Sahlqvist Formulas in Hybrid Polyadic Modal Logics

VALENTIN GORANKO, *Department of Mathematics, Rand Afrikaans University, PO Box 524, Auckland Park 2006, Johannesburg, South Africa. E-mail: vfg@na.rau.ac.za*

DIMITER VAKARELOV, *Faculty of Mathematics and Computer Science, Sofia University, blvd. James Bouchier 5, 1126 Sofia, Bulgaria. E-mail: dvak@fmi.uni-sofia.bg*

Abstract

Building on a new approach to polyadic modal languages and Sahlqvist formulas we define Sahlqvist formulas in *hybrid polyadic modal languages* containing nominals and universal modality or satisfaction operators. Particularly interesting is the case of *reversive* polyadic languages, closed under all 'inverses' of polyadic modalities because the minimal valuations arising in the computation of the first-order equivalents of polyadic Sahlqvist formulae are *definable* in such languages and that makes the proof of first-order definability and canonicity of these formulas a simple syntactic exercise. Furthermore, the first-order definability of Sahlqvist formulas immediately transfers to arbitrary polyadic languages, while the direct transfer of canonicity requires a more involved proof-theoretic analysis.

Keywords: Hybrid polyadic modal logics, Sahlqvist formulas, nominals, universal modality, satisfaction operator, first-order definability, canonicity, completeness.

1 Introduction

In [10] we propose a new treatment of polyadic modal languages as '*purely modal polyadic languages*' where $\vee$ and $\wedge$ are treated as binary modalities (resp. a box and a diamond), boxes are composed as in PDL, and eventually every modal formula is represented as a polyadic box or a diamond. In these languages we defined a class of '*polyadic Sahlqvist formulas*' PSF which substantially extend the previously known class of Sahlqvist formulas [3]. The computation of the first-order equivalents of these formulas extends Sahlqvist-van Benthem's algorithm to an inductive procedure of computing the 'minimal' (first-order definable) valuations of the propositional variables in the formula.

Here we extend polyadic modal languages with nominals and universal modality or satisfaction operators, and define a large class of first-order definable and canonical formulas, extending PSF. In particular, we consider the case of *reversive* hybrid polyadic languages, closed under all 'inverses' of polyadic modalities. In these languages the minimal valuations arising in the computation of the first-order equivalents of polyadic Sahlqvist formulas are *definable*, which makes the proof of Sahlqvist theorem a simple syntactic exercise. As an immediate corollary we obtain first-order definability of Sahlqvist formulas in *arbitrary* polyadic languages, while the direct transfer of completeness apparently requires a more involved proof-theoretic analysis.

The structure of the paper: in the preliminary Section we introduce purely modal polyadic

J. Logic Computat., Vol. 11 No. 5, pp. 737–754 2001

languages and Sahlqvist formulas in them, as defined in [10], and in Section 2 we extend these to the basic hybrid languages. The main result is in Section 3 where we give an essentially syntactic proof of Sahlqvist theorem in reversive polyadic hybrid languages. The paper ends with a discussion on some general problems related to Sahlqvist formulas.

2 Preliminaries: Sahlqvist formulas in purely modal polyadic languages

2.1 *Purely modal polyadic languages*

DEFINITION 2.1
([10]) A *purely modal polyadic language* $\mathcal{L}_\tau$ contains propositional variables, negation $\neg$, and a *modal similarity type* τ consisting of a set of *basic modal terms* (modalities) with pre-assigned finite arities, including a 0-ary modality ι_0, a unary one ι_1, and a binary one ι_2. These distinguished modalities will be interpreted as follows: ι_0 by the constant $\top$ and its dual by $\bot$; ι_1 will be the self-dual identity; ι_2 will be $\vee$, and its dual — $\wedge$.

A *constant formula* will mean a formula containing no variables.

DEFINITION 2.2
By simultaneous mutual induction we define the set of *modal terms* $MT(\tau)$ and their *arity function* ρ, and the set of *(purely) modal formulas* $MF(\tau)$ as follows:

(MT i) Every basic modal term is a modal term of the predefined arity.
(MT ii) Every constant formula is a 0-ary modal term.
(MT iii) If $n > 0$, α, $\beta_1, \ldots, \beta_n$ are modal terms and $\rho(\alpha) = n$, then $\alpha(\beta_1, \ldots, \beta_n)$ is a modal term and $\rho(\alpha(\beta_1, \ldots, \beta_n)) = \rho(\beta_1) + \ldots + \rho(\beta_n)$.

Modal terms of arity 0 will be called *modal constants*.

(MF i) Every propositional variable is a modal formula.
(MF ii) Every modal constant is a modal formula.
(MF iii) If A is a formula then $\neg A$ is a formula.
(MF iv) If $A_1, \ldots, A_n$ are formulas, α is a modal term and $\rho(\alpha) = n > 0$, then $[\alpha](A_1, \ldots, A_n)$ is a modal formula.

Note that constant formulas and 0-ary terms are regarded as both modal terms and formulas.

EXAMPLE 2.3
Let α, β be basic modal terms such that $\rho(\alpha) = 1$, $\rho(\beta) = 2$. Then $\beta(\alpha, \iota_2)$ and $\beta(\alpha(\iota_0), \beta(\alpha, \iota_2))$ are 3-ary modal terms and $[\beta(\alpha, \iota_2)](\neg[\iota_1]q, [\alpha]p, \iota_0)$ is a modal formula. Furthermore, the modality $[\alpha]$ from the second argument can be pulled into the external box resulting into an equivalent formula $[\beta(\alpha, \iota_2(\alpha, \iota_1))](\neg[\iota_1]q, p, \iota_0)$.

Some notation on formulas:

- $\langle\alpha\rangle(A_1, \ldots, A_n) = \neg[\alpha](\neg A_1, \ldots, \neg A_n)$;
- $\top = \iota_0, \bot = \neg\iota_0$;
- $A \vee B = [\iota_2](A, B), A \wedge B = \langle\iota_2\rangle(A, B)$,
  and respectively $A_1 \vee \ldots \vee A_n = [\iota_n](A_1, \ldots, A_n), A_1 \wedge \ldots \wedge A_n = \langle\iota_n\rangle(A_1, \ldots, A_n)$;
- $A \to B = \neg A \vee B$.

Positive and negative occurrences of variables and *positive and negative formulas* are defined as usual.

The *semantics* of purely modal languages is a straightforward combination of the standard Kripke semantics for polyadic modal languages and PDL-type of polymodal languages, taking into account the fact that conjunctions and disjunctions are now treated as modalities. In particular, a τ-*frame* is a structure $\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)}\rangle$ where $R_\alpha \subseteq W^{\rho(\alpha)+1}$ is defined recursively by:

- $R_{\iota_0} = W,\ R_{\iota_1} = \{(x,x)|x \in W\},\ R_{\iota_2} = \{(x,x,x)|x \in W\}$.
- $R_{\alpha(\beta_1,\dots,\beta_n)} = \{(x, x_{11}, \dots, x_{1b_1}, \dots, x_{n1}, \dots, x_{nb_n}) \subseteq W^{b_1+\dots+b_n+1}|$ $\exists y_1 \dots y_n (R_\alpha x y_1 \dots y_n \wedge \bigwedge_{i=1}^n R_{\beta_i} y_i x_{i1} \dots x_{ib_i})\}$ where $\rho(\beta_i) = b_i, i = 1, \dots n$.

Note that $R_{\iota_n} = \{(x, \dots, x) \in W^{n+1}|x \in W\}$.

Now, the *truth definition* of a formula at a point of a Kripke model extends the classical modal case with the clause:

- $M, x \models [\alpha](A_1, \dots, A_n)$ iff $\forall y_1, \dots, y_n (R_\alpha x y_1 \dots y_n \to \bigvee_{i=1}^n M, y_i \models A_i)$.
  In particular, $M, x \models \alpha$ iff $R_\alpha x$ for any modal constant α.

