Modal Logics for Parallelism, Orthogonality, and Affine Geometries

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Abstract

We introduce and study a variety of modal logics of parallelism, orthogonality, and affine geometries, for which we establish several completeness, decidability and complexity results and state a number of related open, and apparently difficult problems. We also demonstrate that lack of finite model property of the modal logics for sufficiently rich affine or projective geometries (incl. the real affine and projective planes) is a rather common phenomenon.

1 Introduction

The formal treatment of geometry goes back to Euclid's epic work "Elements" which was also the first systematic application of the *axiomatic method* in mathematics, which is at the heart of the logical enterprise. Meanwhile, the discovery of non-Euclidean geometries, which showed inter alia the independence of Euclid's 'fifth postulate' from the other axioms of the Euclidean geometry, was an impressive demonstration of the strength and usefulness of formal logical approach in mathematics. Hilbert, the most influential proponent of the axiomatic method in mathematics, illustrated its power by re-doing Euclid's work into a precise and rigorous modern treatment which eventually put geometry on sound axiomatic foundations (see [Hilbert, 71]). The logical aspects of the foundation of geometry were advanced further by Tarski and his students. In particular, Tarski developed the axiomatic theory of Euclidean planes over arbitrary real-closed fields, and embedded the first-order theory of the former into the first-order theory of the latter, using the method of analytic geometry, thus showing the decidability of the Euclidean geometry (see [Tarski, 59]). Furthermore, he showed how the whole analytic geometry can be developed systematically using just two geometric relations, viz. betweenness and equidistance. Also, Tarski and Szczerba obtained a complete elementary characterization of the first-order theory of betweenness alone (see [Szczerba and Tarski, 79]).

While Tarski's methods work for geometric models whose logical languages are rich enough to express metric and ordering properties, there are weaker, yet important geometric structures, viz *affine and projective geometries*, in which this is not possible, and their elementary theories are considerably less studied.

Affine geometry studies the relations of incidence, collinearity, parallelism etc. between points and lines in a real or abstract geometric space. Unlike projective geometry, in which every two lines are incident, the affine geometry admits non-incident, i.e. parallel, lines, and thus comes closer to what we believe to be the "real geometry". Affine geometry also partly deals with other relations on points, such as betweenness (and therefore ordering) of points on lines, but not with distances, angles or related metric notions. Thus, while affine spaces are too general to allow development of analytic geometry in them, provided they satisfy some natural properties they are still amenable to algebraic treatment by means of coordinatization. That enables study of affine models by studying respective algebraic structures called *ternary rings* (see e.g. [Szmielew, 83]). While this algebraic treatment is not instrumental in the present paper, we mention it as a tool for algebraic investigation and characterization of the logical theories of affine spaces which, for instance, can be used to establish various logical properties such as finite model property or the lack thereof, decidability and complexity results of these theories.

The present paper continues and extends the research on modal logics for geometries initiated with [Balbiani et al, 97], [Balbiani, 98] and [Venema, 99], where modal logics for projective geometries have been studied. The transition from projective to affine models turns out to complicate the modal logics substantially, and standard techniques such as canonical model completeness and filtration no longer work readily in the presence of the Euclidean and other natural properties of affine structures.

While the paper does not offer many surprises or technical novelties, it establishes the basic facts in the field and outlines several open problems which seem to be hard to resolve with the currently available techniques. Some of these open problems are related to finite model property and decidability results but, as indicated in the paper, lack of finite model property in such logical systems for rich enough affine or projective geometries becomes a rather common phenomenon.

Finally, a comment on the utility of this enterprise. While the study of weak geometric models and their logical theories has limited mathematical value, it is additionally justified from applied perspective since these logical systems can be used to formalize various aspects and modes of practical spatial reasoning, such as reasoning on street charts, bus or metro schemes, geographic maps, etc. Depending on the level of abstractness or precision, these structures could range from very weak affine models, where little, if anything, from the real Euclidean geometry still holds, to real Euclidean planes and spaces. From this perspective we view the present paper as not only an exercise in modal logic and geometry, but also as an initial effort towards developing suitable modal logical machinery for spatial reasoning in such situations. In that respect the present work is close in spirit to [Aiello and van Benthem, 2001].

The paper is organized as follows: in the preliminary section we introduce the models of incidence, parallelism and orthogonality, and affine geometries which will be considered in the paper. In the following two sections we introduce and study the modal logics of incidence, parallelism (in abstract spaces, as well as $\mathbb{R}^n$) and orthogonality in the real plane, for which we establish complete axiomatizations and decidability results. Section 5 deals with modal logics for affine geometries with strict (irreflexive) parallelism. There we introduce two modal systems, for standard and general affine models, and establish their completeness. It is still unknown to us whether these two systems are equivalent, with respect to standard affine models is still an open problem, neither if they have the finite model property, nor if they are decidable. In section 6, respectively, we study the affine modal logic of weak (reflexive) parallelism. Due to the weaker language, so far we have only been able to establish its completeness with respect to general models (in which two lines may intersect in more than one point), but its completeness for the standard semantics is still open. We have also established the finite model property and decidability of that logic by interpreting it into the modal logic of projective geometry with a distinguished line. In the last section we have demonstrated that lack of finite model property of some modal logic for affine and projective geometry, incl. the modal logics of the real affine and projective planes.

2 Preliminaries: models of incidence, parallelism, orthogonality, and affine geometries

2.1 Affine geometries

Definition 1 *A two-sorted* ***incidence frame*** *is a structure* $< \mathbf{Po}, \mathbf{Li}, \mathbf{I} >$ *where* **Po** *and* **Li** *are non-empty sets and* $\mathbf{I} \subseteq \mathbf{Po} \times \mathbf{Li}$. *A* ***parallelism frame*** *is a structure* $< \mathbf{Li}, \| >$ *where* $\|$ *is a binary relation over a nonempty set* **Li**. *An* ***affine frame*** *is a structure* $< \mathbf{Po}, \mathbf{Li}, \mathbf{I}, \| >$ *where* $< \mathbf{Po}, \mathbf{Li}, \mathbf{I} >$ *is an incidence frame and* $< \mathbf{Li}, \| >$ *is a parallelism frame.*

The elements of **Po** will be called **points** and the elements of **Li** will be called **lines.** We shall use metavariables X, Y, Z, etc. for points, and metavariables x, y, z, etc. for lines. If the relation $\mathbf{I}$ holds for a point X and a line x then we will use expressions like "X and x are **incident**" or "X **lies on** x" or "x **passes through** X". The intended meaning of sentences like "line x **connects** points X and Y" will be that both X and Y lie on x whereas the intended meaning of sentences like "point X is at the **intersection** of lines x and y" will be that both x and y pass through X. We say that "lines x and y are **intersecting**" if there is a point at the intersection of x and y. As for the relation $\|$, if it holds for two lines x and y then we will use the phrases "x **is parallel to** y" or "x **and** y **are parallel**". Recall that, until otherwise assumed, parallelism will be an *irreflexive relation* on lines, and its reflexive version will be referred to as **weak parallelism.**

Let us consider an arbitrary affine frame $<\mathbf{Po},\mathbf{Li},\mathbf{I},\|>$. Clearly, many highly complex affine relations between points and lines can be expressed via $\mathbf{I}$ and $\|$. We introduce special notations for the following relations which will be used further:

- **Collinearity** of 3 points: $Col(X,Y,Z) := \exists x(X\mathbf{Ix} \wedge Y\mathbf{I}x \wedge Z\mathbf{I}x)$;
- **Concurrency** of 3 lines: $Con(x,y,z) := \exists X(X\mathbf{I}x \wedge X\mathbf{I}y \wedge X\mathbf{I}z)$;
- **Incidence** of 2 points with a line: $Inc(X,Y,x) := X\mathbf{I}x \wedge Y\mathbf{I}x$;
- **Intersection** of 2 lines at a point: $Int(x,y,X) := X\mathbf{I}x \wedge X\mathbf{I}y$;
- **Parallelism** relation between 4 points: $\|(X,Y,Z,T) := X = Y \vee Z = T \vee \exists x\exists y(Inc(X,Y,x) \wedge Inc(Z,T,y) \wedge (x = y \vee x\|y))$;
- **Parallelogram** relation of 4 points: $Par(X,Y,Z,T) := \neg Col(X,Y,Z) \wedge \|(X,Y,Z,T) \wedge \|(X,Z,Y,T)$;
- **Trapezoid** relation of 4 points with respect to a point of reference: $Tra(O,X,Y,Z,T) := O \neq X \wedge O \neq Z \wedge \neg Col(X,Y,Z) \wedge Col(O,X,Y) \wedge Col(O,Z,T) \wedge \|(X,Z,Y,T)$.

  For every positive integer $n \geq 0$, let $\mathrm{Dif}_n(X_0,...,X_n)$ be the abbreviation for $\wedge^{i\neq j}_{i,j=0,\dots,n} X_i \neq X_j$, and likewise, $\mathrm{Dif}_n(x_0,...,x_n)$ be the abbreviation for $\wedge^{i\neq j}_{i,j=0,\dots,n} x_i \neq x_j$. Since we look for axioms of affine geometries, we shall express these axioms in terms of $\mathbf{I}$ and $\|$. Firstly, let us consider parallelism frames. There are several degrees of freedom in the choice of the minimal model of parallelism. For a start, let us consider the following conditions.

Definition 2 *A* ***quasi-model of parallelism*** *is a parallelism frame* $<\mathbf{Li},\|>$ *satisfying the following conditions:*

(SYM) *Symmetry:* $\forall x\forall y(x \parallel y \to y \parallel x)$;

(PTRAN) *Pseudo-transitivity:* $\forall x\forall y\forall z(x \parallel y \wedge y \parallel z \to x = z \vee x \parallel z)$.

However we can choose between quasi-models where the notion of parallelism is strict and models where self-parallel lines are allowed.

Definition 3 *A* ***model of (strict) parallelism*** *is an irreflexive quasi-model of parallelism.*

Thus, models of strict parallelism are isomorphic to disjoint unions of relational structures of the form $<\mathbf{W},\neq>$ where $\neq$ is the difference relation over the non-empty set $\mathbf{W}$. In models of strict parallelism, it is still possible for a line to be parallel with no other line.

Definition 4 *Given* $n \geq 0$*, we say that a parallelism frame* $<\mathbf{Li},\|>$ *is* **n-*rich*** *if every line in it is parallel to more than n lines:* $\forall x\exists y_0...\exists y_n(x \parallel y_0 \wedge ... \wedge x \parallel y_n \wedge \mathrm{Dif}_n(y_0,...,y_n))$.

Now, let us consider affine frames. In any affine frame, there might exist self-parallel lines as well as lines having more than one common point with another line. This observation inspires the following definition.

Definition 5 *In an affine frame $< \mathbf{Po}, \mathbf{Li}, \mathbf{I}, \| >$:*

- *a line is **standard** if it is not self-parallel;*
- *a point is **standard** if it is not incident with a non-standard line;*
- *a line is **normal** if it has no more than one common intersection point with every other line;*
- *a point is **normal** if it is not incident with a non-normal line.*

Hence, standard points are those, which are not incident with any self-parallel line and normal points are those which have no more than one common incident line with every other point.

Since we are looking for affine models of space, the relation of parallelism may be defined in terms of incidence in a very simple way: 2 lines x and y should be parallel iff they do not pass through a common point. Whence, the following definition:

Definition 6 *An affine frame $< \mathbf{Po}, \mathbf{Li}, \mathbf{I}, \| >$ is said to be a **basic affine model** if the following conditions are satisfied:*

(BAM1) *$< \mathbf{Li}, \| >$ is a model of parallelism;*

(BAM2) *Parallel lines do not have common points: $\forall x \forall y (x \| y \rightarrow \neg \exists X (Int(x, y, X)))$;*

(BAM3) *every two distinct points have not more than one common incident line (**normality**):*
$\forall X \forall Y \forall x \forall y (Inc(X, Y, x) \wedge Inc(X, Y, y) \rightarrow X = Y \vee x = y)$.

