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Strategic Games and Truly Playable Effectivity Functions

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Abstract A well-known result in the logical analysis of cooperative games states that the so-called playable effectivity functions exactly correspond to strategic games. More precisely, this result states that for every playable effectivity function E there exists a strategic game that assigns to coalitions of players exactly the same power as E, and every strategic game generates a playable effectivity function. While the latter direction of the correspondence is correct, we show that the former does not hold for a number of infinite state games. We point out where the original proof of correspondence goes wrong, and we present examples of playable effectivity functions for which no equivalent strategic game exists. Then, we characterize the class of *truly playable* effectivity functions, that do correspond to strategic games. Moreover, we discuss a construction that transforms any playable effectivity function into a truly playable one while preserving the power of most (but not all) coalitions. We also show that Coalition Logic, a formalism used to reason about effectivity functions, is not expressive enough to distinguish between playable and truly playable effectivity functions, and we extend it to a logic that can make that distinction while still enjoying the good meta-logical properties of Coalition Logic, such as finite axiomatization and decidability via finite model property.

Keywords Strategic games, cooperative games, correspondence, Coalition Logic

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1 Introduction

Several logics for reasoning about coalitional power have been proposed and studied in the last two decades. Eminent examples are: Alternating-time Temporal Logic (ATL) [1], Coalition Logic (CL) [16], and Seeing To It That (STIT) [2], used in computer science and philosophy to reason about properties of multi-agent systems. A crucial feature of these logics is their capacity to reason not only about abstract cooperative games but also about strategic games, that are models of non-cooperative behaviour.

In particular, the connection between the semantics of Coalition Logic and strategic form games relies on *Pauly's representation theorem* [16] which states that for every playable effectivity function E there exists a strategic game that assigns to coalitions of players exactly the same power as E, and every strategic game generates a playable effectivity function. The correspondence has been used to obtain further results for CL: if the semantics can be defined equivalently in terms of strategic games and playable effectivity functions, they can be used interchangeably when proving properties of the logic. A similar remark applies to ATL and STIT, related to Coalition Logic by a number of simulation results [4, 7, 10].

The relevance of Pauly's result goes beyond the logical analysis of interaction as it puts forward a characterization of *strategic games in terms of coalitional games*, therefore establishing a connection between the two families of game models. In this paper, we show that the representation theorem fails in certain cases. More precisely, we show that there are some playable effectivity functions with no corresponding strategic games. We point out where the original proof of correspondence goes wrong and present examples of playable effectivity functions for which no equivalent strategic game exists. Then, we define a more restricted class of effectivity functions, that we call *truly playable*, and show that it corresponds precisely to strategic games. Further, we present a construction that partly recovers the original correspondence in the sense that it transforms any playable function into a truly playable one while preserving the power of most (but not all) coalitions. Finally, we discuss the ramifications for the above mentioned logics. On the one hand we show that the complete axiomatization of Coalition Logic from [16] is not affected if we change the class of models from playable to truly playable. On the other hand, we propose more expressive languages that can characterize the property of true playability, thus drawing a logical distinction with Pauly's original notion of playability.

The paper is structured as follows: in Section 2 we introduce basic definitions and results. In Section 3 we point out the problems with Pauly's representation theorem, and in Section 4 we provide a new representation theorem based on truly playable effectivity functions. In Section 5 we discuss axiomatizations of Coalition Logic and some of its extensions with respect to truly playable models. Section 6 wraps up the paper with concluding remarks and some possible consequences of our result.

Preliminary versions of this work appeared in [8, 9]. This article extends them with detailed proofs, examples and discussion.

1\2	B	S
B	s_1	s_2
S	s_2	s_3

Fig. 1 Battle of Sexes. In the standard account outcomes are associated with vectors of payoffs. Here we consider strategic game forms, that abstract away from players' preferences.

2 Preliminaries

We begin by introducing the game-theoretic and logical notions used in the paper, discussing their relevant features.

2.1 Strategic Games

Strategic games (also: normal form games) are basic models of non-cooperative game theory [15]. Following [16], we focus on abstract game forms, where the effect of strategic interaction between players is represented by abstract outcomes from a given set and players' preferences are not specified. For simplicity we refer to them as strategic games.

Definition 1 (Strategic game) A *strategic game* G is a tuple $(N, \{\Sigma_i | i \in N\}, o, S)$ that consists of a nonempty finite set of players N, a nonempty set of strategies Σ_i for each player $i \in N$, a nonempty set of outcomes S, and an outcome function $o : \prod_{i \in N} \Sigma_i \to S$ which associates an outcome with every strategy profile.

Example 1 The well-known Battle of Sexes/Bach or Stravinsky scenario can be modeled by the game in Figure 1. Normally, the following payoffs are also assigned: $pay_1(s_1) = pay_2(s_3) = 2$, $pay_1(s_3) = pay_2(s_1) = 1$, and $pay_1(s_2) = pay_2(s_2) = 0$. Alternatively, one can specify the players' preferences as $s_2 <_1 s_3 <_1 s_1$ and $s_2 <_2 s_1 <_2 s_3$. As in [16], the definition of strategic game that we use includes only the bare strategic structure, without payoffs or preference relations.

For games where payoffs are given, the outcome function is often assumed a bijection and consequently dispensed with [15]. Some works, mostly aiming at formalizing the condition of *non-imposedness* in social choice theory [14],[1] assume surjectivity. As different scenarios may require different assumptions, we consider games with arbitrary outcome functions, as done in [16].

Additionally, we follow [16] and define coalitional strategies σ_C in G as tuples of individual strategies σ_i for $i \in C$, i.e., $\Sigma_C = \prod_{i \in C} \Sigma_i$. Note that (regardless of possible conceptual interpretations of the empty coalition $\emptyset$, cf. [5] for a discussion) this definition allows for only one strategy $\sigma_\emptyset$ when $C = \emptyset$, namely the empty function.

