definable in $\mathcal{M}$.
- For every $k \in \mathbf{N}$, Γ is complete with respect to $\mathcal{M}_k$. Indeed, every formula of rank n, satisfiable in a tree from $\mathcal{M}$, is satisfiable in a tree from $\mathcal{K}$, and hence, by corollary 4.3, is satisfiable in a pruned subtree (hence, still from $\mathcal{K}$) finitely branched on each level Λ_i, for $i \leq k$. Every such tree is in $\mathcal{M}_k$.

Hence, the lemma follows from theorem 2.2.

Lemma 6.3. *Every tree T from $\mathcal{M}^f$ is elementarily equivalent to a boundedly branching tree from $\mathcal{K}$.*

Proof of the lemma: *Consider the 1-type*

$$\mathbf{t} = \{\neg l_k(x) | k \in \mathbf{N}\}$$

Just like in the proof of theorem 3.5 we show that $\mathbf{t}$ is not principal in T since any generator of that type would be satisfiable in T by a node which belongs to some Λ_k. Hence, $\mathbf{t}$ is omitted in a (countable) model T' such that:

- $T' \equiv T$, and hence $T' \models TH(\mathcal{K})$, so $T' \in \mathcal{K}$;
- T' is boundedly branched at every (definable) level because it satisfies the same formulae $\chi_{k,m}$ as T does, hence it is boundedly branching.

Therefore, Γ is complete with respect to the class of boundedly branching trees from $\mathcal{K}$. $\square$

Theorem 6.4. *The class $\mathcal{T}_{\leq \omega^\omega}$ is level definable by $\Delta = \{\delta_\alpha(x) | \alpha < \omega^\omega\}$.*

Proof. All we need to prove is that every model T of $TH(\mathcal{T}_{\leq \omega^\omega})$ which omits the type $\mathbf{t}_\Delta = \{\neg\delta_\alpha(x) | \alpha < \omega^\omega\}$ is in $\mathcal{T}_{\leq \omega^\omega}$. First, note that T is a definably well-founded tree. Furthermore, for every $t \in T$ there is a unique ordinal $o(t) < \omega^\omega$ such that $T \models \delta_\alpha(t/x)$. The existence follows from T omitting $\mathbf{t}_\Delta$; the uniqueness — from corollary 2.8.

Now, consider any path $\mathbf{p}$ in T. Let γ be the least ordinal such that $\delta_\gamma(x)$ is not satisfied by any node from $\mathbf{p}$, if there is any, otherwise take $\gamma = \omega^\omega$. We shall prove that $\mathbf{p} \cong \gamma$.

First, note that for each $t \in T$, $o(t) < \gamma$. For, otherwise let $o(t) = \alpha > \gamma$. Then $\alpha \models \exists y \delta_\gamma(y)$ by corollary 2.8, hence by proposition 2.7 $\mathbf{p} \models \forall w(\delta_\alpha(w) \to (\exists y \delta_\gamma(y))^{<w})$, so $\mathbf{p} \models (\exists y \delta_\gamma(y))^{<w}(t/w)$. Therefore, by corollary 2.5, $T \models (\exists y \delta_\gamma(y))^{<w}(t/w)$, i.e. $T \models \exists y(y < w \wedge (\varphi_\gamma^{<y}(y))^{<w})(t/w)$, hence by lemma 2.6 $T \models \exists y(y < w \wedge (\varphi_\gamma^{<y}(y))(t/w)$, i.e. there is a node $s < t$ such that $T \models \varphi_\gamma^{<y}(y)(s/w)$, so s satisfies $\delta_\gamma(x)$, contrary to the choice of γ.

Furthermore, if $s < t$ and $o(s) = \alpha$, $o(t) = \beta$, then $T \models (\delta_\beta(x) \wedge \exists y(y < x \wedge \delta_\alpha(y)))(t/x)$, so by lemma 2.6 $T \models (\varphi_\beta^{<x} \wedge \exists y(y < x \wedge (\delta_\alpha(y))^{<x}))(t/x)$, hence by proposition 2.4 $T \downarrow t \models \varphi_\beta \wedge \exists y \delta_\alpha(y)$.Therefore, by proposition 2.7, $\beta \models \exists y \delta_\alpha(y)$, so $\alpha < \beta$.