Note that the latter also implies that every two distinct lines have not more than one common intersection point: $\forall x \forall y \forall X \forall Y (Int(x, y, X) \wedge Int(x, y, Y) \rightarrow x = y \vee X = Y)$.

In other words, in a basic affine model parallel lines in a basic affine model do not intersect, whereas all lines, and hence all points, are standard and normal. Note that basic affine model are defined by *universal conditions* only.

It is still possible, however, in basic affine model that two points are non-incident with a common line and two lines are both non-intersecting and non-parallel, so we have to add some universal-existential conditions, too.

Definition 7 *A basic affine model $< \mathbf{Po}, \mathbf{Li}, \mathbf{I}, \| >$ is:*

- ***Line-connected*** *if every two points are incident with a common line: $\forall X \forall Y \exists x (Inc(X, Y, x))$. In particular, every point in a line-connected model is incident with at least one line.*
- ***Point-connected*** *if every two non-parallel lines are incident with a common point: $\forall x \forall y (\neg \exists X (Int(x, y, X)) \longrightarrow x \| y)$. In particular, every line in a point-connected model is incident with at least one point.*
- ***Connected*** *if it is both line-connected and point-connected.*
- ***Euclidean*** *if every point not incident with a given line is incident with at least one line parallel to the given line: $\forall x \forall X (\neg X \mathbf{I} x \rightarrow \exists y (X \mathbf{I} y \wedge x \| y))$. Note that the uniqueness of y follows in basic affine models from the pseudo-transitivity of the parallelism. This uniqueness property is known as Euclid's axiom of parallelism.*
- ***Point-rich*** *if every line has a non-incident point: $\forall x \exists X \neg (X \mathbf{I} x)$.*
- ***Line-rich*** *if every point has a non-incident line: $\forall X \exists x \neg (X \mathbf{I} x)$.*

- ***Parallel-rich*** *if every line is parallel to some line:* $\forall x \exists y (x \| y)$.

  Note that every connected and parallel-rich model is also point-rich and line-rich. Moreover, every Euclidean and point-rich model is also parallel-rich.

- ***Standard*** *if it is connected, Euclidean, and parallel-rich.*

Our definitions yield the following consequences:

- In every connected model every line is incident with a point and every point is incident with a line. The smallest connected model has just 1 line and 1 point incident with that line.
- Every point-rich, line-rich, and connected model has at least 3 non-collinear points and 3 non-concurrent lines.
- The smallest parallel-rich and connected model has at least 4 points, no 3 of which are collinear, and 6 lines, every line connecting exactly 2 points. It can be visualized by a tetrahedron where the vertices represent points and the edges represent lines. Note that this model is standard.

Remark 8 *Standard models hold the central position among the basic affine models not only due to the consequences considered above. In fact, they represent the most general models of affine space for which the algebraic treatment by means of affine coordinatization is possible, which can be proved by means of representation theorems (see [Szmielew, 83] for details).*

Note also that Hilbert's axioms for incidence in the plane require line-connectedness, normality, existence of at least 3 non-collinear points, and existence of at least 2 points on every line (see [Hilbert, 71]). These properties are satisfied in every standard model.

However, there exist elementary (even, universal) properties concerning incidence and parallelism in the Euclidean affine plane that are not satisfied in some standard models. Notable examples are Desargues' properties*. These properties follow from Hilbert's axioms for incidence in the space, Hilbert's axioms for betweenness and the Euclid's axiom of parallelism. They, however, do not follow from Hilbert's axioms for incidence in the plane alone plus Hilbert's axioms for betweenness and the Euclid's axiom of parallelism. In fact, they are necessary and sufficient conditions for a model of plane geometry to be embeddable in a model of space geometry satisfying the respective axioms.*

Definition 9 *A basic affine model* $<\mathbf{Po}, \mathbf{Li}, \mathbf{I}, \|>$ *is:*

- ***Weakly Desarguesian*** *if it satisfies the property:*

  $$\forall X \forall Y \forall Z \forall T \forall U \forall V (\neg Col(Z,T,U) \wedge Par(X,Y,Z,T) \wedge Par(X,Y,U,V) \rightarrow Par(Z,T,U,V);$$

- ***Strongly Desarguesian*** *if it satisfies the property:*

  $$\forall O \forall X \forall Y \forall Z \forall T \forall U \forall V (\neg Col(Z,T,U) \wedge Tra(O,X,Y,Z,T) \wedge Tra(O,X,Y,U,V) \rightarrow Tra(O,Z,T,U,V).$$

Proposition 10 *([Szmielew, 83]) Every strongly Desarguesian standard model is weakly Desarguesian.*

Another property that is not satisfied in all standard models is the property of Pappus:

Definition 11 *A basic affine model* $<\mathbf{Po}, \mathbf{Li}, \mathbf{I}, \|>$ *is said to be* ***Pappian*** *if it satisfies the following condition:*

$$\forall X \forall Y \forall Z \forall T \forall U \forall V (Col(X,Y,Z) \wedge Col(T,U,V) \wedge \|(X,U,Y,T) \wedge \|(X,V,Z,T) \rightarrow \|(Y,V,Z,U).$$

Proposition 12 *([Szmielew, 83]) Every Pappian standard model is strongly Desarguesian.*

Proposition 13 *([Szmielew, 83]) Every finite strongly Desarguesian standard model is Pappian.*

Important examples of Pappian (hence also Desarguesian) standard affine models are introduced below.

Definition 14 *A model of strict parallelism is **real** if it consists of lines in the real plane with the usual relation of strict parallelism. A basic affine model is **real** if it consists of points and lines in the real plane with the usual incidence and strict parallelism relations. The **Euclidean parallelism plane** is the model of strict parallelism consisting of all lines in the real plane with the usual strict parallelism relation. The **Euclidean affine plane** is the basic affine model consisting of all points and lines in the real plane with the usual incidence and strict parallelism relations.*

Note that every real basic affine model satisfies all universal first-order properties of the real plane. Since on standard models the properties of Desargues and the property of Pappus are equivalent to universal conditions, every real standard model is Desarguesian and Pappian. The first-order theories of the Euclidean parallelism plane and the Euclidean affine plane are decidable, since both of them are embedded in elementary geometry [Tarski, 59]. However, little seems to be known about their axiomatizations[1].

2.2 Orthogonal geometries

Definition 15 *An **orthogonality frame** is a structure $<\mathbf{Li}, \perp>$ where $\perp$ is a binary relation over a non-empty set* **Li**.

Again, the elements of **Li** will be called **lines,** we shall use metavariables x, y, z, etc. for lines, and if the relation $\perp$ holds for lines x and y, we will say that "x is **orthogonal** to y' or that "x and y are **orthogonal**". So far, parallelism frames and orthogonality frames differ only by the notation for the relation.

Definition 16 *A **quasi-model of orthogonality** is an orthogonality frame $<\mathbf{Li}, \perp>$ satisfying the following conditions:*

(SYM) *Symmetry:* $\forall x \forall y (x \perp y \rightarrow y \perp x)$;

(3TRAN) *3-transitivity:* $\forall x \forall y \forall z \forall t (x \perp y \wedge y \perp z \wedge z \perp t \rightarrow x \perp z)$.

Note that quasi-models of orthogonality behave rather simply concerning self-orthogonal lines: if x is self-orthogonal then $\perp$ is the universal relation over the set of lines orthogonal to x.

This observation motivates the following definition.

Definition 17 *A **model of strict orthogonality** is an irreflexive quasi-model of orthogonality.*

Definition 18 *An orthogonality frame $<\mathbf{Li}, \perp>$ is **orthogonal-rich** if every line is orthogonal to some line:* $\forall x \exists y (x \perp y)$.

We introduce the relation of *weak* parallelism by means of orthogonality:

- **Parallelism** of 2 lines: $x \| y := \exists z (x \perp z \wedge z \perp y)$.

[1] To confirm this claim (while revealing our ignorance): we have not been able to find an explicit reference on the first-order theory and the universal theory of the real affine or projective plane. Still, in [Balbiani and Goranko, 2002] we provide an axiomatization of the first-order theory of the parallelism in Euclidean spaces.

In orthogonal-rich models of strict orthogonality $\|$ is an equivalence relation. Thus, these models are disjoint unions of pairs of mutually orthogonal classes of weakly parallel lines.

Definition 19 *An orthogonality frame* $<\mathbf{Li}, \perp>$ *will be called* ***n-rich,*** *where* $n \geq 2$, *if every line is parallel to at least* n *different lines:* $\forall x \exists y_1 ... \exists y_n (x \parallel y_1 \wedge ... \wedge x \parallel y_n \wedge Dif_n(y_1, ..., y_n))$.

Important examples of models of strict orthogonality are introduced below.

Definition 20 *A model of strict orthogonality is* ***real*** *if it consists of lines in the real plane with the usual relation of orthogonality. The* ***Euclidean orthogonality plane*** *is the model of strict orthogonality consisting of all lines in the real plane with the usual orthogonality relation.*

Again, decidability of the elementary theory of the Euclidean orthogonality plane can be obtained via embedding into elementary geometry [Tarski, 59]; its axiomatization is given in [Balbiani and Goranko, 2002].

3 The modal logic of incidence

3.1 Basic logical framework

The modal logic of incidence considered by Balbiani [Balbiani, 98] and Venema [Venema, 99] is based on a two-sorted propositional modal language $\mathcal{L}([\in], [\ni])$ with sorts for points and lines together with unary **modalities of incidence** $[\in]$ and $[\ni]$. The sets of **point formulas PFOR** (with metavariables α, β, γ, etc.) and **line formulas LFOR** (with metavariables σ, τ, ρ, etc.) in this language are defined by mutual recursion over sets of **point variables PV** $=\{P_1, P_2, ...\}$ and **line variables LV** $=\{p_1, p_2, ...\}$ as follows:

- **PFOR** $:= P_i \mid \neg\alpha \mid (\alpha \wedge \beta) \mid [\in]\sigma$;
- **LFOR** $:= p_i \mid \neg\sigma \mid (\sigma \wedge \tau) \mid [\ni]\alpha$.

The diamond modalities $\langle\in\rangle$ and $\langle\ni\rangle$ are defined as duals of $[\in]$ and $[\ni]$.

The semantics of incidence modal logics is based on incidence frames in the expected way. More precisely, an **incidence model** is a structure of the form $<\mathbf{Po}, \mathbf{Li}, \mathbf{I}, \mathbf{m}>$ where $<\mathbf{Po}, \mathbf{Li}, \mathbf{I}>$ is an incidence frame and $\mathbf{m}$ is a **valuation** on $<\mathbf{Po}, \mathbf{Li}, \mathbf{I}>$, i.e. a function associating with each point variable P_i from **PV** a set $\mathbf{m}(P_i)$ of points in **Po**, and with each line variable p_i from **LV** a set $\mathbf{m}(p_i)$ of lines in **Li**. Given an incidence model $\mathbf{M} =<\mathbf{Po}, \mathbf{Li}, \mathbf{I}, \mathbf{m}>$, the **truth-relation** between geometrical entities in **M** and formulas is defined as follows by induction, via the clauses:

- $\mathbf{M}, X \models [\in]\sigma$ iff for all lines x in **Li**, if $X\mathbf{I}x$ then $\mathbf{M}, x \models \sigma$, and
- $\mathbf{M}, x \models [\ni]\alpha$ iff for all points X in **Po**, if $X\mathbf{I}x$ then $\mathbf{M}, X \models \alpha$.

The following definitions are standard. A point formula α (respectively, a line formula σ) is **satisfiable** in an incidence frame if it is true at some point (respectively, some line) in some model based on this frame. Given a class **C** of incidence frames, a point formula α (respectively, a line formula σ) is **C-valid**, in symbols $\models_{\mathbf{C}} \alpha$ (respectively, $\models_{\mathbf{C}} \sigma$), if it is true at every point (respectively, every line) in every model based on a frame of **C**.

The main classes of incidence frames in the papers of Balbiani and Venema are the classes $\mathbf{C}_{QPF}$ and $\mathbf{C}_{PF}$. $\mathbf{C}_{QPF}$ is the class of all incidence frames $<\mathbf{Po}, \mathbf{Li}, \mathbf{I}>$ such that:

- Every two points are incident with a common line. In particular, every point is incident with a line.