It follows immediately from the definitions that $[\alpha(\beta_1, \dots, \beta_n)](A_{11}, \dots, A_{1b_1}, \dots, A_{n1}, \dots, A_{nb_n})$ is logically equivalent to $[\alpha]([\beta_1](A_{11}, \dots, A_{1b_1}), \dots, [\beta_n](A_{n1}, \dots, A_{nb_n}))$, and likewise for the diamonds.

EXAMPLE 2.4

$$
\begin{aligned}
[\beta(\alpha, \iota_2(\alpha, \iota_1))](\neg[\iota_1]q, p, \iota_0) &\equiv [\beta]([\alpha]\neg[\iota_1]q, [\iota_2]([\alpha]p, [\iota_1]\iota_0)) \\
&\equiv [\beta]([\alpha]\neg q, [\iota_2]([\alpha]p, \iota_0)), \\
&\quad \text{also written as } [\beta]([\alpha]\neg q, [\alpha]p \vee \top)) \\
&\equiv [\beta]([\alpha]\neg q, \top)) \\
&\equiv \top.
\end{aligned}
$$

Accordingly, the *standard translation* ST generalizes the one for monadic languages with the clauses:

- $ST(\sigma) = R_\sigma(x)$ for every modal constant σ;
- $ST([\alpha](A_1, \dots, A_n) = \forall y_1, \dots, y_n (R_\alpha x y_1 \dots y_n \to \bigvee_{i=1}^n ST(A_i)(y_i/x))$.

All propositional logical connectives, as defined here, have their standard interpretation, so the purely modal polyadic languages are not really different from the traditional ones regarding expressiveness, but rather more convenient to use.

2.2 Polyadic Sahlqvist formulas

DEFINITION 2.5
Essentially box formula (EBF) of a variable p is a formula $A = [\alpha](p)$ or $A = [\alpha](p, A_1, \dots, A_n)$ where $A_1, \dots, A_n$ are negative formulas not containing p. The variable p in such a formula is called *the essential variable of* A, while all other variables are *inessential* in the formula.

In particular, p, being $[\iota_1]p$, is an EBF of p, too. Intuitively, every EBF in a PSF can be regarded as a *unary* box over its essential variable. Of course, the essential variable in an EBF need not be the first argument, but to keep the notation simple we will be putting it in first position whenever possible.

DEFINITION 2.6
A set of essentially box formulas is:

- *independent*, if no essential variable in a formula from the set occurs as an inessential variable in any formula from the set;
- *separated*, if all EBFs have different essential variables;
- *strongly independent*, if it is independent and separated.

DEFINITION 2.7
A *simple polyadic Sahlqvist formula (SPSF)* is any modal constant σ, or a formula $A = [\alpha](A_1, \ldots, A_n)$ where α is an n-ary modal term and each formula A_i is either positive, or a negation of an essentially box formula, and the set of essentially box formulas whose negations are among $A_1, \ldots, A_n$ is independent. A SPSF $A = [\alpha](A_1, \ldots, A_n)$ is *very simple* if it is a modal constant or all essentially box formulas whose negations are among $A_1, \ldots, A_n$ are variables.

It is proved in [10] that every polyadic Sahlqvist formula as defined in [3] is equivalent to a conjunction of SPSFs.

DEFINITION 2.8
Let $\mathcal{S} = \{B_1, \ldots, B_n\}$ be a set of EBFs with essential variables respectively $\{p_1, \ldots, p_n\}$. A *dependency digraph* $G(\mathcal{S})$ of $\mathcal{S}$ is defined as follows: the vertices of $G(\mathcal{S})$ are the variables $\{p_1, \ldots, p_n\}$ and p_i sends an arc to p_j if p_i occurs as an inessential variable in a formula from $\mathcal{S}$ with an essential variable p_j.

A digraph is called *acyclic* if it does not contain oriented cycles.

DEFINITION 2.9
Polyadic Sahlqvist formula (PSF) is any modal constant σ, or $A = [\alpha](A_1, \ldots, A_n)$ where α is an n-ary modal term and each formula A_i is either positive, or a negation of an essentially box formula, and the dependency digraph of the set of essential variables in A is acyclic.

Furthermore, the class of PSFs can be considered closed under conjunctions. The particular case when there are no arcs in the dependency digraph corresponds to the class of SPSFs.

2.3 Pre-processing of polyadic modal formulas

DEFINITION 2.10
$A \sim B$ means that A and B are valid in the same frames.

A polyadic modal formula can be pre-processed in search of its representation as a PSF, which can then be simplified further, while preserving the class of frames in which it is valid, as follows:

1. The formula can be presented in a purely modal polyadic language as a polyadic box. (Logical equivalences simplifying the formula can be applied in the process.)
2. Elimination of monotone variables: every variable in a formula that has only positive/negative occurrences in that formula can be replaced by $\bot/\top$.

3. Pulling outwards and composing boxes. Eventually the formula can be written as $A = [\alpha](A_1, \dots, A_n)$, where $A_1, \dots, A_n$ are negated boxes or variables.
4. Appropriate substitutions changing the polarity of a variable (i.e. $\neg p/p$) can be applied in order to obtain a PSF.
5. Finally, a PSF can be transformed to one with a *strongly independent* set of EBFs. This can be done by means of successive splittings of an essential variable common for two EBFs into two different variables, using

$$[\alpha](\neg[\beta_1](p, \dots), \neg[\beta_2](p, \dots), \dots, P(p, \bar{q}), \dots) \sim$$
$$[\alpha](\neg[\beta_1](p_1, \dots), \neg[\beta_2](p_2, \dots), \dots, P(p_1 \vee p_2, \bar{q}), \dots),$$

where $\dots$ in the formulas above stands for any string of arguments and $\bar{q}$ is any tuple of variables different from p.

EXAMPLE 2.11
Let $\rho(\alpha) = 3$, $\rho(\beta) = 2$, $\rho(\gamma) = 1$.

$$\begin{array}{rcl} [\alpha]([\gamma]p, \neg[\iota_2](p,q), [\beta](\langle\gamma\rangle p, q)) & \sim & [\alpha(\gamma, \iota_1, \beta)](p, \neg[\iota_2](p,q), \langle\gamma\rangle p, q) \\ & & \text{(substitution: } \neg p/p, \neg q/q\text{)} \\ & \sim & [\alpha(\gamma, \iota_1, \beta)](\neg p, \neg[\iota_2](\neg p, \neg q), \langle\gamma\rangle\neg p, \neg q) \\ & \sim & [\alpha(\gamma, \iota_1, \beta)](\neg p, \langle\iota_2\rangle(p,q), \neg[\gamma]p, \neg q) \\ & \sim & [\alpha(\gamma, \iota_1, \beta)](\neg p_1, \langle\iota_2\rangle(p_1 \vee p_2, q), \neg[\gamma]p_2, \neg q). \end{array}$$

Clearly, these transformations can be performed *deductively* in the minimal polyadic modal logic.

2.4 Canonical form of polyadic Sahlqvist formulas

Every PSF can be transformed by means of pre-processing (and permuting of arguments) to one in *canonical* form:

$$[\alpha](\neg[\beta_1](p_1, \dots), \dots, \neg[\beta_n](p_n, \dots), C_1, \dots, C_k)$$

where $\{[\beta_1](p_1, \dots), \dots, [\beta_n](p_n, \dots)\}$ is a strongly independent set of EBFs with different essential variables respectively $p_1, \dots, p_n$, and $C_1, \dots, C_k$ are positive formulas, such that $p_1, \dots, p_n$ are all variables occurring in the formula. Furthermore, if any of $\beta_1, \dots, \beta_n$ is ι_m then the formula can be further transformed into a conjunction of simpler PSFs:

$$[\alpha](\neg[\iota_m](p_1, \dots, p_m), \dots, \neg[\beta_n](p_n, \dots), C_1, \dots, C_k) \equiv$$
$$[\alpha](\neg p_1, \dots, \neg[\beta_n](p_n, \dots), C_1, \dots, C_k) \wedge \dots \wedge [\alpha](\neg p_m, \dots, \neg[\beta_n](p_n, \dots), C_1, \dots, C_k).$$

In particular, every SPSF can be transformed into

$$[\alpha](\neg[\beta_1]p_1, \dots, \neg[\beta_n]p_n, C_1, \dots, C_k)$$

where $\beta_1, \dots, \beta_n$ are unary modal terms, $p_1, \dots, p_n$ are different propositional variables, and $C_1, \dots, C_k$ are positive formulas not containing any other variables but $p_1, \dots, p_n$.