Thus, o is an order-preserving bijection from $\mathbf{p}$ to γ, hence $\mathbf{p} \cong \gamma$. $\square$

Remark 6.1. *Note that if $\mathcal{K}$ is a level definable class then every subclass of $\mathcal{K}$ which is elementary in $\mathcal{K}$ is level definable, too. Therefore the classes $\mathcal{T}_{<\alpha}$ and $\mathcal{T}_{\approx\alpha}$ for any $\alpha < \omega^\omega$ are level definable.*

Since each of these classes is closed under prunned subtrees we obtain the following.

Corollary 6.5. *The classes $\mathcal{T}_{<\alpha}$ and $\mathcal{T}_{\approx\alpha}$ for any $\alpha < \omega^\omega$, and $\mathcal{T}_{\leq\omega^\omega}$, are bounded-branching reducible.*

7. Finite branching reducibility of $(\omega + n)$-trees.

The question of finite branching reducibility of classes $\mathcal{T}_\alpha$ for $\alpha > \omega$ is still open, though we conjecture affirmative answers. Here we shall prove that for the classes $\mathcal{T}_{\omega+n}$. For technical convenience, we shall consider the case of ω^+-trees.

Note that every ω^+-tree is obtained from an ω-tree by adding *terminal nodes* (on ω level) for *all* paths in that ω-tree.

Theorem 7.1. *The class of ω^+-trees is finite branching reducible.*

Proof. Given an ω^+-tree T we first apply apply corollary 6.5 to obtain a boundedly branching almost ω^+-tree T' which is n-equivalent to T. The tree T', however, need not be a genuine ω^+-tree, and therefore, an additional argument is required to complete the proof.

The idea will be to extend the tree T' to an ω^+-tree T'' by adding terminal nodes for all paths which lack them, while preserving the tree up to N-equivalence, for a large enough N. This, however, cannot be done automatically becuase of the following problem. It may happen that a path without a terminal node cannot be extended with any one without crucially affecting the truth of some formulae in the tree. For instance, it may happen that a formula $\theta(x)$ is satisfied at every node of that path, while every terminal node (i.e. leaf) is a successor of a node which falsifies θ, i.e. the sentence $\forall y(d(y) \to \exists x(x < y \wedge \neg\theta(x)))$ is satisfied in the tree.

To avoid that problem we need some preparation and strengthening of the requirements for the tree T'.

First, some notation. For any finite set S of formulae of one free variable x, we denote:

$\chi_S(x) = \vee_{\theta\in S}\theta(x)$,
$\gamma_S(x) = \forall y(y < x \to \chi_S(y))$,
$\delta_S(x) = \wedge_{\theta\in S}\exists y(y < x \wedge \theta(y))$, and
$\varepsilon_S(x) = \gamma_S(x) \wedge \delta_S(x)$.

Further, we fix a variable x and denote by S_r the finite set of all non-equivalent formulae of quantifier rank r with a free variable x for the language we consider.

Now, let us go back to the original ω^+-tree T and consider any path $\mathbf{P}$ in T. For any $r \in \mathbf{N}$, let $S_r(\mathbf{P}) = \left\{\varphi^r_{T;t}(x) | t \in \mathbf{P},\ \text{and } t \text{ is not terminal}\right\}$. Note that $S_r(\mathbf{P})$ is a subset of S_r. For notational ease, by slight abuse of notation we shall write $\chi^r_{\mathbf{P}}$ instead of $\chi_{S_r(\mathbf{P})}$ and likewise for $\gamma^r_{\mathbf{P}}, \delta^r_{\mathbf{P}}$, and $\varepsilon^r_{\mathbf{P}}$.

Thus, $\chi^r_{\mathbf{P}}$ is the disjunction of all formulae of quantifier rank r realizable at the non-terminal nodes of $\mathbf{P}$; $\delta^r_{\mathbf{P}}(x)$ describes the formulae of quantifier rank r realizable by nodes of $\mathbf{P}$ from the stem of the node assigned to x. We shall call $\varepsilon^r_{\mathbf{P}}(x)$ the *(complete) r-description of* $\mathbf{P}$. Clearly, all nodes from some place further along the path $\mathbf{P}$, including its terminal node, must satisfy $\varepsilon^r_{\mathbf{P}}(x)$, and no other $\varepsilon_S(x)$ for a subset S of S_r.

Let us now notice that, because for every fixed r the set of subsets of S_r is finite, in particular there are finitely many possible sets $S_r(\mathbf{P})$, hence there is a uniform upper bound $K = K(T, r)$ such that all possible r-descriptions of paths in T are already realized by the level K.