[1] The condition of non-imposedness is also referred to as *citizen sovereignty* and it allows players to freely choose among all possible alternatives in a decision process.

$$\begin{array}{ll} E(\{1,2\}) & = \{\{s_1\},\{s_2\},\{s_3\},\{s_1,s_2\},\{s_1,s_3\},\{s_2,s_3\},\{s_1,s_2,s_3\}\} \\ E(\{1\}) = E(\{2\}) & = \{\{s_1,s_2\},\{s_2,s_3\},\{s_1,s_2,s_3\}\} \\ E(\emptyset) & = \{\{s_1,s_2,s_3\}\} \end{array}$$

Fig. 2 Battle of Sexes in a cooperative twist. The effectivity function captures what players can achieve together.

2.2 Effectivity Functions

Effectivity functions have been introduced in cooperative game theory [14] to provide an abstract representation of the powers of coalitions to influence the outcome of the game.

Definition 2 (Effectivity function) An *effectivity function* is a function

$$E : 2^N \to 2^{2^S}$$

that associates a family of sets of states from S with each set of players.

Intuitively, elements of $E(C)$ are *choices* available to coalition C: if $X \in E(C)$ then by choosing X the coalition C can force the outcome of the game to be in X. Effectivity functions are usually required to satisfy additional properties, consistent with this interpretation.

Definition 3 (Playability [16]) An effectivity function E is *playable* iff the following conditions hold:

Outcome monotonicity: $X \in E(C)$ and $X \subseteq Y$ implies $Y \in E(C)$;
N-maximality: $\overline{X} \notin E(\emptyset)$ implies $X \in E(N)$;
Liveness: $\emptyset \notin E(C)$;
Safety: $S \in E(C)$;
Superadditivity: if $C \cap D = \emptyset$, $X \in E(C)$ and $Y \in E(D)$, then $X \cap Y \in E(C \cup D)$.

Intuitively, outcome monotonicity specifies that the effectivity function represents agents' power in a "negative" sense: C having choice X means that the coalition can make sure that *no state outside* X will be the outcome of the game. Thus, if $X \subseteq Y$ and C is effective for X then it must be also effective for Y. N-maximality assumes that the game is determined for the *grand* coalition of players N. Liveness and safety impose two kinds of seriality (every choice leads to an outcome state, and every coalition has at least one choice). Superadditivity expresses the assumption that disjoint coalitions can combine their choices freely. Note that E does not have to be *additive*, i.e., $C \cup D$ can have more power than follows from separate abilities of C and D.

Example 2 The effectivity function in Figure 2 formalizes powers of coalitions in the Battle of Sexes scenario. It is easy to check that the function is playable.

2.2.1 Nonmonotonic Core

Looking at playable effectivity functions, we can observe that their representation contains some redundancy. In particular, the fact that $E(C)$ is outcome monotonic suggests that one could succinctly represent it in terms of minimal sets, i.e., the elements of $E(C)$ that form an antichain under set inclusion. The nonmonotonic core is aimed at providing such a representation.

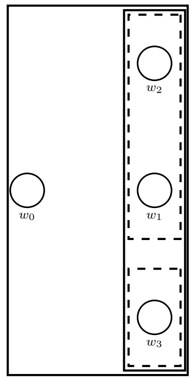

Fig. 3 A choice set $E(C)$ and its nonmonotonic core. The dashed rectangles indicate the choices at the coalition disposal that are minimal, while the thick rectangles indicate the non-minimal ones. As each set $X \in E(C)$ is a superset of some dashed rectangle, the nonmonotonic core of $E(C)$ is also complete.

Definition 4 (Nonmonotonic core [16]) Let E be an effectivity function. The *nonmonotonic core* $E^{nc}(C)$ for $C \subseteq N$ is the set of minimal sets in $E(C)$:

$$E^{nc}(C) = \{X \in E(C) \mid \neg\exists Y (Y \in E(C) \text{ and } Y \subsetneq X)\}.$$

Example 3 The nonmonotonic core of the effectivity function from Example 2 looks as follows: $E^{nc}(\{1,2\}) = \{\{s_1\},\{s_2\},\{s_3\}\}$, $E^{nc}(\{1\}) = E^{nc}(\{2\}) = \{\{s_1,s_2\},\{s_2,s_3\}\}$, $E^{nc}(\emptyset) = \{\{s_1,s_2,s_3\}\}$.

We will show in Section 3.1 that not all effectivity functions have a nonempty nonmonotonic core. Moreover, even when it is nonempty, not all sets in an effectivity function need to contain a set from the nonmonotonic core (cf. Section 4.3). Thus, E^{nc} does not always behave well as a representation of the effectivity function, unless it is *complete* in the following sense.

Definition 5 (Complete nonmonotonic core) The nonmonotonic core $E^{nc}(C)$ is *complete* iff for every $X \in E(C)$ there exists $Y \in E^{nc}(C)$ such that $Y \subseteq X$.

Note that, as illustrated in Figure 3, if $E(C)$ has a complete nonmonotonic core then $E^{nc}(C)$ can be used as a succinct representation of $E(C)$. Complete nonmonotonic cores turn out to be fundamental when establishing the proper correspondence between strategic games and effectivity functions.

The nonmonotonic core of the empty coalition is of particular interest to us. For it, the following holds.

Proposition 1 *For every playable effectivity function E:*

1. *$E(\emptyset)$ is a filter.*[2]

[2] A family F of subsets of Ω is a *filter* if and only if (1) $\Omega \in F$, (2) $\emptyset \notin F$ (3) F is closed under finite intersection, and (4) F is closed under supersets. These structures are sometimes referred to as proper filters, to distinguish them from improper filters, that do not satisfy condition (2) and consequently coincide with 2^{Ω} (cf. e.g. [6]).

2. $E^{nc}(\emptyset)$ *is either empty or a singleton.*

Proof (1) $E(\emptyset)$ is non-empty by safety; closed under supersets by outcome monotonicity, and under intersections by superadditivity (with respect to the empty coalition). Moreover, $\emptyset \notin E(\emptyset)$ by liveness.

(2) Suppose $E^{nc}(\emptyset)$ is non-empty, and let $X, Y \in E^{nc}(\emptyset)$. Then, coalition $\emptyset$ is effective for each of X and Y, hence, by superadditivity, it is effective for $X \cap Y$. By the definition of $E^{nc}(\emptyset)$, it follows that $X = X \cap Y = Y$.