3 Hybrid polyadic modal languages and Sahlqvist formulas

In this section we consider extensions of purely modal polyadic languages with the basic ingredients of hybrid languages, namely *universal modality, nominals, and satisfaction operators*, and expand the class of polyadic Sahlqvist formulas to each of these extensions. The universal modality $[\upsilon]$ is interpreted in a Kripke model by the Cartesian square of its universe. Nominals are a special sort of variables which admit only valuations which assign to them *singleton* sets. The satisfaction operator $@_n A$ expresses the claim that the formula A is true at the state in which the nominal n is evaluated. We note that these extensions boost the expressiveness of the modal languages considerably and many frame conditions non-definable in the classical modal language, such as irreflexivity etc., become definable in hybrid extensions. See [9, 8, 4, 2, 1], and Chapter 7.3 in [3] for detailed studies of hybrid monadic languages.

3.1 The basic normal polyadic modal logic

Suppose a purely modal polyadic language $\mathcal{L}_\tau$ is fixed. The basic normal polyadic logic $\mathcal{K}_\tau$ of $\mathcal{L}_\tau$ is axiomatized as follows:

3.1.1 Axioms:

A0) Enough propositional axioms.

A1.1) The polyadic analogues of the axiom scheme K:

$[\alpha](q_1, \dots, q_{k-1}, p_1 \to p_2, q_{k+1}, \dots, q_n) \to$
$([\alpha](q_1, \dots, q_{k-1}, p_1, q_{k+1}, \dots, q_n) \to [\alpha](q_1, \dots, q_{k-1}, p_2, q_{k+1}, \dots, q_n))$.

A1.2) Composition axiom

$[\alpha(\beta_1, \dots, \beta_n)](p_{11}, \dots, p_{1b_1}, \dots, p_{n1}, \dots, p_{nb_n}) \leftrightarrow$
$[\alpha]([\beta_1](p_{11}, \dots, p_{1b_1}), \dots, [\beta_n](p_{n1}, \dots, p_{nb_n}))$, where $\rho(\beta_i) = b_i, i = 1, \dots n$.

A1.3 $[\iota_1]p \leftrightarrow p$.

3.1.2 Rules of Inference:

R0) Uniform substitution of formulas for variables *SUB*.

R1) Modus ponens *MP*.

R2) Necessitation *NEC*$_k$: If $\vdash A$ then $\vdash [\alpha](B_1, \dots, B_{k-1}, A, B_{k+1}, \dots, B_n)$ for any n-ary modal term α and formulas $A, B_1, \dots, B_{k-1}, B_{k+1}, \dots, B_n$.

PROPOSITION 3.1
Each inference rule above preserves validity in a (general) frame.

PROPOSITION 3.2
$\mathcal{K}_\tau$ is sound and complete.

PROOF. Generalization of the canonical model completeness proof for $\mathcal{K}$. See [3], also [13]. ∎

3.2 Adding universal modality

Given a purely modal polyadic language $\mathcal{L}_\tau$, we denote by $\mathcal{L}_\tau^\upsilon$ its extension with a unary *universal modality* $[\upsilon]$, the semantics of which in a τ-frame $\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)} \rangle$ is given by $R_\upsilon = W^2$.

The sets of modal terms and formulas in $\mathcal{L}_\tau^\upsilon$ accordingly extend those of $\mathcal{L}_\tau$.

The definition of PSF in $\mathcal{L}_\tau^\upsilon$ remains essentially the same, but the universal modality allows for additional preprocessing, using the following equivalences:

$$[\alpha](\neg[\upsilon]A, \dots) \equiv \neg[\upsilon]A \vee [\alpha](\bot, \dots);$$

$$\begin{gathered}[\alpha](\neg[\beta](\neg[\upsilon]A, \dots), \dots) \equiv \\ [\alpha]([\upsilon]A \wedge \neg[\beta](\bot, \dots), \dots) \equiv \\ [\alpha]([\upsilon]A, \dots) \wedge [\alpha](\neg[\beta](\bot, \dots), \dots).\end{gathered}$$

Thus, for instance, the formula $[\alpha](\neg[\beta](p, \neg[\upsilon]q), \neg[\gamma](q, \neg[\upsilon]p), P_1 \dots, P_k)$, where P_1, $\dots$, P_k are positive formulas, is not a PSF but can be transformed to the conjunction of PSFs

$$\begin{gathered}[\alpha]([\upsilon]q, [\upsilon]p, P_1 \dots, P_k) \wedge \\ [\alpha](\neg[\beta](p, \bot), [\upsilon]p, P_1 \dots, P_k) \wedge \\ [\alpha]([\upsilon]q, \neg[\gamma](q, \bot), P_1 \dots, P_k) \wedge \\ [\alpha](\neg[\beta](p, \bot), \neg[\gamma](q, \bot), P_1 \dots, P_k).\end{gathered}$$

The basic normal polyadic logic $\mathcal{K}_\tau^\upsilon$ for $\mathcal{L}_\tau^\upsilon$ extends $\mathcal{K}_\tau$ with the axioms for υ:

A2.1) The S5 axioms for the universal modality $[\upsilon]$.

A2.2) $[\upsilon]p \to [\alpha](q_1, \dots, q_{k-1}, p, q_{k+1}, \dots, q_n)$.

Note that the necessitation rule schema R2 can now be replaced by

R2υ) Necessitation NEC_k : $\vdash A$ implies $\vdash [\upsilon]A$.

PROPOSITION 3.3
$\mathcal{K}_\tau^\upsilon$ is sound and complete.

PROOF. Straightforward combination of the canonical completeness proofs for $\mathcal{K}_\tau$ and the extension of $\mathcal{K}$ with $[\upsilon]$ [9]. ∎

REMARK 3.4

1. We should note that the canonical relation R_υ^c corresponding to υ is *not the universal relation*, but an equivalence relation, so to make the interpretation of υ in the canonical model standard we consider R_υ^c-generated submodels of the whole canonical model.
2. On the other hand, first-order definability of formulas from $\mathcal{L}_\tau^\upsilon$ and the other hybrid languages will always refer to the standard semantics under consideration.

3.3 Adding nominals

We now extend the polyadic modal language $\mathcal{L}_\tau$ to $\mathcal{L}_\tau^n$ by adding *nominals* $c_1, c_2, \dots$. Henceforth by 'variable' we will mean an ordinary propositional variable, not a nominal. The definition of formulas extends accordingly, adding the clause that every nominal is a formula, and extending the set of modal terms as follows.

DEFINITION 3.5
A formula of $\mathcal{L}_\tau^n$ is *pure* if it does not contain propositional variables.

Now the definition of *modal terms* in $\mathcal{L}_\tau^n$ extends the basic one with the clause: every pure formula is a 0-ary modal term, i.e. modal terms can be parametrized with pure formulas.

Further, the definition of a model now accounts for the restriction on the nominals: an $\mathcal{L}_\tau^n$-*model* is a structure $M = \langle W, \{R_\alpha\}_{\alpha\in MT(\tau)}, V\rangle$ where V is a valuation for the propositional variables and the nominals such that $V(c)$ for any nominal c is a singleton. To simplify notation we shall write $V(c) = x$ instead of $\{x\}$. Then:

$$M, x \models c \text{ iff } V(c) = x.$$

Finally, the standard translation ST extends by

$$ST(c_i) := (x = y_i),$$

where $y_1, y_2, \dots$ is a string of reserved variables associated with the nominals $c_1, c_2, \dots$.

PROPOSITION 3.6
Every pure formula is first-order definable.