- Every two lines are intersecting. In particular, every line is incident with a point.
- There are at least 4 points such that no 3 of them are collinear.

The elements of $\mathbf{C}_{QPF}$ are referred to as **quasi-projective frames**, while $\mathbf{C}_{PF}$ is the class of a **projective frames**, i.e. quasi-projective frames $< \mathbf{Po}, \mathbf{Li}, \mathbf{I} >$ satisfying the following *normality* condition: every two distinct points have not more than one common incident line.

Note that in every projective frame:

- Every two distinct lines have not more than one common intersection point.
- There are at least 4 lines such that no 3 of them are intersecting with a common point.

This follows from the well-known duality principle between points and lines in projective frames, which has its counterpart in the incidence modal logic formulated by Balbiani and Venema and presented in section 3.2.

3.2 Modal logic for projective geometry

The main principle of projective geometry states that every two lines are incident with a common point, and it distinguishes projective geometry from affine geometry.

Thus, in a quasi-projective frame the relation $\mathbf{I} \circ \mathbf{I}^{-1}$ is universal over **Po** and the relation $\mathbf{I}^{-1} \circ \mathbf{I}$. is universal over **Li**. Hence, the modal logic of incidence in projective geometry can be axiomatized as the calculus **PG** which is obtained by adding to the minimal normal tense logic in the language $\mathcal{L}([\in],[\ni])$ the **S5** axioms for the modalities $[\in][\ni]$ and $[\ni][\in]$. Derivability of a formula φ in **PG** is denoted by $\vdash_{\mathbf{PG}} \varphi$. Every generated subframe of the canonical frame for **PG** is a quasi-projective frame, but it may not satisfy the normality condition. However, it gives us the starting point to establish the completeness of **PG** with respect to the classes $\mathbf{C}_{QPF}$ and $\mathbf{C}_{PF}$.

Proposition 21 *([Venema, 99]) Every generated subframe of the canonical frame for* **PG** *is a p-morphic image of a projective frame.*

As a consequence, in [Venema, 99] Venema obtains the following completeness result.

Proposition 22 *Let φ be a formula in the language $\mathcal{L}([\in],[\ni])$. Then $\models_{\mathbf{C}_{QPF}} \varphi$ iff $\models_{\mathbf{C}_{PF}} \varphi$ iff $\vdash_{\mathbf{PG}} \varphi$.*

4 Modal logics of parallelism and orthogonality

4.1 Modal logics of parallelism

Our modal logics of parallelism are based on a propositional modal language $\mathcal{L}([||])$ with one unary **modality of parallelism** $[||]$. The formulas of $\mathcal{L}([||])$ are line formulas **LFOR** (with metavariables σ, τ, ρ, etc.) defined by recursion over a set of line variables $\mathbf{LV} = \{p_1, p_2, ...\}$ as follows:

- $\mathbf{LFOR} := p_i \mid \neg\sigma \mid (\sigma \wedge \tau) \mid [||]\sigma$.

The diamond modality $\langle || \rangle$ is defined as the dual of $[||]$. By *length*(σ) we will denote the number of occurrences of symbols in a formula σ.

The semantics is based on parallelism frames in the expected way. More precisely, a **parallelism model**[2] is a structure of the form $< \mathbf{Li}, \|, \mathbf{m} >$ where $< \mathbf{Li}, \| >$ is a parallelism frame and **m** is a **valuation** on $< \mathbf{Li}, \| >$, i.e. a function associating with each line variable p_i from **LV** a set $\mathbf{m}(p_i)$ of lines in **Li**. Given a parallelism model $\mathbf{M} =< \mathbf{Li}, \|, \mathbf{m} >$, the **truth-relation** between lines in **M** and line formulas is defined by induction, via the clause:

- $\mathbf{M}, x \models [\|]\sigma$ iff for all lines y in **Li**, if $x\|y$ then $\mathbf{M}, y \models \sigma$.

In the same way as in section 3.2, we define truth, satisfiability and validity of formulas in parallelism models.

Let $\mathbf{C}_{QMP}$ be the class of all quasi-models of parallelism and $\mathbf{C}_{MSP}$ be the class of all models of strict parallelism.

Predictably, the modal logic of parallelism **PAR** is obtained by adding to **K** the following axioms:

1. $\sigma \rightarrow [\|] \langle \| \rangle \sigma$;
2. $\sigma \wedge [\|]\sigma \rightarrow [\|][\|]\sigma$.

These axioms are also known as the modal axioms for inequality, see [de Rijke, 1992]. Besides the usual rules of modus ponens and necessitation, we can add the irreflexivity rule for $[\|]$:

$$\frac{([\|]p \wedge \neg p) \rightarrow \sigma, \text{ for some line variable } p \text{ not occurring in } \sigma}{\sigma}$$

thus obtaining $\mathbf{PAR}^+$. This rule, however, is admissible in **PAR**, so it will not be used hereafter.

Derivability of a formula σ in **PAR** will be denoted by $\vdash_{\mathbf{PAR}} \sigma$.

Proposition 23 *Let σ be a formula in the language $\mathcal{L}([\|])$. Then $\models_{\mathbf{C}_{QMP}} \sigma$ iff $\models_{\mathbf{C}_{MSP}} \sigma$ iff $\vdash_{\mathbf{PAR}} \sigma$.*

Proof. The proof repeats the completeness proof for the modal logic of inequality **DL**, see [de Rijke, 1992]. ■

Although suggested in [de Rijke, 1992], it is not known whether **PAR** (or just the modal logic of pseudo-transitivity) admits filtration. However, using the fact that in generated models $< \mathbf{Li}, \| >$ of strict parallelism, $\|$ is just the inequality relation, Demri has proved in [Demri, 96] the following complexity result[3].

Proposition 24 *The satisfiability problems in any of the classes $\mathbf{C}_{QMP}$ and $\mathbf{C}_{MSP}$ is NP-complete.*

Remark 25 *Another interesting question, related to the complexity result above, is the complexity of the satisfiability problem in the class $\mathbf{C}_{PT}$ of all frames $< \mathbf{W}, \mathbf{R} >$ where $\mathbf{R}$ is just pseudo-transitive. We know from [Ladner, 77] that the satisfiability problem in the class $\mathbf{C}_{PT}$ is PSPACE-hard. However, there is no upper bound for this complexity problem, known to us. Since the satisfiability problem in $\mathbf{C}_{PT}$ is of rather technical than geometrical interest, we will not study it further.*

Now, we introduce the logic of parallelism in the Euclidean plane $\mathbf{PAR}^E$ by adding to **PAR** the following infinite set of axioms:

($\mathbf{PAR}_n$) For every positive integer $n \geq 0$, $\bigwedge_{i=1}^{n} \langle \| \rangle (\neg p_i \wedge [\|]p_i) \rightarrow \langle \| \rangle \bigwedge_{i=1}^{n} p_i$.

[2] To be distinguished from "model of parallelism" as introduced earlier. This terminological clash is unfortunate, but benign.

[3] This result is also stated without proof for the modal logic of inequality in [de Rijke, 1992], with a reference to an unpublished manuscript of B. de Smit and P. van Emde Boas.

Note that axiom ($\mathbf{PAR}_0$) is the formula $\langle || \rangle \top$, and that for all positive integers $n \geq 0$, ($\mathbf{PAR}_n$) is derivable from ($\mathbf{PAR}_{n+1}$). We also define the calculus $\mathbf{PAR}_n^E$, for each $n \geq 0$, as the extension of **PAR** obtained by adding the axiom ($\mathbf{PAR}_n$).

Let $\mathbf{C}_{QMP}^{\infty}$ (respectively, $\mathbf{C}_{MSP}^{\infty}$) be the class of all quasi-models of parallelism (respectively, all models of strict parallelism) in which every line is parallel to infinitely many lines. For each positive integer $n \geq 0$, let $\mathbf{C}_{QMP}^{n}$ (respectively, $\mathbf{C}_{MSP}^{n}$) be the class of all n-rich quasi-models of parallelism (respectively, all n-rich models of strict parallelism).

Proposition 26 *Let σ be a formula in the language $\mathcal{L}([||])$. Then $\models_{\mathbf{C}_{QMP}^{\infty}} \sigma$ iff $\models_{\mathbf{C}_{MSP}^{\infty}} \sigma$ iff $\vdash_{\mathbf{PAR}^E} \sigma$.*

Proof. The soundness is straightforward. Since all axioms of $\mathbf{PAR}^E$ are of Sahlqvist type, hence canonical, every consistent formula is satisfiable in a quasi-model of parallelism in which every line is parallel to infinitely many lines. The self-parallel lines in such model can be replaced by arbitrarily many mutually parallel but not self-parallel copies, thus producing a model of strict parallelism in which every line is parallel to infinitely many lines. ■

For all positive integers $n \geq 0$, the completeness proof for $\mathbf{PAR}_n^E$ is quite similar.

Proposition 27 *Let $n \geq 0$ and σ be a formula in the language $\mathcal{L}([||])$. Then $\models_{\mathbf{C}_{QMP}^{n}} \sigma$ iff $\models_{\mathbf{C}_{MSP}^{n}} \sigma$ iff $\vdash_{\mathbf{PAR}_n^E} \sigma$.*

For every positive integer $d \geq 2$, the **Euclidean parallelism space of dimension** d is the model $\mathbb{P}^d$ of strict parallelism consisting of all lines in the real space $\mathbb{R}^d$ with the usual strict parallelism relation.

We will show that $\mathbf{PAR}^E$ axiomatizes the strict parallelism in any Euclidean parallelism space $\mathbb{P}^d$ by using the following representation result, proved in [Balbiani and Goranko, 2002].

Lemma 28 *Every model of strict parallelism of cardinality not greater than the continuum is isomorphic to a real one.*

Lemma 29 *For each positive integer $d \geq 2$, every model of strict parallelism in which there are continuum many equivalence classes of parallel lines and each of them has the cardinality of the continuum is isomorphic to the Euclidean parallelism space $\mathbb{P}^d$.*

Let $\mathbf{C}_{RMSP}$ be the class of all real models of strict parallelism. As a consequence of lemmas 28 and 29 we obtain:

Theorem 30 *Let σ be a formula in the language $\mathcal{L}([||])$. Then $\models_{\mathbf{C}_{RMSP}} \sigma$ iff $\vdash_{\mathbf{PAR}} \sigma$.*

Proof. The soundness is straightforward. Completeness follows immediately from proposition 23 and lemma 28 since the canonical frame for **PAR** has the cardinality of the continuum. ■

Theorem 31 *For each positive integer $d \geq 2$, the logic $\mathbf{PAR}^E$ is sound and complete for the Euclidean parallelism space $\mathbb{P}^d$.*

Proof. The soundness is straightforward, again. As for the completeness, it suffices to note the following:

(i) By Löwenheim-Skolem theorems, if a formula σ is satisfiable in some model of strict parallelism by a line parallel to infinitely many distinct lines, then σ is satisfiable in some model of strict parallelism by a line parallel to continuum many distinct lines, since satisfiability of a modal formula in a given model is a first-order property. A generated submodel of such model would consists of one class of continuum many parallel lines.

(ii) A disjoint union of continuum many such models would produce, by lemma 29, a model of strict parallelism isomorphic to the Euclidean parallelism space of dimension d.