2.2.2 α-Effectivity

Each strategic game $\mathcal{G}$ can be canonically associated with an effectivity function, called the α-effectivity function of $\mathcal{G}$ and denoted with $E^{\alpha}_{\mathcal{G}}$.

Definition 6 (α-Effectivity in Strategic Games) For a strategic game G, the (coalitional) *α-effectivity function* $E^{\alpha}_{G} : 2^N \to 2^{2^S}$ *is defined as follows:* $X \in E^{\alpha}_{G}(C)$ *if and only if there exists* σ_C *such that for all* $\sigma_{\overline{C}}$ *we have* $o(\sigma_C, \sigma_{\overline{C}}) \in X$.

Example 4 The effectivity function in Figure 2 is exactly the α-effectivity function of the strategic game in Figure 1.

Proposition 2 *For every α-effectivity function* $E^{\alpha}_{G} : 2^N \to 2^{2^S}$*, the following hold:*

1. *The nonmonotonic core of* $E^{\alpha}_{G}(\emptyset)$ *is the singleton set* $\{Z\}$ *where* $Z = \{x \in S \mid x = o(\sigma_N)$ *for some* $\sigma_N\}$.
2. $E^{\alpha}_{G}(\emptyset)$ *is the principal*[3] *filter generated by* Z.

Proof For both claims it suffices to observe that $Z \in E^{\alpha}_{G}(\emptyset)$ and that $Z \subseteq U$ for every $U \in E^{\alpha}_{G}(\emptyset)$. Therefore, $E^{nc}(\emptyset) = \{Z\}$ for $E = E^{\alpha}_{G}$ and $E^{\alpha}_{G}(\emptyset)$ is the principal filter generated by Z.

2.3 Modal Logic

We assume that the reader has basic familiarity with modal logic.[4] As for the languages to reason about strategic interaction we will use throughout the following terminology. A formula ϕ of a modal language Δ:

- *holds at a state* w of a Kripke model M whenever $M, w \models \phi$;
- is *valid in a Kripke model* M, denoted $\models_M \phi$, if and only if $M, w \models \phi$ for every $w \in W$, where W is the domain of M;
- is *valid in a class of Kripke models* $\mathcal{M}$, denoted $\models_{\mathcal{M}} \phi$, if and only if it is valid in every $M \in \mathcal{M}$;
- is *valid in a Kripke frame* F, denoted $\models_F \phi$, if and only if for every valuation V we have that $\models_{(F,V)} \phi$;
- is *valid in a class of Kripke frames* $\mathcal{F}$, denoted $\models_{\mathcal{F}} \phi$, if and only if it is valid in every $F \in \mathcal{F}$.

[3] Filter F on domain Ω is *principal* iff there exists $X \subseteq \Omega$ such that F is the set of all supersets of X. Then, F is said to be *generated* by X. Filters that are not principal are referred to as *non-principal.*

[4] For a more systematic exposition of basics of modal logic see e.g. [6, 3].

Given a Kripke model M and a formula ϕ, we denote

$$\phi^M = \{w \in M \mid M, w \models \phi\}$$

The set of formulas of Δ that are valid in a class of models $\mathcal{M}$ is denoted $\Delta_{\mathcal{M}}$ (for frames the denotation is $\Delta_{\mathcal{F}}$). For a set of formulas Σ, we write $M, w \models \Sigma$ to say that $M, w \models \sigma$, for all $\sigma \in \Sigma$.

We say that a set of formulas Σ semantically entails a formula ϕ denoted $\Sigma \models \phi$, if for every model M and a state $w \in M$, we have that $M, w \models \Sigma$ implies $M, w \models \phi$.

A modal rule

$$\frac{\phi_1, ..., \phi_n}{\psi} \tag{1}$$

is sound in a class of models $\mathcal{M}$ if $\phi_1, ..., \phi_n \models_{\mathcal{M}} \psi$.

Let us recall, following [6], that a modal logic Δ is called *classical* if it satisfies the rule of equivalence, i.e. for each $\Box$ in the language Δ we have:

$$\frac{\phi \leftrightarrow \psi}{\Box\phi \leftrightarrow \Box\psi} \tag{2}$$

It is called *monotonic* if it is classical and it moreover satisfies the rule of monotonicity, i.e. for each $\Box$ in the language Δ we have:

$$\frac{\phi \rightarrow \psi}{\Box\phi \rightarrow \Box\psi} \tag{3}$$

It is called *normal* if it is monotonic and it moreover satisfies the modal generalization rule and the K axiom, i.e. for each $\Box$ in the language Δ we have

$$\frac{\phi}{\Box\phi} \tag{4}$$

and $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$.

2.4 Coalition Logic

The logical language used to reason about effectivity functions is Coalition Logic [16]. Coalition Logic is multimodal language, where modalities are of the form $[C]\phi$ and represent the fact that a certain coalition C can force a certain formula ϕ to be true. The language of Coalition Logic is denoted $\mathcal{L}_{CL}$ and it is made by formulas that are defined as follows:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid [C]\phi$$

where p ranges over $Prop$ and C ranges over the subsets of N. The other boolean connectives are defined as usual.

The modalities are interpreted in neighbourhood structures [6] induced by the effectivity functions.

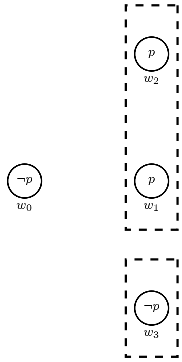

Fig. 4 Coalition Models. The modalities are interpreted in dynamic effectivity functions that specify the neighbourhood relation. In the picture the effectivity function $E^{nc}(w_0)(N) = \{\{w_2, w_1\}, \{w_3\}\}$ — as usual the minimal sets are represented by the dashed lines — and the valuation function $V(p) = \{w_1, w_2\}$ — represented by the atomic proposition assigned to the worlds where it is satisfied —make sure that the following statements hold: $M, w_0 \models [N]p$, i.e. at w_0 coalition N can achieve p and $M, w_0 \models [N]\neg p$, i.e. at w_0 coalition N can achieve $\neg p$.