PROOF. The pure formula $A(c_1, \dots, c_n)$, where $c_1, \dots, c_n$ are all nominals occurring in A, determines the first-order condition $\forall x \forall y_1, \dots, \forall y_n ST(A)$. ∎

REMARK 3.7
A *positive formula* in a language with nominals is a formula in which all occurrences of *variables* are positive, while nominals can occur negatively, too.

DEFINITION 3.8
Polyadic Sahlqvist formulas in $\mathcal{L}_\tau^{\mathbf{n}}$ are defined as before (bearing in mind the extended definitions of modal terms and positive formulas).

In order for nominals to work well in the language, we need an additional mechanism which allows references (access) to the state named by a nominal from anywhere in the model. Such a mechanism can be the universal modality, or the satisfaction operator discussed in the next section.

The extension of $\mathcal{L}_\tau$ with υ and nominals will be denoted by $\mathcal{L}_\tau^{\upsilon,\mathbf{n}}$.

The basic logic $\mathcal{K}_\tau^{\upsilon,\mathbf{n}}$ of $\mathcal{L}_\tau^{\upsilon,\mathbf{n}}$ extends $\mathcal{K}_\tau^{\upsilon}$ with the following axioms for nominals:

A3.1) $\langle\upsilon\rangle\, c$,

A3.2) $\langle\upsilon\rangle\,(c \wedge p) \to [\upsilon](c \to p)$,

and the additional *covering rule* of inference

R3) *COV*: If $\vdash [\alpha](B_1, \dots, B_{k-1}, c \to A, B_{k+1}, \dots, B_n)$ for some nominal c not occurring in $[\alpha](B_1, \dots, B_{k-1}, A, B_{k+1}, \dots, B_n)$ then $\vdash [\alpha](B_1, \dots, B_{k-1}, A, B_{k+1}, \dots, B_n)$.

Note that the rule of uniform substitution in a language with nominals allows substitution of formulas for variables and nominals for nominals.

REMARK 3.9

1. It is easy to see that *COV* is deductively equivalent to the following *infinitary version*:
   COV$^\infty$: If $\vdash [\alpha](B_1, \dots, B_{k-1}, c \to A, B_{k+1}, \dots, B_n)$ for *every* nominal c, then $\vdash [\alpha](B_1, \dots, B_{k-1}, A, B_{k+1}, \dots, B_n)$.

2. It is sufficient to formulate *COV* only for *basic* modal terms α and then it can be proved derivable for all composite terms in a language with υ.

PROPOSITION 3.10
$\mathcal{K}_{\tau}^{\upsilon,\mathbf{n}}$ is sound and complete.

PROOF. Again, straightforward combination of the completeness proofs for $\mathcal{K}_{\tau}$ and the basic modal logic with nominals (see [12] or [8]). ∎

REMARK 3.11
The rule *COV*, which was introduced in [12] and used there to axiomatize PDL with nominals, can be omitted from the basic logic $\mathcal{K}_{\tau}^{\upsilon,\mathbf{n}}$ but is needed for its extensions.

DEFINITION 3.12
A *pure extension* of $\mathcal{K}_{\tau}^{\upsilon,\mathbf{n}}$ is an extension with pure axioms.

DEFINITION 3.13
Given a logic L in $\mathcal{L}_{\tau}^{\upsilon,\mathbf{n}}$, a *named canonical model* for L is every submodel, consisting only of maximal L-consistent sets which are closed under *COV*$^{\infty}$, of a $[\upsilon]$-generated submodel of the ordinary canonical model M_L for L. L is *canonical* if all axioms of L are valid in the underlying frame $F_L^{\mathbf{n}}$ of every named canonical model $M_L^{\mathbf{n}}$.

Note that in every named canonical model the interpretation of υ is standard. As shown for the monadic case in [8] (see also [3]), every named canonical model satisfies the truth lemma and L is complete with respect to the class of named canonical models. Thus, every canonical logic in $\mathcal{L}_{\tau}^{\upsilon,\mathbf{n}}$ is complete. Moreover, every maximal L-consistent set in a named canonical model $M_L^{\mathbf{n}}$, being closed under *COV*$^{\infty}$, contains a nominal, which is therefore evaluated at the corresponding state of $M_L^{\mathbf{n}}$, i.e. it can be used as a 'name' labelling that state. Thus, a modal language with nominals can refer to each state in a named canonical model, which entails the following fact.

PROPOSITION 3.14
Every pure extension of $\mathcal{K}_{\tau}^{\upsilon,\mathbf{n}}$ is canonical.

PROOF. Since L is closed under substitution of nominals for nominals, all substitution instances of each pure axiom are valid in every named canonical model, hence each pure axiom is valid in the underlying frame. For a more detailed proof for the monadic case see [8] or [3]. ∎

REMARK 3.15
It is known (see [8]) that the modal language with υ and nominals has the same expressive power regarding frame definability as the language with *difference operator* D (see [7, 3]). The same applies for polyadic modal languages, and thus results about Sahlqvist formulas in $\mathcal{L}_{\tau}^{\upsilon,\mathbf{n}}$ can be transferred to the polyadic language $\mathcal{L}_{\tau}^{D}$ and supplement related results in [14] where the Sahlqvist theorem has been proved for a class of Sahlqvist formulas in *versatile* polyadic languages with D, also allowing certain additional rules of inference resembling *COV*.

3.4 Satisfaction operators

Blackburn and Seligman proposed in [4] the use of the so-called *satisfaction operator* @ in hybrid modal languages instead of universal modality, in order to keep the language local and reduce its complexity. The satisfaction operator works together with nominals and has the following semantics:

$M, s \vDash @_c A$ iff $M, V(c) \vDash A$ where c is a nominal.

Thus, $@$ introduces the truth at a state of a model explicitly in the language.

The language extending $\mathcal{L}_\tau$ with nominals and $@$ will be denoted by $\mathcal{L}_\tau^{@,\mathbf{n}}$. Its basic normal logic $\mathcal{K}_\tau^{@,\mathbf{n}}$ is axiomatized over $\mathcal{K}_\tau$ by adding the following axioms, where c, d denote nominals (see [5, 1, 3] for the monadic case):

A@1 $@_c(p \to q) \to (@_c p \to @_c q)$,

A@2 $@_c p \leftrightarrow \neg @_c \neg p$,

A@3 $c \wedge p \to @_c p$,

A@4 $@_c c$,

A@5 $@_c d \leftrightarrow @_d c$,

A@6 $@_c @_d p \leftrightarrow @_d p$,

A@7 $\langle \alpha \rangle (\dots, @_c p, \dots) \to @_c p$,

and the following rules:

R@0 *@-GEN*: If $\vdash A$ then $\vdash @_c A$.

R@1 *PASTE*: If $\vdash @_d \langle \alpha \rangle (\dots, c, \dots) \wedge @_c B \to A$ for some nominal c distinct from d and not occurring in A and B then $\vdash @_d \langle \alpha \rangle (\dots, B, \dots) \to A$.

The rule *PASTE* can be rewritten by contraposition as follows:

If $\vdash A \wedge @_c B \to @_d[\alpha](\dots, \neg c, \dots)$ for some nominal c distinct from d and not occurring in A and B then $\vdash A \to @_d[\alpha](\dots, \neg B, \dots)$.

The following rule is easily derivable from *PASTE*, taking $\alpha = \iota_1$ and $B = d$:

R@2 *NAME*: If $\vdash @_c A$ for some nominal c not occurring in A then $\vdash A$.

Clearly, the rules *NAME* and *PASTE* are analogues of particular cases of *COV*, and it is interesting to note that they suffice.

Combining the completeness proofs for $\mathcal{K}_\tau$ and for the basic hybrid modal logic with nominals and $@$ (see [3]) we obtain:

PROPOSITION 3.16
$\mathcal{K}_\tau^{@,\mathbf{n}}$ is sound and complete.

REMARK 3.17
In a language with υ and nominals, $@$ is definable as follows:
$@_c A := \langle \upsilon \rangle (c \wedge A)$.