■

Proposition 32 *The logic* $\mathbf{PAR}^E$ *is not finitely axiomatizable.*

Proof. Tarski's argument applies here. Assuming that $\mathbf{PAR}^E$ has a complete finite set of axioms $\mathcal{A}$, then all formulas from $\mathcal{A}$ can be derived from finitely many axioms of $\mathbf{PAR}^E$, hence they can be derived in some $\mathbf{PAR}^E_n$ for a large enough positive integer $n \geq 0$. But this is clearly impossible, because the formula ($\mathbf{PAR}_{n+1}$), being false in the model of strict parallelism consisting of exactly $n+2$ parallel lines, is not derivable in $\mathbf{PAR}^E_n$. ■

Of course, there is no finite model property of $\mathbf{PAR}^E$ with respect to the classes $\mathbf{C}^\infty_{QMP}$ and $\mathbf{C}^\infty_{MSP}$. However it can be proved that:

Proposition 33 *The satisfiability problem in each of the classes* $\mathbf{C}^\infty_{QMP}$ *and* $\mathbf{C}^\infty_{MSP}$ *is NP-complete.*

Proof. Let σ be a line formula and $\mathbf{M} =< \mathbf{Li}, \|, \mathbf{m} >$ be a model of strict parallelism in which every line is parallel to infinitely many lines. Suppose that for some line x, $\mathbf{M}, x \models \sigma$. We can assume that $\mathbf{M}$ is generated and therefore $\|$ is nothing but the inequality relation over $\mathbf{Li}$. Now, let $Sf(\sigma)$ be the set of all subformulas of σ. Note that $Card(Sf(\sigma)) \leq length(\sigma)$. Let $\cong_{Sf(\sigma)}$ be the binary relations over $\mathbf{Li}$ defined as follows:

- $y \cong_{Sf(\sigma)} z$ iff for all formulas τ in $Sf(\sigma)$, $\mathbf{M}, y \models \tau$ iff $\mathbf{M}, z \models \tau$.

Notice that $\cong_{Sf(\sigma)}$ is an equivalence relation. For every line y, we will denote its equivalence class by $Cl(y)$. Since $Sf(\sigma)$ is finite, there are finitely many equivalence classes modulo $\cong_{Sf(\sigma)}$. Since $\mathbf{Li}$ is infinite, there exists a line y such that its equivalence class $Cl(y)$ is infinite. Let:

- $\Delta = \{[\|]\tau$: $[\|]\tau$ is in $Sf(\sigma)$ and there is exactly one line z such that $\mathbf{M}, z \not\models \tau\}$;
- $\Lambda = \{[\|]\tau$: $[\|]\tau$ is in $Sf(\sigma)$ and there are at least two different lines z and t with $\mathbf{M}, z \not\models \tau$ and $\mathbf{M}, t \not\models \tau\}$.

Notice that Δ and Λ are disjoint subsets of $Sf(\sigma)$. For every $[\|]\tau$ in Δ, let z_τ be the unique line in $\mathbf{Li}$ with $\mathbf{M}, z_\tau \not\models \tau$, and for every $[\|]\tau$ in Λ, let z_τ and t_τ be two different lines in $\mathbf{Li}$ with $\mathbf{M}, z_\tau \not\models \tau$ and $\mathbf{M}, t_\tau \not\models \tau$. Let

$\mathbf{Li}' = \{x\} \cup Cl(y) \cup \{z_\tau$: $[\|]\tau$ is in $\Delta\} \cup \{z_\tau$: $[\|]\tau$ is in $\Lambda\} \cup \{t_\tau$: $[\|]\tau$ is in $\Lambda\}$.

If $\|'$ and $\mathbf{m}'$ are the restrictions of $\|$ and $\mathbf{m}$ to $\mathbf{Li}'$ then, obviously, the structure $\mathbf{M}' = < \mathbf{Li}', \|', \mathbf{m}' >$ is a model of strict parallelism in which every line is parallel to infinitely many lines. By induction on τ one can prove that if τ is in $Sf(\sigma)$ then for every line z in $\mathbf{Li}'$ the two following conditions are equivalent:

- $\mathbf{M}, z \models \tau$;
- $\mathbf{M}', z \models \tau$;

Therefore, $\mathbf{M}', x \models \sigma$. Hence, the satisfiability problem in each of $\mathbf{C}^\infty_{QMP}$ and $\mathbf{C}^\infty_{MSP}$ can be solved in non-deterministic polynomial time. ■

4.2 Modal logics for line orthogonality

Unlike parallelism, the modal logics of line orthogonality are different for Euclidean spaces of different dimensions. In fact, the dimension of the space can be determined as the greatest number of pairwise orthogonal lines. Here we will only introduce the modal logics of line orthogonality in the plane (i.e. 2-dimensional space). They are based on a propositional modal language $\mathcal{L}([\perp])$ with a unary **modality of orthogonality** $[\perp]$. Line formulas are interpreted in **orthogonality models**, i.e. structures of the form $<\mathbf{Li}, \perp, \mathbf{m}>$, in the expected way. Let $\mathbf{C}_{QMO}$ be the class of all quasi-models of orthogonality and $\mathbf{C}_{MSO}$ be the class of all models of strict orthogonality. Predictably again, the modal logic of line orthogonality **ORT** in the plane is obtained by adding to **K** the following axioms:

1. $\sigma \to [\perp]\langle\perp\rangle\sigma$;
2. $[\perp]\sigma \to [\perp][\perp][\perp]\sigma$.

Besides the usual rules of modus ponens and necessitation, we can add the irreflexivity rule for $[\perp]$:

$$\frac{([\perp]p \wedge \neg p) \to \sigma, \text{ for some line variable } p \text{ not occurring in } \sigma}{\sigma}$$

which, however, is admissible again, as seen from the completeness result below.

Proposition 34 *Let σ be a formula in the language $\mathcal{L}([\perp])$. Then $\models_{\mathbf{C}_{QMO}} \sigma$ iff $\models_{\mathbf{C}_{MSO}} \sigma$ iff $\vdash_{\mathbf{ORT}} \sigma$.*

Proof. The soundness is straightforward. Concerning completeness, one can prove using a canonical model argument that **ORT** is complete with respect to the class $\mathbf{C}_{QMO}$. Every quasi-model $<\mathbf{Li}, \perp>$ of orthogonality is a p-morphic image of a model $<\mathbf{Li}', \perp'>$ of strict orthogonality. To prove this, one has just to consider the new set of lines $\mathbf{Li}' = \mathbf{Li} \times \{0,1\}$ and the new relation $\perp'$ defined over $\mathbf{Li}'$ by $(x,i)\perp'(y,j)$ iff $x \perp y$ and $i \neq j$. ■

Proposition 35 **ORT** *has the finite model property and is decidable.*

Proof. Let σ be a formula and $\mathbf{M} = <\mathbf{Li}, \perp, \mathbf{m}>$ be a quasi-model of orthogonality. Suppose that for some line x, $\mathbf{M}, x \models \sigma$. We can assume that $\mathbf{M}$ is generated. Now, let $Sf(\sigma)$ be the smallest set of line formulas containing the set of all subformulas of σ and such that for every formula τ, if $[\perp]\tau$ is in $Sf(\sigma)$ and τ is not a formula of the form $[\perp]\rho$, then $[\perp][\perp]\tau$ is also in $Sf(\sigma)$. Note that $Card(Sf(\sigma)) \leq 2 \times length(\sigma)$. Let $\cong_{Sf(\sigma)}$ be the equivalence relation over **Li** defined as in the proof of proposition 33. Again, for every line y, we will denote its equivalence class by $Cl(y)$. And again, since $Sf(\sigma)$ is finite, there are only finitely many equivalence classes modulo $\cong_{Sf(\sigma)}$. Now we define $\mathbf{Li}'$ to be the set of all equivalence classes modulo $\cong_{Sf(\sigma)}$. For all lines y, z in **Li**, let us say that $Cl(y)\perp' Cl(z)$ iff for all formulas $[\perp]\tau$ in $Sf(\sigma)$, the two following conditions are satisfied:

- If $\mathbf{M}, y \models [\perp]\tau$ then $\mathbf{M}, z \models \tau \wedge [\perp][\perp]\tau$;
- If $\mathbf{M}, z \models [\perp]\tau$ then $\mathbf{M}, y \models \tau \wedge [\perp][\perp]\tau$.

Notice that the binary relation $\perp'$ is symmetric and 3-transitive over $\mathbf{Li}'$ and that $<\mathbf{Li}', \perp', \mathbf{V}'>$ is a filtration of $\mathbf{M}$ through $Sf(\sigma)$. Hence, **ORT** has the finite model property and, being finitely axiomatizable, is decidable. ■

As usual, the filtration argument presented above implies that the satisfiability problem in each of the classes $\mathbf{C}_{QMO}$ and $\mathbf{C}_{MSO}$ is in NEXPTIME. However, we obtain below a stronger complexity result.

Proposition 36 *The satisfiability problem in each of the classes $\mathbf{C}_{QMO}$ and $\mathbf{C}_{MSO}$ is NP-complete.*

Proof. Indeed, we show that every formula σ satisfiable in a model of strict orthogonality is also satisfiable in a model of strict orthogonality of size $\mathcal{O}(length(\sigma))$. Let σ be a line formula and $\mathbf{M} =< \mathbf{Li}, \perp, \mathbf{m} >$ be a model of strict orthogonality. We can assume that $\mathbf{M}$ is orthogonal-rich, otherwise the proof is immediate. Suppose that for some line x, $\mathbf{M}, x \models \sigma$. If $\mathbf{M}$ is generated, the sets:

- $\mathbf{Li}_1 = \{y$: y is in $\mathbf{Li}$ and $x \perp y\}$, and
- $\mathbf{Li}_2 = \{x\} \cup \{y$: y is in $\mathbf{Li}$ and $x \| y\}$;

constitute a partition of $\mathbf{Li}$ where $\|$ is the relation over $\mathbf{Li}$ defined by $y \| z$ iff there is a line t such that $y \perp t$ and $t \perp z$. Notice that for all lines y, z in $\mathbf{Li}$, $y \perp z$ iff either $y \in \mathbf{Li}_1$ and $z \in \mathbf{Li}_2$ or $y \in \mathbf{Li}_2$ and $z \in \mathbf{Li}_1$. If $Sf(\sigma)$ is the set of all subformulas of σ, then let:

- $\Delta_1 = \{[\perp]\tau$: $[\perp]\tau$ is in $Sf(\sigma)$ and there exists y in $\mathbf{Li}_1$ such that $\mathbf{M}, y \not\models [\perp]\tau\}$;
- $\Delta_2 = \{[\perp]\tau$: $[\perp]\tau$ is in $Sf(\sigma)$ and there exists $y \in \mathbf{Li}_2$ such that $\mathbf{M}, y \not\models [\perp]\tau\}$.

Note that $Card(\Delta_1) < length(\sigma)$ and $Card(\Delta_2) < length(\sigma)$. Moreover, for all formulas $[\perp]\tau \in \Delta_1$, there exists at least one line y in $\mathbf{Li}_2$ (let us choose and denote y_τ one such line) such that $\mathbf{M}, y \not\models \tau$, and for all formulas $[\perp]\tau \in \Delta_2$, there exists at least one line y in $\mathbf{Li}_1$ (again, let us choose and denote y_τ one such line) such that $\mathbf{M}, y \not\models \tau$.

Let $\mathbf{Li}' = \{x\} \cup \{y_\tau$: $[\perp]\tau \in \Delta_1\} \cup \{y_\tau$: $[\perp]\tau \in \Delta_2\}$. Note that $Card(\mathbf{Li}') < 2 \times length(\sigma)$. Let $\perp'$ and m' be the restriction of $\perp$ and m over Li'.It can be easily verified that $\mathbf{M}' =< \mathbf{Li}', \perp', \mathbf{m}' >$ is a model of strict orthogonality. By induction on τ one can prove that if τ is in $Sf(\sigma)$ then for every line y in $\mathbf{Li}'$, the two following conditions are equivalent:

- $\mathbf{M}, y \models \tau$;
- $\mathbf{M}', y \models \tau$;

Therefore, $\mathbf{M}', x \models \sigma$. Hence, the satisfiability problem in $\mathbf{C}_{QMO}$ and $\mathbf{C}_{MSO}$ can be solved in non-deterministic polynomial time. ■

Let $\mathbf{C}^{ser}_{QMO}$ be the class of all orthogonal-rich quasi-models of orthogonality and $\mathbf{C}^{ser}_{MSO}$ be the class of all orthogonal-rich models of strict orthogonality. Recall that the relation of weak parallelism is defined by means of orthogonality in the following way:

- $x \| y$ iff there is a line z such that $x \perp z$ and $z \perp y$.