Definition 7 (Coalition Models) A *Coalition Model* is a triple

$$(W, E, V)$$

where:

- W is a nonempty set of states;
- $E : W \longrightarrow (2^N \longrightarrow 2^{2^W})$ is a *dynamic* effectivity function [16], that associates an effectivity function to each state;
- $V : W \longrightarrow 2^{Prop}$ is a valuation function.

The satisfaction relation of the formulas of the form $[C]\phi$ with respect to a pair M, w is defined as follows:

$$M, w \models [C]\phi \text{ iff } \phi^M \in E(w)(C)$$

where, $\phi^M = \{w \in W \mid \ M, w \models \phi\}$. As outcome monotonicity is taken to be a property of all effectivity functions, the rule of monotonicity is valid in Coalition Logic, which is therefore a monotonic modal logic [11]. Figure 4 gives an example of Coalition Model.

The rule of monotonicity takes this form for each $C \subseteq N$:

$$\frac{\phi \rightarrow \psi}{[C]\phi \rightarrow [C]\psi} \tag{5}$$

As usual with neighbourhood structures, relations between set theoretical and logical properties are fairly immediate to spot. Standard correspondence results between class of frames and neighbourhood relations [6] can be automatically used for Coalition Logic.

Proposition 3 *Let $F = (W, E)$ be a coalition frame, and C, C' arbitrary coalitions. The following equivalences hold:*

- $\models_F [C]\top$ *if and only if for all* $w \in W$, $E(w)(C)$ *has safety;*
- $\models_F \neg[\emptyset]\neg\phi \to [N]\phi$ *if and only if for all* $w \in W$, $E(w)$ *is N-maximal;*
- $\models_F \neg[C]\bot$ *if and only if for all* $w \in W$, $E(w)(C)$ *has liveness;*
- $\models_F [C']\phi \wedge [C'']\psi \to [C' \cup C''](\phi \wedge \psi)$ *if and only if for all* $w \in W$, $E(w)$ *is superadditive;*
- $\phi \to \psi \models_F [C]\phi \to [C]\psi$ *if and only if for all* $w \in W$, $E(w)$ *is outcome monotonic.*

Proof The proof is standard and given in [16].

3 Problem with the Correspondence Result

The representation theorem given in [16][Theorem 2.27], known as Pauly's Representation Theorem, states that an effectivity function is playable if and only if it corresponds to a strategic game. It is a generalization of already existing correspondence results in [14, 17] for strategic games with arbitrary outcome functions. Its claim is the following:

Claim ([16], Theorem 2.27) A coalitional effectivity function E is playable if and only if there exists a strategic game G such that $E^{\alpha}_{G} = E$.

Specifically, the correspondence (called α-correspondence [16]) is formulated in two directions:

- every playable effectivity function is the α-effectivity function of some strategic game,
- each strategic game has an α-effectivity function that is playable.

While the proof of the latter claim is an easy check of the definitions, the former turns out not be correct.

3.1 A Counterexample to Pauly's Representation Theorem

We will now show a counterexample to Pauly's Representation Theorem, obtained by constructing an effectivity function that is playable but cannot correspond to any strategic game.

Proposition 4 *There is a playable effectivity function* E *for which* $E \neq E^{\alpha}_{G}$ *for all strategic games* G.

Proof Consider a coalitional game frame with a single player 'a' that has the set of natural numbers $\mathbb{N}$ as the domain (i.e., $N = \{a\}, W = \mathbb{N}$), and the effectivity defined as follows:

- $E(\{a\}) = \{X \subseteq \mathbb{N} \mid X \text{ is infinite}\}$;
- $E(\emptyset) = \{X \subseteq \mathbb{N} \mid \overline{X} \text{ is finite}\}$.

In other words, the grand coalition $\{a\}$ is effective for all infinite subsets of the natural numbers, while the empty coalition can enforce all its cofinite subsets.

E is playable and it does not correspond to any strategic game. To see this let us first verify the playability conditions. Outcome monotonicity, N-maximality, liveness and safety are straightforward to check. For superadditivity, notice that we only have two cases to verify:

1. $C = \{a\}, C' = \emptyset$;
2. $C = \emptyset, C' = \emptyset$.

For the first case, consider a set $X \in E(\{a\})$ and a set $Y \in E(\emptyset)$. To show that $X \cap Y \in E(\{a\} \cup \emptyset) = E(\{a\})$ we only need to observe that $X = (X \cap Y) \cup (X \cap \overline{Y})$. As $X \cap \overline{Y}$ is a finite set and Y cofinite, we must have that $X \cap Y$ is infinite, so $X \cap Y \in E(\{a\})$. For the second case it is sufficient to note that the intersection of two cofinite sets is cofinite.

On the other hand, $E^{nc}(\emptyset) = \emptyset$ because there are no minimal cofinite sets. This implies, by Proposition 2, that $E \neq E^{\alpha}_{G}$ for all strategic games G.

The counterexample constructs a playable effectivity function that assigns no minimal set to the empty coalition. Using Proposition 2, that states that α-effectivity functions have a minimal set, we are able to conclude that there are playable effectivity functions that do not correspond to any strategic games.

Given this fact, it is to be expected that the rather technical argument provided in [16] fails at some point. As its proof will be readapted for an alternative characterization result, it is useful to have a look at it.

3.2 Tracing the Problem

When showing that playable effectivity functions exactly correspond to strategic games, the difficult direction is from effectivity functions to games. Below, we summarize the relevant part of the proof of Theorem 2.27 from [16], and show where it goes wrong. We first outline the construction of a strategic game $\mathcal{G}$ given a playable effectivity function E (Steps 1–4); then, the argument is supposed to show that E α-corresponds to $\mathcal{G}$ (Steps 5–6).

Step 1: the players and the domain remain the same. The game $\mathcal{G} = (N, S, \Sigma_i, o)$ inherits the set of outcomes and the set of players as in the effectivity function E.

Step 2: coalitions choose a set from their effectivity function. Now, for each $i \in N$ a family of functions F_i is defined:

$$F_i = \{f_i : \mathcal{C}_i \to 2^S \mid f_i(C) \in E(C) \text{ for all } C \in \mathcal{C}_i\}$$

where $\mathcal{C}_i = \{C \subseteq N \mid i \in C\}$. Each function f_i assigns choices to all coalitions of which i is a member. F_i simply collects all such assignments.