4 Sahlqvist theorem in reversive languages with nominals

In [10] we show that every polyadic Sahlqvist formula is first-order definable and canonical, and give an algorithm, extending Sahlqvist – van Benthem's algorithm, for computing their first-order equivalents. Here we will show that in sufficiently rich hybrid polyadic languages the first-order equivalents can be computed *deductively within the modal logic* and the completeness can be obtained as a corollary to Proposition 3.14.

4.1 Reversive polyadic languages

DEFINITION 4.1
A purely modal polyadic language is *reversive* if together with every n-ary modal term α it contains its *inverses* $\alpha^{-1}, \dots, \alpha^{-n}$, where for each $k = 1, \dots, n$:

$$xR_{\alpha^{-k}}y_1 \dots y_k \dots y_n \text{ iff } y_k R_\alpha y_1 \dots x \dots y_n.$$

REMARK 4.2

1. It *does not* suffice to require this condition only for the basic modal terms from the type τ, because reversiveness is apparently not preserved by compositions. For instance, the inverse δ^{-1} of $\delta = \gamma(\beta, \alpha)$ where α is a unary term and β, γ are binary terms seems not definable in terms of α, β, γ and their inverses.
2. Note that $(\alpha^{-k})^{-k} = \alpha$.

An arbitrary purely modal polyadic language $\mathcal{L}_\tau$ can be extended to a reversive one $\mathcal{L}_{\tau r}$ by adding the following clause to the definition of modal terms.

DEFINITION 4.3
(MT iv) If $n > 0$, $k \leq n$ and α is an n-ary modal term then α^{-k} is an n-ary modal term, too.

The semantics extends accordingly:

$$\begin{aligned} R_{\alpha^{-k}} \quad = \quad & \{(x_0, x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_n) \subseteq W^{n+1} | \\ & (x_k, x_1, \dots, x_{k-1}, x_0, x_{k+1}, \dots, x_n) \in R_\alpha\}. \end{aligned}$$

Thus, $R_{\alpha^{-k}}$ is obtained from R_α by transposing the 0th and the kth arguments. In particular, for a unary term α, $R_{\alpha^{-1}}$ is the usual inverse of α, as expected.

As we will see further, reversiveness provides the language with the degree of expressiveness necessary for the explicit definition of the minimal valuations of the variables, used in the computation of the first-order equivalents of PSFs.

4.2 Reversive vs. versatile languages

In [14] Venema introduces *versatile* polyadic languages, in which the set of modal terms is closed under *cyclic* permutations of the arguments of the corresponding relations, i.e. with every n-ary modal term α a versatile language contains its *conjugates* $\alpha^{(1)}, \dots, \alpha^{(n)}$, where for each $k = 1, \dots, n$:

$$xR_{\alpha^{(k)}}y_1 \dots y_k \dots y_n \text{ iff } y_k R_\alpha y_{k+1} \dots y_n, x, y_1, \dots, y_{k-1};$$

in particular $xR_{\alpha^{(n)}}y_1 \dots y_k \dots y_n$ iff $y_n R_\alpha x, y_1, \dots, y_{n-1}$.

Ostensibly, versatile languages are weaker than reversive ones since only the modalities corresponding to cyclic permutations of the relational arguments can be defined in them, while in a reversive language by composing α and $\alpha^{-1}, \dots, \alpha^{-n}$ one can construct all modal terms whose relations are obtained by permutations of the arguments of R_α.

Still, the difference in expressiveness between versatile and reversive languages is only apparent: every inverse term α^{-k} to an n-ary term α can be obtained from a conjugate of α by permuting appropriately the arguments of the box, and hence both types of languages have *the same* expressive power, and moreover are interdefinable. Indeed,

$$xR_{\alpha^{-k}}y_1 \dots y_k \dots y_n \text{ iff } y_k R_\alpha y_1 \dots y_{k-1} x y_{k+1} \dots y_n \text{ iff } xR_{\alpha^{(k)}}y_{k+1} \dots y_n y_1 \dots y_{k-1}.$$

Therefore, $[\alpha^{-k}](A_1, \ldots, A_k \ldots A_n) = [\alpha^{(k)}](A_{k+1}, \ldots, A_n, A_1, \ldots, A_{k-1})$.

Hence, the difference between the two types is only technical and not essential. What are the pros and cons of preferring one to another? On one hand, it is easy to see that versatility of polyadic languages is preserved under compositions, so it only needs to be required for the basic modal terms, while this is apparently not the case for reversive languages. On the other hand, working with versatile languages would require either extending the language with operators permutating the arguments of boxes, or keeping track of these permutations, which is technically inconvenient. That is why we choose to work with reversive languages hereafter, but all that follows holds accordingly in versatile languages, too.

4.3 The minimal logics of reversive polyadic languages

The minimal normal modal logic $\mathcal{K}_{\tau r}$ of a reversive polyadic language $\mathcal{L}_{\tau r}$ is axiomatized over $\mathcal{K}_{\tau r}$ by adding the following *axiom schemes for inverse modalities*:

A4.1) $A \to [\alpha](\neg B_1, \ldots, \neg B_{k-1}, \langle \alpha^{-k} \rangle (B_1, \ldots, B_{k-1}, A, B_{k+1}, \ldots, B_n), \neg B_{k+1}, \ldots, \neg B_n)$,

A4.2) $[(\alpha^{-k})^{-k}](\ldots, A, \ldots) \leftrightarrow [\alpha](\ldots, A, \ldots)$.

In particular, in a tense language axiom A4.1 becomes $A \to [\alpha] \langle \alpha^{-1} \rangle A$.

Likewise, the schemes above added to $\mathcal{K}_{\tau}^{\upsilon}$ and $\mathcal{K}_{\tau}^{\upsilon,\mathbf{n}}$ produce the minimal logics $\mathcal{K}_{\tau r}^{\upsilon}$ and $\mathcal{K}_{\tau r}^{\upsilon,\mathbf{n}}$ in the respective reversive languages with υ and nominals..

REMARK 4.4

1. The following are easily derivable:
   (a) $[\upsilon^{-1}]p \leftrightarrow [\upsilon]p$,
   (b) $[\iota_1^{-1}]p \leftrightarrow p$,
   (c) $[\iota_2^{-i}](p, q) \leftrightarrow [\iota_2](p, q)$, $i = 1, 2$.
2. Axiom A4.2 can be replaced by the dual of A4.1:
   $A \to [\alpha^{-k}](\neg B_1, \ldots, \neg B_{k-1}, \langle \alpha \rangle (B_1, \ldots, B_{k-1}, A, B_{k+1}, \ldots, B_n), \neg B_{k+1}, \ldots, \neg B_n)$.
   These two are inter-derivable using A4.1.
3. In a reversive language with universal modality, the scheme A4.1 can be replaced by:
   A4.1.υ) $[\upsilon](A \to [\alpha](B_1, \ldots, B_{k-1}, B, B_{k+1}, \ldots, B_n)) \leftrightarrow$
   $[\upsilon](\neg B \to [\alpha^{-k}](B_1, \ldots, B_{k-1}, \neg A, B_{k+1}, \ldots, B_n))$.

THEOREM 4.5
Each of $\mathcal{K}_{\tau r}$, $\mathcal{K}_{\tau r}^{\upsilon}$ and $\mathcal{K}_{\tau r}^{\upsilon,\mathbf{n}}$ is sound and complete.

DEFINITION 4.6
Formulas A and B of a modal language $\mathcal{L}$ are *axiomatically equivalent* if each of them is derivable in the logic axiomatized with the other over the basic normal logic for $\mathcal{L}$.

PROPOSITION 4.7
Axiomatically equivalent formulas define the same classes of frames.

PROOF. Immediate from Proposition 3.1. ∎

4.4 Computing first-order equivalents of Sahlqvist formulas

Here we show that every PSF in $\mathcal{L}^{\upsilon,\mathbf{n}}_{\tau r}$ is axiomatically equivalent in $\mathcal{K}^{\upsilon,\mathbf{n}}_{\tau r}$ to a pure formula, which is obtained by systematic computation and substitution of the minimal valuations for the essential variables. This can be done deductively within $\mathcal{K}^{\upsilon,\mathbf{n}}_{\tau r}$ because the minimal valuations here are definable by means of pure formulas. Thus we obtain a *modal calculus of Sahlqvist equivalents* which computes the first-order Sahlqvist equivalents of PSFs.