Obviously, in orthogonal-rich quasi-models of orthogonality, $\|$ is reflexive. Since $[\|]\sigma$ is the abbreviation for $[\perp][\perp]\sigma$, every formula of the form $[\|]\sigma \rightarrow \sigma$ is valid in the classes $\mathbf{C}^{ser}_{QMO}$ and $\mathbf{C}^{ser}_{MSO}$. Therefore, if $n \geq 1$, the axiom $(\mathbf{PAR}_n)$ is also valid in $\mathbf{C}^{ser}_{QMO}$ and $\mathbf{C}^{ser}_{MSO}$, hence the only extension of $\mathbf{ORT}$ that makes sense is the logic $\mathbf{ORT}^E$ obtained by adding to $\mathbf{ORT}$ the axiom $(\mathbf{PAR}_0)$, i.e. the seriality axiom $\langle\perp\rangle\top$. As proved below, the logic $\mathbf{ORT}^E$ is the logic of line orthogonality in the Euclidean plane.

Let $\mathbf{C}^{\infty}_{QMO}$ (respectively, $\mathbf{C}^{\infty}_{MSO}$) be the class of all quasi-models of orthogonality (respectively, all models of strict orthogonality) in which every line is parallel to infinitely many lines.

Proposition 37 *Let σ be a line formula in the language $\mathcal{L}([\perp])$. Then $\models_{\mathbf{C}^{\infty}_{QMO}} \sigma$ iff $\models_{\mathbf{C}^{\infty}_{MSO}} \sigma$ iff $\vdash_{\mathbf{ORT}^E} \sigma$.*

Proof. Soundness is straightforward. The completeness can be proved using a canonical model argument that $\mathbf{ORT}^E$ is complete with respect to the class $\mathbf{C}^{ser}_{QMO}$. Furthermore, every orthogonal-rich quasi-model $< \mathbf{Li}, \perp >$ of orthogonality is a p-morphic image of a model $< \mathbf{Li}', \perp' >$ of strict orthogonality in which every line is parallel to infinitely many lines. To prove this, one has just to consider the new set of lines $\mathbf{Li}' = \mathbf{Li} \times \mathbb{Z} \times \{0,1\}$ and the new relation $\perp'$ defined over $\mathbf{Li}'$ by $(x,m,i)\perp'(y,n,j)$ iff $x \perp y$ and $i \neq j$. ■

The **Euclidean orthogonality plane** is the model of strict orthogonality consisting of all lines in the real plane $\mathbb{R}^2$ with the usual orthogonality relation. We will show that $\mathbf{ORT}^E$ axiomatizes orthogonality in the Euclidean orthogonality plane by using the following representation results, proved in [Balbiani and Goranko, 2002].

Lemma 38 *Every model of strict orthogonality of cardinality not greater than the continuum is isomorphic to a real one.*

Lemma 39 *Every model of strict orthogonality in which there are continuum many equivalence classes of parallel lines and each of them has the cardinality of the continuum is isomorphic to the Euclidean orthogonality plane.*

Let $\mathbf{C}_{RMSO}$ be the class of all real models of strict orthogonality. As a consequence of lemmas 38 and 39 we obtain:

Theorem 40 *Let σ be a line formula in the language $\mathcal{L}([\perp])$. Then $\vdash_{\mathbf{C}_{RMSO}} \sigma$ iff $\vdash_{\mathbf{ORT}} \sigma$.*

Proof. The soundness is straightforward. Completeness follows immediately from proposition 34 and lemma 38 since the canonical frame for **ORT** has the cardinality of the continuum. ■

Theorem 41 *The logic $\mathbf{ORT}^E$ is sound and complete for the Euclidean orthogonality plane.*

Proof. The soundness is straightforward. As for the completeness, the argument runs as in the proof of theorem 31, using lemma 39. ■

By simple modifications of the proofs of propositions 35 and 36, one can easily obtain the following results.

Proposition 42 $\mathbf{ORT}^E$ *has the finite model property and is decidable.*

Proposition 43 *The satisfiability problem in the classes $\mathbf{C}^{ser}_{QMO}$ and $\mathbf{C}^{ser}_{MSO}$ is NP-complete.*

5 Modal logics of affine geometries

The logic **AFF** introduced here is meant to axiomatize validity in the class of standard affine models, while **GAFF** captures the validity in a slightly more general models, viz. those described by the axioms.

5.1 Syntax and semantics

The syntax of **GAFF** and **AFF** is based on a two-sorted propositional language $\mathcal{L}([\in],[\ni],[||])$ with unary **modalities of incidence** $[\in]$ and $[\ni]$ and a unary **modality of parallelism** $[||]$. The sets of **point formulas PFOR** and **line formulas LFOR** of $\mathcal{L}([\in],[\ni],[||])$ are defined by recursion in the expected way:

- **PFOR** $:= P_i \mid \neg\alpha \mid (\alpha \wedge \beta) \mid [\in]\sigma$.
- **LFOR** $:= p_i \mid \neg\sigma \mid (\sigma \wedge \tau) \mid [\ni]\alpha \mid [||]\sigma$.

Definition 44 *An affine frame* $< \mathbf{Po}, \mathbf{Li}, \mathbf{I}, || >$ *is said to be a* ***general affine model*** *if it satisfies the following conditions:*

1. Line-connectedness: $\forall X \forall Y \exists x (Inc(X, Y, x))$.

2. Point-connectedness: $\forall x \forall y (\neg \exists X (Int(x, y, X)) \to x \parallel y)$.

3. Point-line connectedness: $\forall X \forall x \exists y \exists Y (X\mathbf{I}y \wedge Int(x, y, Y))$.

4. Parallel-richness: $\forall x \exists y (x \| y)$.

5. Symmetry of $\|$: $\forall x \forall y (x \| y \to y \| x)$.

6. Pseudo-transitivity of $\|$: $\forall x \forall y \forall z (x \| y \wedge y \| z \to x = z \vee x \| z)$.

7. Distinct points are incident with parallel lines: $\forall X \forall Y (X = Y \vee \exists x \exists y (X\mathbf{I}x \wedge Y\mathbf{I}y \wedge x \| y))$.

8. For every two distinct lines, either one is parallel to a line incident with the other: $\forall x \forall y (x = y \vee \exists z \exists X (z \| x \wedge Int(y, z, X)))$.

9. If a point is not incident with a line then it is incident with a line parallel to it: $\forall X \forall x (X\mathbf{I}x \vee \exists y (X\mathbf{I}y \wedge y \| x))$.

10. *(Weak normality)* Every standard point is normal.

Obviously every standard affine model is also a general affine model. Let $\mathbf{C}_{SM}$ be the class of all standard affine models (i.e. structures of the form $< \mathbf{Po}, \mathbf{Li}, \mathbf{I}, \| >$ satisfying the conditions formulated in definition 7), and $\mathbf{C}_{GM}$ be the class of all general affine models.

The semantics of **AFF** is based on standard affine models, while the semantics of **GAFF** is based on general affine models, combining the semantics of the modal logic of incidence and the modal logics of parallelism.

Some modalities definable in affine frames:

- $[\between]\sigma := [\|]\sigma \wedge \sigma$.
- $\mathbf{A_P}\alpha := [\in][\ni]\alpha$, and its dual $\mathbf{E_P}\alpha := \langle\in\rangle\langle\ni\rangle\alpha$. In line-connected affine models these represent the **universal** modality and the **existential** modality on points.
- $\mathbf{A_L}\sigma := [\|]\sigma \wedge [\ni][\in]\sigma$, and its dual $\mathbf{E_L}\sigma := \langle\|\rangle\sigma \vee \langle\ni\rangle\langle\in\rangle\sigma$. In point-connected affine models these represent the **universal** modality and the **existential** modality on lines.
- $\mathbf{A_{PL}}\sigma := [\in][\ni][\in]\sigma$ and $\mathbf{A_{LP}}\alpha := [\ni][\in][\ni]\alpha$. In connected models these represent accordingly the universal modality between points and lines and the universal modality between lines and points.

In every Euclidean affine model:

- $\mathbf{D_P}\alpha := [\in][\|][\ni]\alpha$ represents the **difference** modality on points, saying that "α holds of every point different from the current one".
- Likewise, $\mathbf{D_L}\sigma := [\|][\ni][\in]\sigma$ represents the **difference** modality on lines.
- Accordingly, $\mathbf{O_P}\alpha := \alpha \wedge \mathbf{D_P}\neg\alpha$ says that "α only holds of the current point", and $\mathbf{O_L}\sigma := \sigma \wedge \mathbf{D_L}\neg\sigma$ says that "σ only holds of the current line".
- Finally, $\mathbf{S_P}\alpha := \mathbf{E_P}\mathbf{O_P}\alpha$ says that "α holds of a single point" and $\mathbf{S_L}\sigma := \mathbf{E_L}\mathbf{O_L}\sigma$ says that "σ holds of a single line".

5.2 Axiomatic system for GAFF and AFF

The modal logic **GAFF** is obtained by adding to $\mathbf{K}([\in],[\ni],[\|])$ the following axioms.

1. The seriality axioms: $\langle\in\rangle\top$, $\langle\ni\rangle\top$ and $\langle\|\rangle\top$.
2. The tense axioms: $\alpha \to [\in]\langle\ni\rangle\alpha$ and $\sigma \to [\ni]\langle\in\rangle\sigma$.

   The **S5** axioms for $[\between]$, which reduce to:

3. $\sigma \to [\|]\langle\|\rangle\sigma$, and
4. $\sigma \wedge [\|]\sigma \to [\|][\|]\sigma$. (Note that on standard models this axiom expresses the uniqueness part of the Euclidean property.)

   The axioms of the universal modality (see [Goranko and Passy, 91]) for $\mathbf{A_P}$ with respect to point formulas, which reduce to:

5. $\mathbf{A_P}\alpha \to \mathbf{A_P}\mathbf{A_P}\alpha$, (This implies line-connectedness).
6. $\mathbf{A_P}\alpha \to \mathbf{D_P}\alpha$.

   The axioms for the difference modality for $\mathbf{D_P}$, which reduce to:

7. $\alpha \wedge \mathbf{D_P}\alpha \to \mathbf{A_P}\alpha$

   The axioms of the universal modality for $\mathbf{A_L}$ with respect to line formulas, which reduce to:

8. $\mathbf{A_L}\sigma \to \mathbf{D_L}\sigma$.

   The axioms for the difference modality for $\mathbf{D_L}$, which reduce to:

9. $\sigma \wedge \mathbf{D_L}\sigma \to [\ni][\in]\sigma$.
10. $\mathbf{A_{LP}}\alpha \leftrightarrow ([\ni]\alpha \wedge [\|][\ni]\alpha)$.
11. $\mathbf{A_{PL}}\sigma \leftrightarrow ([\in]\sigma \wedge [\in][\|]\sigma)$.
    (The two axioms above express the existence part of the Euclidean property.)
12. $\alpha \wedge \langle\in\rangle(\sigma \wedge \langle\ni\rangle\mathbf{O_P}\neg\alpha) \to [\in]([\ni]\alpha \vee \sigma)$. (Normality.)

The rules of inference for **GAFF** are the standard ones: modus ponens and necessitation.

The logic **AFF**extends **GAFF** with the following pair of rules for the difference modalities for point formulas and for line formulas:

$$\frac{\mathbf{O_P}P \to \alpha \text{ for some } P \text{ not occurring in } \alpha}{\alpha};\ \frac{\mathbf{O_L}p \to \sigma \text{ for some } p \text{ not occurring in } \sigma}{\sigma}.$$

These rules obviously preserve validity in the class $\mathbf{C}_{SM}$.