Step 3: coalitions are partitioned according to their choices. Let $f = (f_i)_{i \in N}$, be a tuple of such assignments, one per player, where $f_i \in F_i$ for each $i \in N$. The next step is to define the set $P_\infty(f)$ which is the fixed point of iterative partitioning of the set of players in a coarsest possible way [5] such that players in the same part are assigned same coalitional choices. The partitioning goes along the following

[5] A partition $P(X)$ of a set X is a *coarsest partition* of X satisfying property $\mathcal{P}$ if it has some cardinality n and there is no partition of X with cardinality less than n satisfying $\mathcal{P}$. Notice that in our case the coarsest partition of a coalition, given a choice function f, is unique.

procedure:

$$\begin{aligned}
P_0(f) &:= \langle N \rangle \\
P_1(f) &:= P(f, N) = \langle C_1^1, \ldots, C_{k_1}^1 \rangle \\
P_2(f) &:= \langle P(f, C_1^1), \ldots, P(f, C_{k_1}^1) \rangle = \langle C_2^2, \ldots, C_{k_2}^2 \rangle \\
&\ldots \\
P_\infty(f) &:= P_r(f) \text{ where } r \text{ is the least index such that } P_r(f) = P_{r+1}(f),
\end{aligned}$$

where each $P(f, C)$ returns the coarsest partitioning $\langle C_1, \ldots, C_m \rangle$ of coalition C such that for all $l \leq m$ and for all $i, j \in C_l$ it holds that $f_i(C) = f_j(C)$. That is, a subset of C belongs to the partition $P(f, C)$ iff its members agree on their choices for C.

Step 4: an outcome is chosen in the intersection of coalitional choices. Let $P_\infty(f) = \langle C_1, \ldots, C_k \rangle$. Strategies and the outcome function are defined as follows. Each player in N is given a set of strategies of the form (f_i, t_i, h_i) where $f_i \in F_i$ is an assignment of coalitional choices for player i (see Step 2), $t_i \in N$ points out to a player (possibly different from i), and $h_i : 2^S \setminus \emptyset \to S$ is a *selector* function that picks up an arbitrary element from each nonempty subset of S.

The outcome of strategy σ_N is now defined as:

$$o(\sigma_N) = h_{i_0}(\mathcal{G}(f))$$

where:

- $\mathcal{G}(f) = \bigcap_{l=1}^{k} f(C_l)$, (note that $\mathcal{G}(f) \neq \emptyset$ due to superadditivity and liveness of E.)
- $i_0 = ((t_1 + \cdots + t_{|N|}) \bmod |N|) + 1$ is a unique choice of a player that depends on all t_i's, and
- h_{i_0} is the outcome selector from i_0's strategy.

This concludes the construction of a game $\mathcal{G}$ which is claimed to α-correspond to the effectivity function E. Steps 5–6 are to prove that $E = E_{\mathcal{G}}^{\alpha}$.

Step 5: choices are not removed by the construction. First, an attempt to prove $E(C) \subseteq E_{\mathcal{G}}^{\alpha}(C)$ for arbitrary coalition C is presented:

> For the inclusion from left to right, assume that $X \in E(C)$. Choose any C-strategy $\sigma_C = (f_i, t_i, h_i)_{i \in C}$ such that for all $i \in C$ and for all $C' \supseteq C$ we have $f_i(C') = X$. (*)
> By coalition monotonicity, such f_i exists.(**) Take now any $\overline{C}$-strategy, $\sigma_{\overline{C}} = (f_i, t_i, h_i)_{i \in \overline{C}}$. We need to show that $o(\sigma_C, \sigma_{\overline{C}}) \in X$. To see this, note that C must be a subset of one of the partitions C_l in $P_\infty(f)$. Hence, $o(\sigma_N) = h_{i_0}(\mathcal{G}(f)) = h_{i_0}\left(\bigcap_{l=1}^{k} f(C_l)\right) \in X$ [16, p.29].[6]

The deduction of the last sentence is where the proof goes wrong. The problem is that, for $C = \emptyset$, the only available strategy is the empty strategy $\sigma_\emptyset$ which vacuously satisfies condition (*). And, for any agent i, a choice assignment f_i satisfying the condition must exist. However, *there is no guarantee that any i will indeed choose f_i as its strategy* since the coalition C for which we can fix its strategy does not

[6] Technically speaking, no nonempty coalition can be a subset of a partition of N, as the first is a subset of N and the latter is a subset of 2^N.

include any players. In consequence, one cannot deduce that $h_{i_0}\big(\bigcap_{l=1}^{k} f(C_l)\big) \in X$; this could be only concluded if the intersection contains at least one player whose choice $f_i(C_l)$ is X (or a subset of X). To see this more clearly, let us consider the effectivity function from the proof of Proposition 4. Note that $\sigma_{\overline{C}} = \sigma_{\{a\}} = (f_a, a, h_a)$ such that $f_a(\{a\}) \in E(\{a\})$. Let us now take $X = \mathbb{N} \setminus \{1\}$, $f_a(\{a\}) = \mathbb{N}$, and $h_a(\mathbb{N}) = 1$. Now, $o(\sigma_N) = o(\sigma_{\{a\}}) = 1 \notin X$, which invalidates the argument from [16] quoted above.

Step 6: choices are not added by the construction. The proof of the other direction ($E^{\alpha}_{\mathcal{G}}(C) \subseteq E(C)$) fails too, because in order to establish the inclusion for $C = N$, it is reduced to inclusion in step 5 for $C = \emptyset$. In fact, a direct argument shows that, too. Consider a state space S with $x \in S$, and an effectivity function E such that $\{x\} \notin E(N)$. Now, let strategy profile σ_N consist of $\sigma_i = (f_i, t_i, h_i)$ where everybody assumes choosing the whole state space in all circumstances (i.e., $f_i(C) = S$ for all i and C) and applies the same selector h_i such that $h_i(S) = x$. Now we get that $o(\sigma_N) = x$, so $\{x\} \in E^{\alpha}_{\mathcal{G}}(N)$, and hence $E(N) \neq E^{\alpha}_{\mathcal{G}}(N)$.