Hereafter in this section $\vdash A$ will refer to derivability in the basic logic of the minimal language containing the formula A.

LEMMA 4.8
(Monotonicity lemma) Let $\bar{p} = p^1, \dots p^m$ be a list of positive occurrences of a variable p in a formula A and B, C be any formulas. Denote by $A(Q/\bar{p})$ the result of the uniform substitution of a formula Q for the occurrences $\bar{p}$ in A.

1. If $\vdash B \to C$ then $\vdash A(B/\bar{p}) \to A(C/\bar{p})$.
2. In a language with υ : $\vdash [\upsilon](B \to C) \to (A(B/\bar{p}) \to A(C/\bar{p}))$.
3. In a language with @ : $\vdash @_c B \to (A(c/\bar{p}) \to A(B/\bar{p}))$.

PROOF. Easy structural induction on A, using the axioms for the universal modality (respectively for @). ∎

LEMMA 4.9
The formula $A = [\alpha](\neg[\gamma](p, \neg Q_1, \dots, \neg Q_n), P_1, \dots, P_k)$, where $P_1, \dots, P_k$ are formulas positive in p, is axiomatically equivalent over $\mathcal{K}_{\tau r}$ to $A^q = [\alpha](\neg q, P_1^q, \dots, P_k^q)$ where $P_i^q = P_i(\langle \gamma^{-1} \rangle (q, Q_1, \dots, Q_n)/p)$, for any variable q not occurring in A and $i = 1, \dots, k$.

PROOF. First, suppose $\vdash A$ and substitute $\langle \gamma^{-1} \rangle (q, Q_1, \dots, Q_n)$ for p in A. Then

$$\vdash [\alpha](\neg[\gamma](\langle \gamma^{-1} \rangle (q, Q_1, \dots, Q_n), \neg Q_1, \dots, \neg Q_n), P_1^q, \dots, P_k^q). \qquad (*)$$

From axiom A4.1 we obtain by contraposition

$$\vdash \neg[\gamma](\langle \gamma^{-1} \rangle (q, Q_1, \dots, Q_n), \neg Q_1 \dots, \neg Q_n) \to \neg q.$$

Now, from $(*)$ by the monotonicity Lemma 4.8 we get $\vdash [\alpha](\neg q, P_1^q, \dots, P_k^q)$.

Conversely, suppose $\vdash [\alpha](\neg q, P_1^q, \dots, P_k^q)$. Let $Q = \langle \gamma^{-1} \rangle ([\gamma](p, \neg Q_1, \dots, \neg Q_n), Q_1, \dots, Q_n)$. Then, substituting $[\gamma](p, \neg Q_1, \dots, \neg Q_n)$ for q we get

$$\vdash [\alpha](\neg[\gamma](p, \neg Q_1, \dots, \neg Q_n), P_1(Q/p), \dots, P_k(Q/p)). \qquad (**)$$

From axioms A4.1 and A4.2, we obtain by contraposition

$$\vdash \langle \gamma^{-1} \rangle ([\gamma](p, \neg Q_1, \dots, \neg Q_n), Q_1, \dots, Q_n) \to p,$$

whence by the monotonicity Lemma 4.8

$$\begin{aligned}\vdash & [\alpha](\neg[\gamma](p, \neg Q_1, \dots, \neg Q_n), P_1(Q/p), \dots, P_k(Q/p)) \\ & \to [\alpha](\neg[\gamma](p, \neg Q_1, \dots, \neg Q_n), P_1, \dots, P_k),\end{aligned}$$

hence $\vdash [\alpha](\neg[\gamma](p, \neg Q_1, \dots, \neg Q_n), P_1, \dots, P_k)$ by (**). ∎

Likewise the lemma applies to any $A = [\alpha](P_1, \dots, \neg[\gamma](p, \neg Q_1, \dots, \neg Q_n), \dots, P_k)$.

THEOREM 4.10
Every PSF is axiomatically equivalent in $\mathcal{K}_{\tau r}$ to a very simple PSF.

PROOF. Let $A = [\alpha](\neg B_1, \dots, \neg B_n, C_1, \dots, C_k)$ be a pre-processed PSF with EBFs $B_1, \dots, B_n$ and different essential variables resp. $q_1, \dots, q_n$. Let the dependency digraph of A determine a precedence order $\prec$ on these variables. We transform A into A° by a sequence of intermediate formulas $A = A_1, \dots, A_s = A^\circ$ obtained by successive replacement of all EBFs by variables one by one inductively on $\prec$, by applying Lemma 4.9. When there are more than one EBFs to be eliminated at a time, they are dealt with one by one in arbitrary order. Note that the definition of $\prec$ ensures that at every step the condition requiring the essential variable of the EBF which is being eliminated to be positive in all other arguments, needed for the application of Lemma 4.9, will be maintained.

The resulting formula A° is $[\alpha](\neg q_1, \dots, \neg q_n, D_1, \dots, D_k)$ where $D_1, \dots, D_k$ are positive formulas. ∎

LEMMA 4.11
The formula $A = [\alpha](\neg q, P_1, \dots, P_m)$ where $P_1, \dots, P_m$ are formulas positive in q is axiomatically equivalent over either of $\mathcal{K}_\tau^{\upsilon,\mathbf{n}}$ or $\mathcal{K}_\tau^{@,\mathbf{n}}$ to a $\mathcal{L}_\tau^{\mathbf{n}}$-formula $A^c = [\alpha](\neg c, P_1^c, \dots, P_m^c)$ where $P_i^c = P_i(c/q)$ for any nominal c not occurring in A and $i = 1, \dots, m$.

PROOF. If $\vdash A$ then $\vdash A^c$ by substitution of c for q in A. Conversely, let $\vdash A^c$.

1. The derivation in $\mathcal{K}_\tau^{\upsilon,\mathbf{n}}$. First, we shall prove $\vdash [\alpha](c \to \neg q, P_1, \dots, P_m)$. It is easy to see that $\vdash [\upsilon](c \to A) \vee [\upsilon](c \to \neg A)$, so it suffices to prove

$$\vdash ([\upsilon](c \to q) \vee [\upsilon](c \to \neg q)) \to [\alpha](c \to \neg q, P_1, \dots, P_m),$$

i.e.

$$\vdash [\upsilon](c \to q) \to [\alpha](c \to \neg q, P_1, \dots, P_m) \text{ and} \qquad (i)$$
$$\vdash [\upsilon](c \to \neg q) \to [\alpha](c \to \neg q, P_1, \dots, P_m). \qquad (ii)$$

For (i) first note that $\vdash [\alpha](\neg c, P_1^c, \dots, P_m^c) \to [\alpha](c \to \neg q, P_1^c, \dots, P_m^c)$ since $\vdash \neg c \to (c \to \neg q)$, by monotonicity. Thus, $\vdash [\alpha](c \to \neg q, P_1^c, \dots, P_m^c)$. Then, again by monotonicity, $\vdash [\upsilon](c \to q) \to ([\alpha](c \to \neg q, P_1^c, \dots, P_m^c) \to [\alpha](c \to \neg q, P_1, \dots, P_m))$, hence $[\upsilon](c \to q) \to [\alpha](c \to \neg q, P_1, \dots, P_m)$.

(ii) is immediate from axiom A2.2.

Now, from $\vdash [\alpha](c \to \neg q, P_1, \dots, P_m)$ applying *COV* we get $\vdash [\alpha](\neg q, P_1, \dots, P_m)$.