5.3 Completeness

Theorem 45 *Let φ be a formula in the language $\mathcal{L}([\in],[\ni],[||])$. Then $\models_{\mathbf{C}_{GM}} \varphi$ iff $\vdash_{\mathbf{GAFF}} \varphi$.*

Proof. *We only need to note that all axioms of* **GAFF** *are Sahlqvist formulas and that their correspondent first-order conditions are exactly the conditions defining general affine models.* ∎

Theorem 46 *Let φ be a formula in the language $\mathcal{L}([\in],[\ni],[||])$. Then $\models_{\mathbf{C}_{SM}} \varphi$ iff $\vdash_{\mathbf{AFF}} \varphi$.*

Proof. Soundness is straightforward. As for completeness, we build a special canonical model $\mathbf{M} = <\mathbf{Po}, \mathbf{Li}, \mathbf{I}, ||, \mathbf{m}>$ from maximal consistent sets of point formulas (respectively, maximal consistent sets of line formulas) which are closed under the infinitary versions of the respective irreflexivity rules (for details see e.g. [Gargov and Goranko, 93] or [Blackburn, de Rijke, and Venema]). Let, for any box-modality, i.e. string of boxes, $\mathbf{B}$ in the language, $\mathcal{R}(\mathbf{B})$ denote the relation in $\mathbf{M}$ corresponding to $\mathbf{B}$. Since all axioms are Sahlqvist formulas and the language is versatile, they are all di-persistent, and since the language contains difference modalities, the underlying frame of $\mathbf{M}$ is discrete, hence all axioms are valid in it. For more details, see [Venema, 93] or [Blackburn, de Rijke, and Venema][4].

Now, given a consistent set of point formulas (respectively, a consistent set of line formulas) we choose a maximal consistent (MC) set Γ which contains it and then we take the submodel $\mathbf{M}_\Gamma$ of $\mathbf{M}$ generated by Γ as follows:

- If Γ is a point set, we take the family $\mathbf{Po}(\Gamma)$ of all MC point sets in $\mathbf{M}$ reachable from Γ via $\mathcal{R}(\mathbf{A_P})$ and then the family $\mathbf{Li}(\Gamma)$ of all MC line sets in $\mathbf{M}$ reachable from any of the points in $\mathbf{Po}(\Gamma)$ via the relation $\mathcal{R}(\mathbf{A_{PL}})$.

The axioms guarantee that the $\mathbf{Li}(\Gamma)$ will contain all line sets reachable from Γ via $\mathcal{R}([\in])$ and that $\mathcal{R}(\mathbf{A_L})$ will be the universal relation on $\mathbf{Li}(\Gamma)$, while $\mathcal{R}(\mathbf{A_{LP}})$ will be $\mathbf{Li}(\Gamma) \times \mathbf{Po}(\Gamma)$. The construction in the case when Γ is a line set is analogous. Furthermore, the relations corresponding to $\mathbf{D_P}$ and $\mathbf{D_L}$ in $\mathbf{M}_\Gamma$ are the standard inequality relations on the respective sets, due to the axioms and the irreflexivity rules. Therefore, $\mathcal{R}([||])$ is irreflexive and all first-order conditions corresponding to the axioms in standard affine models, including the normality condition, hold in $\mathbf{M}_\Gamma$, hence $\mathbf{M}_\Gamma$ is a standard affine model. ∎

Remark 47 *We do not know yet whether the rules for the difference modalities are admissible in* **GAFF** *and the completeness of* **GAFF** *with respect to standard affine models is still an open problem, as well as the finite model property and decidability of* **GAFF** *and* **AFF** *(but see further, and section 7).*

5.4 Some variations of AFF

The logic **AFF** axiomatizes the class of standard models. For various practical reasons (see the introduction) one may be interested in weaker systems, such as the logics of basic, connected, parallel-rich or Euclidean basic affine models. These can be axiomatized by selecting the relevant axioms from **AFF** and modifying appropriately the completeness argument. On the other hand, since the language $\mathcal{L}([\in],[\ni],[||])$ contains difference modalities, and hence it can simulate nominals, for any universal condition Φ on the relations $\mathbf{I}$ and $||$ there is a uniformly definable formula φ_Φ in the language such that the logic $\mathbf{AFF} + \varphi_\Phi$, obtained by adding to **AFF** the axiom φ_Φ, is sound and complete with respect to the class $\mathbf{C}_{SM}^{\Phi}$ of all standard models verifying Φ (see e.g. [Gargov and Goranko, 93] or [Blackburn, de Rijke, and Venema] for more details).

Since the properties of Desargues and Pappus are equivalent to universal conditions on standard affine models, there are formulas φ_{wDe}, φ_{sDe} and φ_{Pa} such that:

Theorem 48 *1. The logic $\mathbf{AFF} + \varphi_{wDe}$ is sound and complete for the class $\mathbf{C}_{SM}^{wDe}$ of all weakly Desarguesian standard models.*

[4] Although the relevant results there have been established for one-sorted modal languages, their extension to two-sorted languages is straightforward.

2. *The logic* $\mathbf{AFF} + \varphi_{sDe}$ *is sound and complete for the class* $\mathbf{C}_{SM}^{sDe}$ *of all strongly Desarguesian standard models.*

3. *The logic* $\mathbf{AFF} + \varphi_{Pa}$ *is sound and complete for the class* $\mathbf{C}_{SM}^{Pa}$ *of all Pappian standard models.*

Important further extensions of **AFF** are the logic of all real basic affine models and the logic of the Euclidean affine plane, which are yet to be axiomatized.

Finally, a natural extension of $\mathcal{L}([\in], [\ni], [||])$ involves an additional modality $[\times]$ of **line intersection**, with semantics:

- $\mathbf{M}, x \models [\times]\sigma$ iff for every line y, if x and y are different and intersecting then $\mathbf{M}, y \models \sigma$.

The new modality is *locally definable* within the class $\mathbf{C}_{SM}$ in the language of **AFF** by the following formula (see [Gargov and Goranko, 93]):

$$\mathbf{O_L}p \to ([\times]\sigma \leftrightarrow [\ni][\in](\sigma \vee p))$$

Therefore, the above formula, added as an axiom to **AFF**, axiomatizes completely the validity of $\mathcal{L}([\in], [\ni], [||], [\times])$-formulas in the class $\mathbf{C}_{SM}$.

5.5 On the complexity of AFF

The truth is that we do not know yet whether the satisfiability problems in the classes $\mathbf{C}_{GM}$ and $\mathbf{C}_{SM}$ are decidable or not. The best of our current knowledge is that the satisfiability problem in the class $\mathbf{C}_{SM}$ is NEXPTIME-hard. To demonstrate this, we use the following argument, suggested by Venema in [Venema, 99]. Let us consider the one-sorted language $\mathcal{L}(\Box_1, \Box_2)$ with unary modalities $\Box_1$ and $\Box_2$. The semantics is based on frames of the form $< \mathbf{W}, \mathbf{R}_1, \mathbf{R}_2 >$ in the expected way. Let $\mathbf{C}_{12}$ be the class of all such frames where $\mathbf{R}_1$ and $\mathbf{R}_2$ are two commuting equivalence relations over $\mathbf{W}$. The satisfiability problem $\mathbf{SAT}(\mathcal{L}(\Box_1, \Box_2), \mathbf{C}_{12})$ of $\mathcal{L}(\Box_1, \Box_2)$-formulas in the class $\mathbf{C}_{12}$, is NEXPTIME-hard (see [Marx and Venema, 97] for details). Marx and Venema also show that if a formula in $\mathcal{L}(\Box_1, \Box_2)$ is satisfiable in the class $\mathbf{C}_{12}$, it is also satisfiable in a model based on the rational square $< \mathbb{Q}^2, \equiv_1, \equiv_2 >$ where $\mathbb{Q}^2$ is the set of all pairs of rational numbers and $\equiv_1$ and $\equiv_2$ are the binary relations over $\mathbb{Q}^2$ defined in the following way:

- $(q_1, q_2) \equiv_1 (q_1', q_2')$ iff $q_1 = q_1'$;
- $(q_1, q_2) \equiv_2 (q_1', q_2')$ iff $q_2 = q_2'$.

Our aim is to demonstrate that $\mathbf{SAT}(\mathcal{L}(\Box_1, \Box_2), \mathbf{C}_{12})$ is reducible to $\mathbf{SAT}(\mathcal{L}([\in], [\ni], [||]), \mathbf{C}_{SM})$. To this end, we associate with every $\mathcal{L}(\Box_1, \Box_2)$-formula α a point formula $\alpha^\diamond$ in $\mathcal{L}([\in], [\ni], [||])$. To simulate the relations $\mathbf{R}_1$ and $\mathbf{R}_2$, we use two line variables, p_x and p_y. The definition of $\alpha^\diamond$ is by induction on the construction of α:

- $P_i^\diamond := P_i$;
- $(\neg\alpha)^\diamond := \neg\alpha^\diamond$;
- $(\alpha \vee \beta)^\diamond := \alpha^\diamond \vee \beta^\diamond$;
- $(\Box_1\alpha)^\diamond := [\in](p_x \vee \langle || \rangle p_x \to [\ni]\alpha^\diamond)$;
- $(\Box_2\alpha)^\diamond := [\in](p_y \vee \langle || \rangle p_y \to [\ni]\alpha^\diamond)$.

Lemma 49 *If the $\mathcal{L}(\Box_1, \Box_2)$-formula α is satisfiable in a model based on the rational square $<\mathbb{Q}^2, \equiv_1, \equiv_2>$ then the point formula $\alpha^\diamond \wedge \mathbf{E_P}(\langle\in\rangle \mathbf{O_L} p_x \wedge \langle\in\rangle \mathbf{O_L} p_y) \wedge \mathbf{A_P}[\in]\neg(p_x \wedge p_y)$ in $\mathcal{L}([\in], [\ni], [||])$ is satisfiable in the class $\mathbf{C}_{SM}$.*

Proof. Let $\mathbf{m}$ be a valuation on the rational square $<\mathbb{Q}^2, \equiv_1, \equiv_2>$. Suppose that α is satisfiable in the model $\mathbf{M} =<\mathbb{Q}^2, \equiv_1, \equiv_2>$. Let $<\mathbf{Po}, \mathbf{Li}, \mathbf{I}, ||>$ be the standard model defined as follows:

- $\mathbf{Po} = \mathbb{Q}^2$;
- $\mathbf{Li}$ is the set of all lines with rational coefficients in the real plane;
- $\mathbf{I}$ is the restriction to $\mathbf{Po}$ and $\mathbf{Li}$ of the usual incidence relation in the real plane;
- $||$ is the restriction to $\mathbf{Li}$ of the usual parallelism relation in the real plane.

Let $\mathbf{m}'$ be a valuation on $<\mathbf{Po}, \mathbf{Li}, \mathbf{I}, ||>$ such that:

- For every point variable P, $\mathbf{m}'(P)$ is $\mathbf{m}(P)$;
- $\mathbf{m}'(p_x)$ is the line with equation $x = 0$ in the real plane;
- $\mathbf{m}'(p_y)$ is the line with equation $y = 0$ in the real plane.