This concludes our analysis of the proof of Pauly's representation theorem in [16]. The construction of the strategic game corresponding to a given effectivity function fails because the game might endow the empty coalition and the grand coalition of players with inappropriate powers. We consider the analysis important for two reasons. First, we have identified precisely what was wrong with the construction of the proof. Second, we will reuse the sound parts of the original construction when proving a revised version of the correspondence in Section 4.2 and to obtain some additional results in Section 4.4.

3.3 A Look at Consequences

We have seen that playability conditions are not sufficient to characterize strategic games. This raises some relevant issues for studying game models and logics for reasoning about games:

1. What are the "truly playable" effectivity functions that really correspond to strategic games? How can we characterize these functions in an abstract way? This issue is discussed in Section 4.
2. Conversely, how can we generalize the counterexample from Section 3.1 in order to characterize the class of playable but not truly playable effectivity functions? Section 4.3 deals with this question.
3. Is it possible to "reconstruct" playable effectivity functions into truly playable ones, without modifying the coalitional abilities much? We propose such a procedure in Section 4.4, and show that it preserves the powers of most coalitions.
4. Finally, what is the impact on logics for strategic ability, Coalition Logic in particular? Does changing from playable to truly playable models yields a different notion of validity? Are axiomatizations from [16, 10] sound and complete for truly playable models? What logical constructs are needed to distinguish between playable and truly playable structures? These questions are addressed in Section 5.

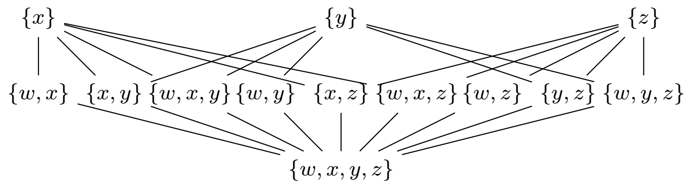

Fig. 5 A crown

4 Truly Playable Effectivity Functions

In this section we introduce an additional constraint on playable effectivity functions, that will enable us to restore the correspondence with strategic games in Section 4.2.

4.1 Characterizing True Playability

The subset of playable effectivity functions that α-correspond to strategic games can be characterized in terms of the nonmonotonic core of the empty coalition. Alternatively, it can be characterized in terms of effectivity of the grand coalition of all the agents.

Definition 8 An effectivity function E is *truly playable* iff it is playable and $E(\emptyset)$ has a complete nonmonotonic core.

We will formally prove the correspondence between strategic games and truly playable functions in Section 4.2.

Several equivalent characterizations of truly playable effectivity functions are given in Proposition 5. For one of them, we will need the additional notion of a *crown*. Intuitively, an effectivity function is a crown if every choice of the agents in the grand coalition includes at least one state that the grand coalition can enforce precisely. Formally, this means that N can only force some singleton sets and all their supersets. By forming a set of singleton outcomes and drawing the cones we obtain a "crown" as in Figure 5, hence the term.

Definition 9 An effectivity function $E : 2^N \to 2^{2^S}$ is a *crown* iff $X \in E(N)$ implies $\{x\} \in E(N)$ for some $x \in X$.

Proposition 5 *The following are equivalent for every playable effectivity function* $E : 2^N \to 2^{2^S}$.

1. *E is truly playable.*
2. *$E(\emptyset)$ has a non-empty nonmonotonic core.*
3. *$E^{nc}(\emptyset)$ is a singleton and $E(\emptyset)$ is a principal filter, generated by $E^{nc}(\emptyset)$.*
4. *E is a crown.*

Proof **(1) ⇒ (2):** immediate, by safety.

(2) ⇒ (3): Let $Z \in E^{nc}(\emptyset)$ and let $X \in E(\emptyset)$. Then, by superadditivity, $Z \cap X \in E(\emptyset)$, and $Z \cap X \subseteq Z$, hence $Z \cap X = Z$ by definition of $E^{nc}(\emptyset)$. Thus, $Z \subseteq X$. So, $E(\emptyset)$ is the principal filter generated by Z, hence $E^{nc}(\emptyset) = \{Z\}$.

(3) ⇒ (1): immediate from the definitions.

(3) ⇒ (4): Let $E^{nc}(\emptyset) = \{Z\}$ and suppose $\{x\} \notin E(N)$ for all $x \in X$ for some $X \in E(\emptyset)$. Then, by N-maximality, $S \setminus \{x\} \in E(\emptyset)$, i.e. $Z \subseteq S \setminus \{x\}$ for every $x \in X$. Then $Z \subseteq S \setminus X$, hence $S \setminus X \in E(\emptyset)$. Therefore, $X \notin E(N)$ by superaditivity and liveness. By contraposition, E is a crown.

(4) ⇒ (3): Let $Z = \{z \mid \{z\} \in E(N)\}$ and let $X \in E(\emptyset)$. Take any $z \in Z$, which is nonempty by liveness and the fact that E is a crown. By superadditivity we obtain that $\{z\} \cap X \in E(\emptyset)$, hence $z \in X$ by liveness. Thus, $Z \subseteq X$. Moreover, $Z \in E(\emptyset)$, for else $S \setminus Z \in E(N)$ by N-maximality, hence $\{x\} \in E(N)$ for some $x \in S \setminus Z$, which contradicts the definition of Z. Therefore, $E(\emptyset)$ is the principal filter generated by Z, hence $E^{nc}(\emptyset) = \{Z\}$.

We also observe that on finite domains playability and true playability coincide.

Proposition 6 *Every playable effectivity function $E : 2^N \to 2^{2^S}$ on a finite domain S is truly playable.*

Proof Straightforward, by Proposition 5.3 and the fact that every filter on a finite set is principal.

Finally, note that in a truly playable function the nonmonotonic core for coalitions different from $\emptyset, N$ does not have to be complete, and neither does it have to be nonempty, as Example 5 demonstrates.

Example 5 Consider the following effectivity function for $N = \{a, b\}, S = \mathbb{N}$:

- $E(\emptyset) = \{\mathbb{N}\}$,
- $E(\{a\}) = E(\{b\}) =$ all cofinite subsets of $\mathbb{N}$,
- $E(\{a, b\}) = 2^{\mathbb{N}} \setminus \emptyset$.