2. The derivation in $\mathcal{K}_\tau^{@,\mathbf{n}}$. Let d be a nominal not occurring in $[\alpha](\neg c, P_1, \dots, P_m)$. From $\vdash [\alpha](\neg c, P_1^c, \dots, P_m^c)$ we get $\vdash @_d[\alpha](\neg c, P_1^c, \dots, P_m^c)$ by *@-GEN*. Then, by the monotonicity lemma, $\vdash @_c q \to (@_d[\alpha](\neg c, P_1^c, \dots, P_m^c) \to @_d[\alpha](\neg c, P_1, \dots, P_m))$, so $\vdash @_c q \to @_d[\alpha](\neg c, P_1, \dots, P_m)$. Now applying the contrapositive version of *PASTE* we get $\vdash @_d[\alpha](\neg q, P_1, \dots, P_m)$, hence $\vdash [\alpha](\neg q, P_1, \dots, P_m)$ by applying the rule *NAME*. ∎

LEMMA 4.12
Every very simple PSF $A = [\alpha](\neg q_1, \dots, \neg q_n, Q_1, \dots, Q_k)$ in $\mathcal{L}_\tau^{\upsilon,\mathbf{n}}$ or $\mathcal{L}_\tau^{@,\mathbf{n}}$ is axiomatically equivalent to the pure formula $A^{\mathbf{P}} = [\alpha](\neg c_1, \dots, \neg c_n, P_1, \dots, P_k)$, where $P_1, \dots, P_k$ are obtained from $Q_1, \dots, Q_k$ by substitutions $c_1/q_1, \dots, c_n/q_n$ where $c_1, \dots, c_n$ are fresh nominals.

PROOF. We can assume that no other variables but $q_1, \dots, q_n$ occur in $Q_1, \dots, Q_k$ (otherwise they can be replaced by $\bot$). Applying several times Lemma 4.11 we consecutively replace the variables $q_1, \dots, q_n$ by new nominals $c_1, \dots, c_n$ and eventually obtain $A^{\mathbf{p}} = [\alpha](\neg c_1, \dots, \neg c_n, P_1, \dots, P_k)$. ∎

THEOREM 4.13
Every PSF $A = [\alpha](\neg B_1, \dots, \neg B_n, Q_1, \dots, Q_k)$ in $\mathcal{L}^{v,\mathbf{n}}_{\tau r}$ or $\mathcal{L}^{@,\mathbf{n}}_{\tau r}$ is axiomatically equivalent to a pure formula $A^{\mathbf{p}} = [\alpha](\neg c_1, \dots, \neg c_n, P_1, \dots, P_k)$ where $P_1, \dots, P_k$ are obtained from $Q_1, \dots, Q_k$ by appropriate pure substitutions.

PROOF. Immediate from Theorem 4.10 and Lemma 4.12. ∎

Now, the fist-order equivalent of the pure formula $A^{\mathbf{p}}$ is obtained immediately. Let $d_1, \dots, d_m$ be all nominals occurring in $P_1, \dots, P_k$ and different from $c_1, \dots, c_n$. Further, let $\bar{y} = y_1, \dots, y_n$ be the variables used in the standard translation ST of the nominals $c_1, \dots, c_n$, $\bar{x} = x_1, \dots, x_m$ be the variables used for $d_1, \dots, d_m$, and $x, z_1, \dots, z_k$ be a string of variables disjoint from $\bar{y}$ and $\bar{x}$. Then

$$FO(A^{\mathbf{p}}) = \forall x \bar{x} \bar{y} \bar{z} \left(R_\alpha x y_1 \dots y_n z_1 \dots z_k \to \bigvee_{j=1}^{k} ST(P_j)(z_j/x) \right).$$

EXAMPLE 4.14
The formula

$$A = [\alpha](\neg[\beta]p_1, \neg[\gamma](p_2, \neg[\delta](p_1, p_3)), \neg[\gamma](p_3, \neg\langle\delta\rangle(p_1, p_1)), [\beta]p_1 \wedge p_2)$$

has essential variables p_1, p_2, p_3 and the ordering on them induced by the dependency graph is: $p_1 \prec p_2,\ p_1 \prec p_3,\ p_3 \prec p_2$, so their essentially box formulas should be eliminated in order: p_1, p_3, p_2, applying Lemma 4.9. That elimination transforms the formula into a very simple PSF in three steps as follows:

$A_1 = [\alpha](\neg q_1, \neg[\gamma](p_2, \neg[\delta](\langle\beta^{-1}\rangle q_1, p_3)), \neg[\gamma](p_3, \neg\langle\delta\rangle(\langle\beta^{-1}\rangle q_1, \langle\beta^{-1}\rangle q_1)),$
$[\beta]\langle\beta^{-1}\rangle q_1 \wedge p_2),$

$A_2 = [\alpha](\neg q_1, \neg[\gamma](p_2, \neg[\delta](\langle\beta^{-1}\rangle q_1, \langle\gamma^{-1}\rangle(q_2, \langle\delta\rangle(\langle\beta^{-1}\rangle q_1, \langle\beta^{-1}\rangle q_1)))), \neg q_2,$
$[\beta]\langle\beta^{-1}\rangle q_1 \wedge p_2),$

$A^\circ = A_3 = [\alpha](\neg q_1, \neg q_3, \neg q_2,$
$[\beta]\langle\beta^{-1}\rangle q_1 \wedge \langle\gamma^{-1}\rangle(q_3, [\delta](\langle\beta^{-1}\rangle q_1, \langle\gamma^{-1}\rangle(q_2, \langle\delta\rangle(\langle\beta^{-1}\rangle q_1, \langle\beta^{-1}\rangle q_1))))).$

Now,
$A^{\mathbf{p}} = [\alpha](\neg c_1, \neg c_3, \neg c_2,$
$[\beta]\langle\beta^{-1}\rangle c_1 \wedge \langle\gamma^{-1}\rangle(c_3, [\delta](\langle\beta^{-1}\rangle c_1, \langle\gamma^{-1}\rangle(c_2, \langle\delta\rangle(\langle\beta^{-1}\rangle c_1, \langle\beta^{-1}\rangle c_1))))).$

The first-order equivalent of $A^{\mathbf{p}}$, and hence of A, is:
$FO(A^{\mathbf{p}}) = \forall x \forall y_1 \forall y_2 \forall y_3 \forall z (R_\alpha x y_1 y_2 y_3 z \to \theta_1 \wedge \theta_2)$, where
$\theta_1 = \forall u(R_\beta z u \to \exists v(R_\beta v u \wedge v = y_1))$ and
$\theta_2 = \exists v_1 \exists v_2 (R_\gamma z v_1 v_2 \wedge v_1 = y_3 \wedge \forall w_1 \forall w_2 (R_\delta v_2 w_1 w_2 \to \theta_{12} \wedge \theta_{22}))$,
where
$\theta_{12} = \exists t(R_\beta t w_1 \wedge t = y_1)$, and
$\theta_{22} = \exists t_1 \exists t_2 (R_\gamma t_1 w_2 t_2 \wedge t_1 = y_2 \wedge \exists s_1 \exists s_2 (R_\delta t_2 s_1 s_2 \wedge \theta_{221} \wedge \theta_{222}))$,

where $\theta_{221} = \exists t(R_\beta t s_1 \wedge t = y_1)$, and $\theta_{222} = \exists t(R_\beta t s_2 \wedge t = y_1)$.

This formula simplifies to:
$\forall x \forall y_1 \forall y_2 \forall y_3 \forall z(R_\alpha x y_1 y_2 y_3 z \to \varphi_1 \wedge \exists v(R_\gamma z y_3 v \wedge \forall w_1 \forall w_2(R_\delta v w_1 w_2 \to \varphi_2))$,
where
$\varphi_1 = \forall u(R_\beta z u \to R_\beta y_1 u)$, and
$\varphi_2 = R_\beta y_1 w_1 \wedge \exists t(R_\gamma y_2 w_2 t \wedge \exists s_1 \exists s_2(R_\delta t s_1 s_2 \wedge R_\beta y_1 s_1 \wedge R_\beta y_1 s_2))$.