$\langle\in\rangle \mathbf{O_L} p_x$ is obviously true at the point $(0, 0)$ in model $\mathbf{M}' =<\mathbf{Po}, \mathbf{Li}, \mathbf{I}, ||, \mathbf{m}'>$, since the unique line in $\mathbf{Li}$ satisfying line variable p_x is the line with equation $x = 0$. Likewise, one can verify that $\mathbf{M}', (0, 0) \models \langle\in\rangle \mathbf{O_L} p_y$. Since no line in $\mathbf{Li}$ satisfies both p_x and p_y, $[\in]\neg(p_x \wedge p_y)$ is true at any point (q_1, q_2). By induction on β one can prove that if β is a subformula of α then for every point (q_1, q_2), the two following conditions are equivalent:

- $\mathbf{M}, (q_1, q_2) \models \beta$;
- $\mathbf{M}', (q_1, q_2) \models \beta^\diamond$;

Since α is satisfiable in $\mathbf{M}$, then $\alpha^\diamond$ is satisfiable in $\mathbf{M}'$. ■

Lemma 50 *If the point formula $\alpha^\diamond \wedge \mathbf{E_P}(\langle\in\rangle \mathbf{O_L} p_x \wedge \langle\in\rangle \mathbf{O_L} p_y) \wedge \mathbf{A_P}[\in]\neg(p_x \wedge p_y)$ in $\mathcal{L}([\in], [\ni], [||])$ is satisfiable in the class $\mathbf{C}_{SM}$ then the $\mathcal{L}(\Box_1, \Box_2)$-formula α is satisfiable in the class $\mathbf{C}_{12}$.*

Proof. Suppose that $\alpha^\diamond \wedge \mathbf{E_P}(\langle\in\rangle \mathbf{O_L} p_x \wedge \langle\in\rangle \mathbf{O_L} p_y) \wedge \mathbf{A_P}[\in]\neg(p_x \wedge p_y)$ is satisfiable in the class $\mathbf{C}_{SM}$ in the model $\mathbf{M} =<\mathbf{Po}, \mathbf{Li}, \mathbf{I}, ||, \mathbf{m}>$. Then there is a point O in $\mathbf{Po}$ lying on two distinct lines x_O and y_O such that these lines are the unique lines in $\mathbf{Li}$ satisfying, respectively, p_x and p_y. Let $<\mathbf{W}, \mathbf{R}_1, \mathbf{R}_2>$ be the frame defined as follows:

- $\mathbf{W} = \mathbf{Po}$;
- $\mathbf{R}_1$ is the binary relation over $\mathbf{W}$ defined by $X\mathbf{R}_1Y$ iff there is a line x such that $Inc(X, Y, x)$ and $x = x_O$ or $x||x_O$;
- $\mathbf{R}_2$ is the binary relation over $\mathbf{W}$ defined by $X\mathbf{R}_2Y$ iff there is a line y such that $Inc(X, Y, y)$ and $y = y_O$ or $y||_{Oy}$.

Let $\mathbf{m}'$ be the restriction to $\mathbf{W}$ of $\mathbf{m}$. $\mathbf{R}_1$ and $\mathbf{R}_2$ are obviously two commuting equivalence relations over $\mathbf{W}$. By induction on β one can prove that if β is a subformula of α then for every point X in $\mathbf{Po}$, the two following conditions are equivalent:

- $\mathbf{M}, X \vDash \beta^{\diamond}$;
- $\mathbf{M}', X \vDash \beta$.

Since $\alpha^{\diamond}$ is satisfiable in $\mathbf{M}$, then α is satisfiable in $\mathbf{M}'$. ■

Corollary 51 *The satisfiability problem in the class* $\mathbf{C}_{SM}$ *is NEXPTIME-hard.*

Remark 52 *The translation used above does not work for* **GAFF** *and at present we have no complexity results about it.*

6 Modal logics of projective geometry and the affine modal logic of weak parallelism

We now turn to study the affine modal logic of weak (reflexive) parallelism. Using the connection between projective and affine structures, it can be interpreted into an appropriate projective modal logic, and thus some ideas and results from [Balbiani, 98] and [Venema, 99] can be transferred here.

6.1 The modal logic $\mathbf{PG}_\infty$ of projective geometry with distinguished infinite line

A natural extension of the two-sorted propositional modal language $\mathcal{L}([\in],[\ni])$ introduced by Balbiani [Balbiani, 98] and Venema [Venema, 99] within the context of projective geometry involves an additional designated variable (actually, nominal) ∞ of line type, the **infinity line nominal**. We denote the extended language by $\mathcal{L}([\in],[\ni],\infty)$.

Definition 53 *An* ***incidence model with infinity*** *is a structure of the form* $\mathbf{M} = <\mathbf{Po},\mathbf{Li},\mathbf{I},\omega,\mathbf{m}>$ *where* $<\mathbf{Po},\mathbf{Li},\mathbf{I}>$ *is an incidence frame,* ω *is a designated line in* $\mathbf{Li}$ *(called the* ***infinite line****) and* $\mathbf{m}$ *is a valuation on* $<\mathbf{Po},\mathbf{Li},\mathbf{I}>$ *such that* $\mathbf{m}(\infty) = \{\omega\}$*. A* ***point in*** M ***is finite*** *if it is not incident with the infinite line. A* ***line in*** M ***is finite*** *if it is different from the infinite line.*

Therefore, for every line x in $\mathbf{Li}$, $\mathbf{M}$,$x \vDash \infty$ iff $x = \omega$.

Definition 54 *Let* $\mathbf{C}^{\infty}_{QPM}$ *be the class of all incidence models* $<\mathbf{Po},\mathbf{Li},\mathbf{I},\omega>$ *with infinity such that:*

- $<\mathbf{Po},\mathbf{Li},\mathbf{I}>$ *is a quasi-projective frame;*
- *Every finite line is incident with at least one finite point.*
- *The infinite line* ω *is normal in* $<\mathbf{Po},\mathbf{Li},\mathbf{I}>$*.*

Elements of $\mathbf{C}^{\infty}_{QPM}$ are called **quasi-projective models with infinity**.

Definition 55 *Let* $\mathbf{C}^{\infty}_{PM}$ *be the class of all* ***projective models with infinity****, i.e. incidence models* $<\mathbf{Po},\mathbf{Li},\mathbf{I},\omega>$ *with infinity such that* $<\mathbf{Po},\mathbf{Li},\mathbf{I}>$ *is a projective frame.*

The axiomatic system $\mathbf{PG}_\infty$ extends the one for $\mathbf{PG}$ from [Balbiani, 98] and [Venema, 99] with the following additional axioms for ∞:

1. $\langle\ni\rangle\langle\in\rangle\infty$;

2. $\infty \wedge \sigma \rightarrow [\ni][\in](\infty \rightarrow \sigma)$;

3. $[\ni]\langle\in\rangle\infty \rightarrow \infty$;

4. $\langle\ni\rangle(\alpha \wedge \langle\in\rangle\infty) \wedge \langle\ni\rangle(\neg\alpha \wedge \langle\in\rangle\infty) \rightarrow \infty$.

The first two are the well-known axioms for nominals (see [Gargov and Goranko, 93]), while the last one claims normality of the infinite line. Canonical completeness of $\mathbf{PG}_\infty$ with respect to the class $\mathbf{C}^\infty_{QPM}$ is straightforward.

Proposition 56 *Let φ be a formula in the language $\mathcal{L}([\in],[\ni],\infty)$. Then $\models_{\mathbf{C}^\infty_{QPM}} \varphi$ iff $\vdash_{\mathbf{PG}_\infty} \varphi$.*

Remark 57 *Completeness of $\mathbf{PG}_\infty$ with respect to the class $\mathbf{C}^\infty_{PM}$, however, remains so far open.*

Decidability of $\mathbf{PG}_\infty$, on the other hand, follows easily by filtration.

Proposition 58 *$\mathbf{PG}_\infty$ has the finite model property and is decidable.*

Proof. Let φ be a formula and $\mathbf{M} = < \mathbf{Po}, \mathbf{Li}, \mathbf{I}, \omega, \mathbf{m} >$ be a quasi-projective model with infinity. Now, let $Sf(\varphi)$ be the smallest set of formulas containing φ, closed for the subformulas and such that for every point formula α, if α is in $Sf(\varphi)$, then $[\ni](\alpha \rightarrow [\in]\neg\infty)$ and $[\ni](\neg\alpha \rightarrow [\in]\neg\infty)$ are also in $Sf(\varphi)$. Note that $Card(Sf(\varphi)) = \mathcal{O}(length(\varphi))$. Let $\cong_{Sf(\varphi)}$ be the equivalence relation over $\mathbf{Po} \cup \mathbf{Li}$ defined as follows:

- $X \cong_{Sf(\varphi)} Y$ iff for all point formulas α in $Sf(\varphi)$, $\mathbf{M}, X \models \alpha$ iff $\mathbf{M}, Y \models \alpha$;
- $x \cong_{Sf(\varphi)} y$ iff for all line formulas σ in $Sf(\varphi)$, $\mathbf{M}, x \models \sigma$ iff $\mathbf{M}, y \models \sigma$.

Note that $Cl(\omega) = \{\omega\}$ since ∞ belongs to $Sf(\varphi)$. Now we define $\mathbf{Po}'$ to be the set of all equivalence classes of points and $\mathbf{Li}'$ to be the set of all equivalence classes of lines. For all points X in $\mathbf{Po}$ and for all lines x in $\mathbf{Li}$, we put $Cl(X)\mathbf{I}'Cl(x)$ iff there exists a point Y in $\mathbf{Po}$ and there exists a line y in $\mathbf{Li}$ such that $X \cong_{Sf(\alpha)} Y$, $x \cong_{Sf(\alpha)} y$ and $Y\mathbf{I}y$. Note that the incidence model $< \mathbf{Po}', \mathbf{Li}', \mathbf{I}', Cl(\omega) >$ with infinity is a filtration of $\mathbf{M}$ through $Sf(\alpha)$. It can be easily proved that it verifies all the conditions required to be a quasi-projective model with infinity. Hence, $\mathbf{PG}_\infty$ has the finite model property and is therefore decidable. ■

As usual, the filtration argument presented above implies that the satisfiability problem in the class $\mathbf{C}^\infty_{QPM}$ is in NEXPTIME. Using Venema's result in [Venema, 99] about the complexity of $\mathbf{PG}$,we obtain:

Corollary 59 *1. The satisfiability problem in the class $\mathbf{C}^\infty_{QPM}$ is NEXPTIME-complete.*

2. The satisfiability problem in the class $\mathbf{C}^\infty_{PM}$ is NEXPTIME-hard.

Remark 60 *The precise complexity, and even decidability, of the satisfiability problem in the class $\mathbf{C}^\infty_{PM}$, remain open.*

6.2 The affine modal logic of weak parallelism WAFF

Now, we consider again the two-sorted propositional modal language $\mathcal{L}([\in],[\ni],[||])$ with formulas interpreted in standard affine models, and introduce the affine modal logic of weak parallelism **WAFF**, in which, unlike **AFF**, the line formulas of the form $[||]\sigma$ are now interpreted in the following way:

- $\mathbf{M}, x \models [||]\sigma$ iff for all lines y in $\mathbf{Li}$, if $x = y$ or $x||y$ then $\mathbf{M}, y \models \sigma$.

Hence, the modality $[|||]$, corresponding to the reflexive closure of parallelism, is an **S5** modality. This change makes the language essentially weaker, as the difference modalities are no longer definable and their deductive machinery cannot be applied here. In fact, the logic **WAFF** can be regarded as the affine counterpart of $\mathbf{PG}_\infty$ in a sense which will be made precise further.

The axiomatic system for **WAFF** is obtained by adding to $\mathbf{K}([\in],[\ni],[|||])$ the following axioms

1. The seriality axioms: $\langle\in\rangle\top$ and $\langle\ni\rangle\top$;
2. The tense axioms: $\alpha \to [\in]\langle\ni\rangle\alpha$ and $\sigma \to [\ni]\langle\in\rangle\sigma$;
3. The **S5** axioms for $[|||]$:
4. $\mathbf{A_P}\alpha \to \mathbf{A_P}\mathbf{A_P}\alpha$;
5. $\mathbf{A_P}\alpha \to \mathbf{D_P}\alpha$.
6. $\mathbf{A_L}\sigma \to \mathbf{A_L}\mathbf{A_L}\sigma$.
7. $[\in][|||]\sigma \to \mathbf{A_{PL}}\sigma$.

The rules of inference now only include modus ponens and necessitation. Every axiom of **WAFF** is clearly $\mathbf{C}_{SM}$-valid, hence **WAFF** is sound with respect to its standard semantics.

Remark 61 *The completeness of* **WAFF** *with respect to* $\mathbf{C}_{SM}$*, is still open.*

Definition 62 *An affine frame* $<\mathbf{Po},\mathbf{Li},\mathbf{I},\|>$ *is a* ***weak affine model*** *whenever the following conditions are satisfied:*

- Seriality of $\mathbf{I}$: $\forall X\exists x(X\mathbf{I}x)$ *and* $\forall x\exists X(X\mathbf{I}x)$.
- Reflexivity of $\|$: $\forall x(x\|x)$.
- Symmetry of $\|$: $\forall x\forall y(x\|y \to y\|x)$.
- Transitivity of $\|$: $\forall x\forall y\forall z(x\|y \wedge y\|z \to x\|z)$.
- Line-connectedness: $\forall X\forall Y\exists x(Inc(X,Y,x))$.
- Point-connectedness: $\forall x\forall y(\neg\exists X(Int(x,y,X)) \to x \parallel y)$.
- Every point is incident with a line parallel to a given line: $\forall X\forall x\exists y(X\mathbf{I}y \wedge x\|y)$.