It is easy to see that E is truly playable, but the nonmonotonic core of $E(\{a\})$ is empty, and hence also not complete.

Remark 1 Out of the 4 alternative characterizations of true playability, condition (4) is perhaps most revealing conceptually. It says that, whenever the grand coalition N can enforce the outcome state to be in X, they can do it by singling out a particular state from X. Note that choices in effectivity functions have slightly "negative" flavor: $X \in E(A)$ means that A have the power to rule out all states from $S \setminus X$. However, it does not say which states from X can be actually achieved in the next moment. This can be attributed to either imprecision of the game description or inherent nondeterminism of the game. The "crown" condition imposes that, for every enforceable property, N can achieve it by selecting the right outcome precisely and with no nondeterminism. By Proposition 6, we know that the playability conditions from [16] rule out imprecision and/or nondeterminism of N's choices in case of finite games. However, those conditions prove too weak for infinite games (Proposition 4).

Game models are usually deterministic. It is often claimed that this is not a conceptual limitation because one can always add an extra player (be it chance, nature, or Some External Modeler) whose choices are used to resolve the non-determinism (cf. for instance the discussion in [1]). We will use the idea for our reconstruction of playable – but not truly playable – effectivity functions in Section 4.4.

4.2 Truly Playable Functions Correspond to Strategic Games

The proof of Theorem 2.27 from [16] fails when we consider the effectivity function of the empty coalition or the grand coalition. However the proof *is* correct for the other cases. We will now show that the additional condition of true playability yields correctness of the original construction from [16].

Theorem 1 *A coalitional effectivity function E α-corresponds to a strategic game if and only if E is truly playable.*

Proof By Propositions 2 and 5, for any strategic game $\mathcal{G}$ its α-effectivity function $E^{\alpha}_{\mathcal{G}}$ is truly playable.

For the other direction, given a truly playable effectivity function E, we slightly change Pauly's procedure outlined in Section 3.2 (Steps 1–4). We impose an additional constraint on players' strategies $\sigma_i = (f_i, t_i, h_i)$, namely, we require that $h_i(X) = x$ for some $\{x\} \in E(N)$. In other words, the selector functions only select from the "jewels" in the crown. Note that for $C \notin \{\emptyset, N\}$ the new procedure yields game $\mathcal{G}$ with exactly the same $E^{\alpha}(C)$ as the original construction $\mathcal{G}'$ from [16] because:

1. We do not add any new choice sets to $E^{\alpha}_{\mathcal{G}'}(C)$. Indeed, that could only happen because the selectors chosen by agents outside C are restricted to $\{x \mid \{x\} \in E(N)\}$, and hence we can have that $X \cap \{x \mid \{x\} \in E(N)\} \in E^{\alpha}_{\mathcal{G}}(C)$ in the new construction for some $X \in E^{\alpha}_{\mathcal{G}'}(C)$ from the previous construction. However, by true playability of E and Proposition 5 we have that $\{x \mid \{x\} \in E(N)\} \in E(\emptyset)$, and thus by superadditivity all the states $y \notin \{x \mid \{x\} \in E(N)\}$ can be removed from C's strategies that yielded X in $\mathcal{G}'$. But then these states will also be removed from the intersection $\bigcap_{l=1}^{k} f(C_l)$, and so $X \cap \{x \mid \{x\} \in E(N)\} \in E^{\alpha}_{\mathcal{G}'}(C)$ already in the previous construction.
2. We do not remove any choice sets from $E^{\alpha}_{\mathcal{G}'}(C)$. Indeed, that could only happen because of removing an $X \in E^{\alpha}_{\mathcal{G}'}(C)$ which contains "superfluous" elements and replacing it with $X \cap \{x \mid \{x\} \in E(N)\}$. But then, X must also be in $E^{\alpha}_{\mathcal{G}}(C)$ because $E^{\alpha}_{\mathcal{G}}(C)$ is closed under supersets.

It remains now to show that the procedure constructs a strategic game $\mathcal{G}$ such that $E(C) = E^{\alpha}_{\mathcal{G}}(C)$ for $C = \emptyset$ and $C = N$, that is, to show that steps 5 and 6 work well for these coalitions in truly playable structures.

Additional Step 5. We show that $E(C) \subseteq E^{\alpha}_{\mathcal{G}}(C)$ for $C = \emptyset$ and $C = N$, the only cases in which the original proof failed for playable structures.

Assume that $X \in E(\emptyset)$. We need to prove that $X \in E^{\alpha}_{\mathcal{G}}(\emptyset)$. By true playability we know that there exists $Y \in E^{nc}(\emptyset)$ such that $Y \subseteq X$. By Proposition 5, $E^{nc}(\emptyset) = \{Y\}$ and $E(\emptyset) = \{Z \mid Y \subseteq Z\}$. We will show now that $Y = \{x \mid \{x\} \in E(N)\}$ (*).

First, suppose that $x \in Y$ and $\{x\} \notin E(N)$, then by N-maximality $S \setminus \{x\} \in E(\emptyset)$, a contradiction. Second, let $\{x\} \in E(N)$ and $x \notin Y$, then by superadditivity $\emptyset \in E(N)$ which contradicts liveness.

Now, consider any strategy profile σ_N. We have $o(\sigma_N) = h_{i_0}\big(\bigcap_{l=1}^{k} f(C_l)\big) \in Y$ because every h_i returns only elements in Y by construction.

For the case $C = N$, assume that $X \in E(N)$. We need to prove that $X \in E^{\alpha}_{\mathcal{G}}(N)$. By true playability, there exists $x \in X$ such that $\{x\} \in E(N)$. Now, let σ_N consist of strategies $\sigma_i = (f_i, t_i, h_i)$ such that $f_i(N) = \{x\}$ for every i. It is easy to see that $o(\sigma_N) = x$, and hence $\{x\} \in E^{\alpha}_{\mathcal{G}}(N)$. Thus, $X \in E^{\alpha}_{\mathcal{G}}(N)$ because $E^{\alpha}_{\mathcal{G}}(N)$ is closed under supersets.