Therefore, for each set S of formulae in x of quantifier rank r, $T \models \Phi^r_T(S)$, where

$$\Phi^r_T(S) = \forall y(\exists z(z < y \wedge l(z, K) \wedge \varepsilon_S(z)) \rightarrow \forall z((z < y \wedge l(z, K)) \rightarrow \varepsilon_S(z))).$$

Also, because every path in T has a terminal node, then $T \models \Psi^r_T(S)$ where

$$\Psi^r_T(S) = \forall y(\exists z(z < y \wedge l(z, K) \wedge \varepsilon_S(z)) \rightarrow \exists t(y < t \wedge d(t) \wedge \varepsilon_S(z))).$$

Thus, we have defined finitely many sentences $\Phi^r_T(S) \wedge \Psi^r_T(S)$, covering all paths in T. Let $q(r)$ be the maximum of their quantifier ranks.

Finally, let $N = \max(q(n), n)$.

Let us now strengthen the requirement for T' : we want it to be N-equivalent to T, and therefore all sentences $\Phi^n_T(S) \wedge \Psi^n_T(S)$ will be true in T''.

We are ready to prove that for every path in T'' with no terminal node, such a node can be added without affecting $\varphi^n_{T''}$, and hence the satisfiability of any sentence of a quantifier rank not greater than n.

Let $\mathbf{U} = \{u_0, u_1, ...\}$ be a non-terminating ω-path in T'. We add a new terminal node u to the tree and place it as a successor of all (and only) nodes from $\mathbf{U}$. It remains to define the truth of the additional (unary) predicates at u in a way which will not change $\varphi^n_{T'}$. In order to do that it is sufficient to prove that there is a terminal node w in T' the path of which has the same n-description as $\mathbf{U}$ and that every node from $\mathbf{U}$ has a successor which is a terminal node with the same n-description as w.

So, let $S = S_n(\mathbf{U})$. Then, by $\Phi^n_T(S) \wedge \Psi^n_T(S)$, all nodes of $\mathbf{U}$ at levels higher than K satisfy the n-description of $\mathbf{U}$ and see terminal nodes of path in T' wth the same n-description as $\mathbf{U}$. Let R be the set of all such terminal nodes.

Let t_1 be any terminal node from R. Since that node is not terminal for $\mathbf{U}$, it will not be a successor of any node of $\mathbf{U}$ after some level k_1. Then the node of $\mathbf{U}$ on level $k_1 + 1$ will see another terminal node t_2 from R. At some level k_2 the path $\mathbf{U}$ will diverge from t_2, etc. Thus, an infinite subset R' of R is defined (using the axiom of dependent choice). At least one of the possible n-descriptions of terminal nodes will be satisfied by infinitely many nodes from R' and therefore terminal nodes with that n-description φ will be seen by *every* node from $\mathbf{U}$. We now colour (i.e. define the additional predicates at) u according to φ. Again, due to $\Psi^n_T(S)$, the n-description of $\mathbf{U}$ will be satisfied at u, which completes the argument.

We can carry out this construction simultaneously (or in a transfinite sequence of consecutive steps) for all paths in $\mathbf{U}$ without terminal nodes. The resulting ω^+-tree T' will be n-equivalent to T' and hence to T.

Acknowledgements: This work was partly supported by a research grant GUN 2034353 of the Foundation for Research Development of South Africa. I thank Michael Dorfling for some corrections and useful remarks. □

References

[Backofen et al, 95] Backofen R., J. Rogers, & K. Vijay-Shanker, A First-order Axiomatization of the Theory of Finite Trees, *Journal of Logic, Language and Information*, **4**, 1995, 5-39.

[Doets, 87] Doets K., Completeness and Decidability, Ph. D. Thesis, Univ. of Amsterdam, 1987.

[Ebbinghaus et al, 94] Ebbinghaus H.-D., J. Flum, W. Thomas, **Mathematical Logic**, Springer-Verlag, 2nd ed., 1994.

[Goranko, 99] Goranko V., Logical approximation of structures: a topological approach to proving semantic completeness, manuscript, 1999.

[Hodges, 93] Hodges W., **Model Theory,** Cambridge Univ. Press, 1993.

[Rosenstein, 82] Rosenstein J., **Linear Orderings,** Academic Press, New York, 1982.

Department of Mathematics, Rand Afrikaans University, PO Box 524, Auckland Park 2006, Johannesburg, South Africa