Let $\mathbf{C}_{WM}$ be the class of all weak affine models.

Proposition 63 *Let* φ *be a formula in the language* $\mathcal{L}([\in],[\ni],[|||])$*. Then* $\vDash_{\mathbf{C}_{WM}} \varphi$ *iff* $\vdash_{\mathbf{WAFF}} \varphi$.

Proof. Straightforward, since every axiom of **WAFF** is a Sahlqvist formula. ■

Proposition 64 **WAFF** *has the finite model property and is decidable in NEXPTIME.*

Proof. Let $\mathbf{M} = <\mathbf{Po},\mathbf{Li},\mathbf{I},\|,\mathbf{m}>$ be a weak model. Our aim is to obtain the finite model property of **WAFF** using the finite model property of $\mathbf{PG}_\infty$ and the well-known correspondence between affine geometry and projective geometry. To this end, we associate to every $\mathcal{L}([\in],[\ni],[|||])$-formula φ a formula φ^* in $\mathcal{L}([\in],[\ni],\infty)$. The infinity line variable ∞ is needed to simulate the relation $\|$. The definition of φ^* is by induction on the construction of φ:

- $P_i^* := P_i$ and $p_i^* := p_i$;
- $(\neg\alpha)^* := \neg\alpha^*$ and $(\neg\sigma)^* := \neg\sigma^*$;
- $(\alpha \wedge \beta)^* := \alpha^* \wedge \beta^*$ and $(\sigma \wedge \tau)^* := \sigma^* \wedge \tau^*$;
- $([\in]\sigma)^* := [\in]\sigma^*$;
- $([\ni]\alpha)^* := [\ni]([\in]\neg\infty \rightarrow \alpha^*)$;
- $([\|\|]\sigma)^* := [\ni](\langle\in\rangle\infty \rightarrow [\in](\neg\infty \rightarrow \sigma^*))$.

The **projective extension** of $\mathbf{M}$ is the quasi-projective models $\mathbf{M}^+ =< \mathbf{Po}^+, \mathbf{Li}^+, \mathbf{I}^+, \omega, \mathbf{m}^+>$ with infinity defined as follows:

- $\mathbf{Po}^+ = \mathbf{Po} \cup \{Cl(x)\colon x \text{ is in } \mathbf{Li}\}$ where $Cl(x)$ denotes the equivalence class of x modulo $\|$;
- $\mathbf{Li}^+ = \mathbf{Li} \cup \{\omega\}$ where ω is a new line;
- $\mathbf{I}^+ = \mathbf{I} \cup \{(Cl(x), y)\colon x \text{ and } y \text{ are in } \mathbf{L} \text{ and } x \parallel y\} \cup \{(Cl(x), \omega)\colon x \text{ is in } \mathbf{L}\}$;
- $\mathbf{m}^+(P_i) = \mathbf{m}(P_i)$, $\mathbf{m}^+(p_i) = \mathbf{m}(p)$ and $\mathbf{m}^+(\infty) = \{\omega\}$.

The reader can easily prove by mutual induction on the $\mathcal{L}([\in],[\ni],[\|\|])$-formulas α and σ that for every point X in $\mathbf{Po}$, the following conditions are equivalent:

- $\mathbf{M}, X \models \alpha$;
- $\mathbf{M}^+, X \models \alpha^*$;

and for every line x in $\mathbf{Li}$, the following conditions are equivalent:

- $\mathbf{M}, x \models \sigma$;
- $\mathbf{M}^+, x \models \sigma^*$.

  Now take any $\mathcal{L}([\in],[\ni],[\|\|])$-formula φ which is true in $\mathbf{M}$. Suppose, for the sake of the argument, that φ is a point formula. Then the point $\mathcal{L}([\in],[\ni],\infty)$-formula, $[\in]\neg\infty \wedge \varphi^*$ is true in $\mathbf{M}^+$. Now let us now apply filtration to $\mathbf{M}^+$ with respect to $Sf([\in]\neg\infty \wedge \varphi^*)$ as was done in the proof of proposition 58. We obtain a finite quasi-projective models $\mathbf{M}^\ddagger =< \mathbf{Po}^\ddagger, \mathbf{Li}^\ddagger, \mathbf{I}^\ddagger, \omega^\ddagger, \mathbf{m}^\ddagger>$ with infinity in which the point $\mathcal{L}([\in],[\ni],\infty)$-formula, $[\in]\neg\infty \wedge \varphi^*$ is still true. Now, using again the correspondence between affine geometry and projective geometry, we associate to $\mathbf{M}^\ddagger$ the weak model $\mathbf{M}' =< \mathbf{Po}', \mathbf{Li}', \mathbf{I}', \|', \mathbf{m}'>$ defined as follows:

- $\mathbf{Po}' = \mathbf{Po}^\ddagger \setminus \{X\colon X \text{ is in } \mathbf{Po}^\ddagger \text{ and } X\mathbf{I}^\ddagger\omega^\ddagger\}$;
- $\mathbf{Li}' = \mathbf{Li} \setminus \{\omega^\ddagger\}$;
- $\mathbf{I}' = \mathbf{I}^\ddagger \cap (\mathbf{Po}' \times \mathbf{Li}')$;
- $\|' = \{(x,y)\ \colon x \text{ and } y \text{ are in } \mathbf{Li}' \text{ and there is a point } X \text{ in } \mathbf{Po}^\ddagger \text{ such that } X\mathbf{I}^\ddagger\omega^\ddagger,\ X\mathbf{I}^\ddagger x \text{ and } X\mathbf{I}^\ddagger y\}$;
- $\mathbf{m}'(P_i) = \mathbf{m}^\ddagger(P_i) \cap \mathbf{Po}'$ and $\mathbf{m}'(p_i) = \mathbf{m}^\ddagger(p_i) \cap \mathbf{Li}'$.

Again, the reader can easily prove by mutual induction on the $\mathcal{L}([\in],[\ni],[\|\|])$-formulas α and σ that for every point X in $\mathbf{Po}'$, the following conditions are equivalent:

- $\mathbf{M}^\ddagger, X \models \alpha^*$;

- $\mathbf{M}', X \models \alpha$;

and for every line x in $\mathbf{Li}'$, the following conditions are equivalent:

- $\mathbf{M}^{\ddagger}, x \models \sigma^*$;
- $\mathbf{M}', x \models \sigma$.

Hence **WAFF** has the finite model property and, being finitely axiomatizable, is decidable. Since the translation is polynomial, it preserves the complexity of $\mathbf{PG}_\infty$. ■

7 On the failure of finite model property in affine and projective modal logics

As we have already seen, the decidability of a number of systems introduced here is still open. The most common method for proving decidability of a finitely axiomatizable modal logic is by establishing its finite model property, by means of filtration or otherwise. It seems, however, that finite model property for logics of affine and projective geometries can be easily lost, and here we will illustrate that phenomenon with a few negative results using two different ideas of the same spirit.

The first idea (also used in [Stebletsova, 2000] to establish lack of finite model property for logics of projective geometries of dimension ≥ 3) revolves around the relationship between algebraic and geometric structures, as already discussed. Recall proposition 13 that every finite strongly Desarguesian standard model is Pappian. Its proof, based on a well known algebraic result, can be sketched as follows. A standard affine model is strongly Desarguesian iff the corresponding ternary ring obtained by affine coordinatization is a skew (i.e. non-commutative) field, while the model is Pappian, iff that ring is a (commutative) field. Now, conversely, with any skew field $< \Gamma, +, \times, \mathbf{0}, \mathbf{1} >$ one can associate an affine frame $< \mathbf{Po}, \mathbf{Li}, \mathbf{I}, \| >$ in the same way as one associates the Euclidean affine plane with the field of reals. Then, $< \mathbf{Po}, \mathbf{Li}, \mathbf{I}, \| >$ is strongly Desarguesian and if $\times$ is commutative over Γ, then it is Pappian as well. A well known algebraic result is Wedderburn-Artin's theorem that every finite skew field is commutative, i.e. a field, which implies that every finite strongly Desarguesian standard affine model is Pappian. On the other hand, every infinite non-commutative skew field produces an infinite strongly Desarguesian standard affine model which is not Pappian. Thus, we have established the following:

Proposition 65 *The logic* $\mathbf{AFF} + \varphi_{sDe}$ *does not have the finite model property with respect to standard affine models.*

Proof. *The formula* $\neg\varphi_{Pa}$ *is satisfiable in the class* $\mathbf{C}_{SM}^{sDe}$ *but not in any finite model from that class.* ■

Remark 66 *The result above does not imply that* $\mathbf{AFF} + \varphi_{sDe}$ *does not have the finite model property with respect to the class of general affine models. This question is still open.*

The second idea uses some combinatorial-geometric results implying that certain configurations of points and lines in a plane (space) can only be infinite. Two of the most popular such results are the following:

> **Sylvester's theorem** (see e.g. [Chakerian, 70]): Given a finite number of non-collinear points in the plane, there is a line in the plane which passes through exactly two of them.

and

> **Motzkin's theorem** (see [Motzkin, 67] or [Chakerian, 70]): Given a finite non-collinear set of points in the plane, each coloured in either black or white, there is a monochrome line (i.e. containing only points of the same colour) passing through at least two of these points.

It is know that both theorems hold in the real affine and projective planes, and in a number of planes over fields of finite characteristics, but for instance, Sylvester's theorem fails in $\mathbb{C}^2$ (where $\mathbb{C}$ is the complex field), while the truth of Motzkin's theorem there is still open. Most of the known proofs use metric notions, but there are ones (e.g. in [Chakerian, 70]) using only combinatorial or topological characteristics such as Euler's formula, which suggests that these results may still hold in very general affine or projective models.

Properties like those state above could be expressed in appropriate modal languages. For instance, Motzkin's theorem can be expressed in the language of incidence as follows: Consider the point formula

$$\mu = \langle\in\rangle\, p \wedge \langle\in\rangle \neg p \wedge [\in]\, [\ni]\, [\in]\, (\langle\ni\rangle\, P \wedge \langle\ni\rangle \neg P).$$

It is easy to see that μ is satisfied in an affine or projective model (of a most general nature) iff that model contains a point-line configuration contradicting the one described in Motzkin's theorem (where P and $\neg P$ represent the two colours). Therefore, in geometries where Motzkin's theorem holds, the formula μ can only be satisfied in an infinite model. Thus, in particular, we have established the following:

Proposition 67 *Let* **C** *be a class of real basic affine or projective models which contains at least one model satisfying Motzkin's property. Then the modal logic of incidence of the valid formulae in* **C** *does not have the finite model property.*

Corollary 68 *The following modal logics of incidence do not have the finite model property:*

- *The logic of all real basic affine models.*
- *The logic of all real projective models.*
- *The logic of the real affine plane.*
- *The logic of the real projective plane.*

8 Concluding remarks

In this paper, which is a natural continuation of the research initiated in [Balbiani et al, 97], [Balbiani, 98], and [Venema, 99] we have introduced and studied various modal logics for parallelism, orthogonality, affine and projective models and have discussed their metamathematical properties. We hope to have established the basic results and provided a good background for further development in the topic. On the other hand, we have left a number of open, and seemingly difficult to resolve (at least with the currently available techniques) problems, indicating that the area is at least technically challenging. We have also left unexplored a wide range of more or less expressive combinations of the basic geometric relations of incidence, parallelism, orthogonality, betweenness, equidistance etc. and the subject is still to take off the plane and fly into the 3-dimensional space and beyond.

Finally, we wish to reiterate that we see the value of the this research not only within the abstract realm of logic and geometry, but hope that it will find applications to practical spatial reasoning in the Real World.

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