As a corollary to Propositions 3.6 and 3.14 and Theorem 4.13 we obtain the main result:

THEOREM 4.15 (Sahlqvist theorem in reversive polyadic hybrid languages)

1. Every PSF in $\mathcal{L}^{\upsilon;\mathbf{n}}_{\tau r}$ or $\mathcal{L}^{@;\mathbf{n}}_{\tau r}$ is first-order definable.
2. Every extension of $\mathcal{K}^{\upsilon;\mathbf{n}}_{\tau r}$ or $\mathcal{K}^{@;\mathbf{n}}_{\tau r}$ with PSFs is complete.

COROLLARY 4.16
Every PSF in any polyadic modal language of the types introduced here is first-order definable.

5 Concluding remarks

5.1 Transfer of Sahlqvist theorem

We are interested in obtaining a proof of Sahlqvist theorem for hybrid polyadic languages not covered by Theorem 4.15 by transferring the completeness part of Theorem 4.15 down to such languages by means of a *proof-theoretic* argument based on conservativeness of the extensions to reversive hybrid languages. There are two main cases to consider: extensions from non-reversive to reversive languages and extensions with nominals. In the case of polyadic languages with υ, but without nominals, Sahlqvist theorem can be proved directly, by adapting the proof for basic purely modal polyadic languages in [10]. However, in arbitrary polyadic languages $\mathcal{L}^{\upsilon,\mathbf{n}}_{\tau}$ or $\mathcal{L}^{D}_{\tau}$ certain complications arise (see Section 6 in [14] for details) and, although some partial results still hold, the full Sahlqvist theorem for these languages is still open and that justifies the quest for transfer results. In general, neither of the types of extensions mentioned above is always conservative, but here we only consider extensions of logics axiomatized with Sahlqvist formulas. Moreover, in cases when the Sahlqvist theorem holds in the weaker languages we *do know that these extensions are conservative*. Besides, reverting the argument, i.e. proving completeness via conservativeness, would have an independent value because it would shed light both on how Sahlqvist formulas work as axioms and how the hybrid mechanisms act over standard modal languages.

5.2 On Kracht calculus

Kracht has identified in [11] a class of first-order formulas which can be algorithmically translated into Sahlqvist formulas in the classical modal language, and every classical Sahlqvist formula has an equivalent Kracht formula. In a sense, Theorem 4.15 provides a modal analogue of Kracht calculus: the first-order equivalents of pure formulas in reversive hybrid languages can be easily described syntactically, hence the first-order equivalents of *all* Sahlqvist formulas in these languages can be described and their respective Sahlqvist formulas can be computed effectively by reverting the derivations outlined in the previous section.

5.3 On the scope of the Sahlqvist theorem

Theorem 4.15 tells that in reversive hybrid languages all logics axiomatized by Sahlqvist formulas, as defined here, can be axiomatized by pure formulas as well, and for such logics the claims of the Sahlqvist theorem are much more transparent. This holds accordingly for all stronger hybrid languages, e.g. those involving *binders* or *quantifiers* over nominals [4, 1]: i.e. the pure formulas in those languages subsume, in terms of frame definability and axiomatizability, the respective classes of Sahlqvist formulas.

On the other hand, it is known that this is generally not the case for non-reversive languages. For instance (see [14]), Church-Rosser's formula $\Diamond\Box p \to \Box\Diamond p$, which is a Sahlqvist formula, is *not equivalent to a pure formula in the basic modal language, because it is not di-persistent in that language*. (A formula is di-persistent if it preserves validity from *discrete general frames* (in which all singletons are definable) to their underlying Kripke frames.) Same formula, however, is di-persistent in a *tense language* and, in fact, it is equivalent to the *pure tense formula* $Fc \to GFPc$.

Since every pure formula is di-persistent, we have thus re-proved and extended Venema's result about di-persistence of Sahlqvist tense formulas to reversive hybrid polyadic languages.

A fundamental question in this topic is: *what is the largest class of Sahlqvist formulas*? To make this a little more precise, let us emphasize again that all explicit definitions of Sahlqvist formulas given so far are *syntactic* and, in fact, very syntactically sensitive, while the idea behind both parts of Sahlqvist theorem is *semantic*, and it hinges on two basic observations:

- For every Sahlqvist formula A there is a *first-order definable minimal valuation* associated with every Kripke frame F which defines a 'minimal' model M_A^F over F such that $M_A^F \models A$ implies $F \models A$.
- These minimal valuations are *closed* with respect to the topologies associated with descriptive frames, which enforces d-persistence, hence canonicity, of Sahlqvist formulas.

These two properties can be formulated more precisely to give a purely semantic definition of the maximal class of Sahlqvist formulas, simply being the class of those formulas for which the method of proof of Sahlqvist theorem works. However, this definition gives little insight on the syntactic shape of these formulas.

Thus, in conclusion, there is still a gap between the syntactic form and semantic essence of Sahlqvist formulas. Some undecidability results (see [6]) indicate that this gap cannot be closed completely, but still there is justified hope that in the case of reversive hybrid languages syntax and semantics can meet in a large and natural class of formulas as stated in the following conjecture.

CONJECTURE 5.1
Every di-persistent and locally first-order definable formula in a reversive language with nominals is axiomatically equivalent to a pure formula, and therefore the classes of Sahlqvist, pure, and locally first-order definable di-persistent formulas in such languages coincide, up to frame equivalence.

Acknowledgements

We thank Carlos Areces and an anonymous referee for useful comments and corrections. The work of Valentin Goranko was partly done during his visit to the University of Sofia and was supported by the SASOL research fund of the Faculty of Natural Sciences of Rand Afrikaans

University and by a research grant GUN 2034353 of the National Research Foundation of South Africa.

References

[1] C. Areces. *Logic Engineering. The case of Description and Hybrid Logics, ILLC Dissertation Series DS-2000-5*. PhD thesis, ILLC, University of Amsterdam, 2000.

[2] P. Blackburn. Internalizing labeled deduction. *Journal of Logic and Computation*, **10**, 137–168, 2000.

[3] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Number 53 in Cabridge Tracts in Theoretical Computer Science. Cambridge University Press, 2001.

[4] P. Blackburn and J. Seligman. What are hybrid languages? In *Advances in Modal Logic, vol. 1*, M. Kracht, M. de Rijke, H. Wansing, and M. Zakcharyaschev, eds. pp. 41–62, CSLI Publications, Stanford, 1998.

[5] P. Blackburn and M. Tzakova. Hybrid completeness. *Logic Journal of the IGPL*, **4**, 625–650, 1998.

[6] A. Chagrov and M. Zakharyaschev. Sahlqvist formulas are not so elementary even above S4. In *Logic Colloquium'92*, L. Csirmaz, D. Gabbay, and M. de Rijke, eds. pp. 61–73, CSLI Publications, Stanford, 1995.

[7] M. de Rijke. *Extending Modal Logic*. PhD thesis, ILLC, University of Amsterdam, 1993.

[8] G. Gargov and V. Goranko. Modal logics with names. *Journal of Philosophical Logic*, **22**, 607–636, 1993.

[9] V. Goranko and S. Passy. Using the universal modality: Gains and questions. *Journal of Logic and Computation*, **2**, 5–30, 1992.

[10] V. Goranko and D. Vakarelov. Sahlqvist formulas unleashed in polyadic modal languages. In *Advances in Modal Logic, vol. 3*, F. Wolter, H. Wansing, M. de Rijke, and M. Zakcharyaschev, eds. CSLI Publications, Stanford, 2001, to appear.

[11] M. Kracht. How completeness and correspondence theory got married. In *Diamonds and Defaults*, M. de Rijke, ed., pp. 175–214. Kluwer, Dordrecht, 1993.

[12] S. Passy and T. Tinchev. An essay in combinatory dynamic logic. *Information and Computation*, **93**, 263–332, 1991.

[13] D. Vakarelov. Rough polyadic modal logics. *Journal of Applied Non-classical Logics*, **1**, 9–35, 1991.

[14] Y. Venema. Derivation rules as anti-axioms in modal logic. *Journal of Symbolic Logic*, **58**, 1003–1034, 1993.