Additional Step 6. Dually to Step 5, we show that $E^{\alpha}_{\mathcal{G}}(C) \subseteq E(C)$ for $C = \emptyset$ and $C = N$. That is, assuming $X \notin E(C)$ we show that $X \notin E^{\alpha}_{\mathcal{G}}(C)$. We do it by a slight modification of the original proof from [16].

Suppose first that $C = N$. Then, $X \notin E(N)$ implies that $\overline{X} \in E(\emptyset)$ by N-maximality, and by Step 5 we have $\overline{X} \in E^{\alpha}_{\mathcal{G}}(\emptyset)$. Since $E^{\alpha}_{\mathcal{G}}$ is truly playable, we have also that $X \notin E^{\alpha}_{\mathcal{G}}(N)$.

Now, for $C = \emptyset$ (and in fact for every $C \neq N$), we choose an arbitrary $j_0 \in \overline{C}$. Let σ_C be any strategy for coalition C. We must show that there is a strategy $\sigma_{\overline{C}}$ such that $o(\sigma_C, \sigma_{\overline{C}}) \notin X$. To show this, we take $\sigma_{\overline{C}} = (f_i, t_i, h_i)_{i \in \overline{C}}$ such that for all $C' \supseteq \overline{C}$ and for all $i \in \overline{C}$ we have $f_i(C') = S$. We also choose t_{j_0} such that $((t_1 + \ldots + t_n) \bmod n) + 1 = j_0$. Note that $\overline{C}$ must be a subset of one of the partitions C_l in $P_\infty(f)$, say C_{l_0}. Moreover, there must be a partitioning $\langle C_1, \ldots, C_k \rangle$ of $N \setminus C_{l_0}$ such that $\mathcal{G}(f) = f(C_{l_0}) \cap \bigcap_{l=1}^{k} f(C_l) = \bigcap_{l=1}^{k} f(C_l)$. Since $f(C_l) \in E(C_l)$ we get that $\mathcal{G}(f) \in E(N) \setminus C_{l_0}$ by superadditivity. By coalition-monotonicity and the fact that $N \setminus C_{l_0} \subseteq C$, we also have $\mathcal{G}(f) \in E(C)$. Finally, by (*) and superadditivity we obtain $\mathcal{G}(f) \cap \{x \mid \{x\} \in E(N)\} \in E(C)$.

Since $X \notin E(C)$ and $E(C)$ is closed under supersets, it must hold that $\mathcal{G}(f) \cap \{x \mid \{x\} \in E(N)\} \not\subseteq X$. Thus, there is some $s_0 \in S$ such that: $s_0 \in \mathcal{G}(f)$, $\{s_0\} \in E(N)$, and $s_0 \notin X$. Now we fix h_{j_0} so that $h_{j_0}(\mathcal{G}(f)) = s_0$. Then, $o(\sigma_C, \sigma_{\overline{C}}) = h_{j_0}(\mathcal{G}(f)) = s_0 \notin X$ which concludes the proof.

4.3 Non-Truly Playable Structures

In this section we focus on the class of playable but not truly playable effectivity functions, hereafter called "non-truly playable". From Proposition 5 we know that a playable effectivity function E is truly playable if and only if the filter $E(\emptyset)$ is principal and generated by $E^{nc}(\emptyset)$. Hence, playability and true playability coincide on finite domains. There exist, however, non-truly playable effectivity functions on infinite domains, and we have already discussed an example of such a function in Section 3.1.

Non-truly playable effectivity functions have a simple abstract characterization:

Proposition 7 *Effectivity function $E : 2^N \to 2^{2^S}$ is non-truly playable if and only if it is playable and $E(\emptyset)$ is a non-principal filter.*

Proof Straightforward, by Proposition 5.

For a generic class of examples, consider an infinite domain S, and let $\mathcal{F}$ be any non-principal filter on S. Then we define an effectivity function $E_{\mathcal{F}}$ on S as follows.

- $E_{\mathcal{F}}(\emptyset) = \mathcal{F}$.
- $E_{\mathcal{F}}(N) = \{X \mid \overline{X} \notin \mathcal{F}\}$
- For each C with $\emptyset \subsetneq C \subsetneq N$ take $E_{\mathcal{F}}(C)$ to be any set of sets such that $E_{\mathcal{F}}(\emptyset) \subseteq E_{\mathcal{F}}(C) \subseteq E_{\mathcal{F}}(N)$ that is closed under outcome monotonicity and that are pairwise closed under regularity and superadditivity.

Proposition 8 *$E_{\mathcal{F}}$ is playable but not truly playable.*

Proof That $E_{\mathcal{F}}$ is not truly playable follows by the fact that $E_{\mathcal{F}}(\emptyset)$ is a non-principal filter. In order to check that $E_{\mathcal{F}}$ is playable we only need to check superaddivity for $\emptyset$ and N, as the other conditions follow by construction. Assume $X \in E_{\mathcal{F}}(\emptyset)$ and $Y \in E_{\mathcal{F}}(N)$. We have to prove that $X \cap Y \in E_{\mathcal{F}}(N)$. Suppose that $X \cap Y \notin E_{\mathcal{F}}(N)$. But then, by definition of $E_{\mathcal{F}}(N)$ we have that $\overline{X \cap Y} \in E_{\mathcal{F}}(\emptyset)$. By de Morgan's law we have that $\overline{X} \cup \overline{Y} \in E_{\mathcal{F}}(\emptyset)$. But as $Y \in E_{\mathcal{F}}(N)$ we know that $\overline{Y} \notin E_{\mathcal{F}}(\emptyset)$. However $E_{\mathcal{F}}(\emptyset)$ is a filter so $X \cap (\overline{X} \cup \overline{Y}) \in E_{\mathcal{F}}(\emptyset)$. From this follows that $\overline{Y} \in E_{\mathcal{F}}(\emptyset)$. Contradiction.

Here are some examples of non-principal filters on $\mathbb{N}$:

- For any $k \in \mathbb{N}$ let $E_k(\emptyset) = \{X \subseteq \mathbb{N} \mid X \text{ is cofinite in } \mathbb{N} \text{ and } k \in X\}$.
- More generally, for any $K \subseteq \mathbb{N}$ which is not cofinite in $\mathbb{N}$ let $E_K(\emptyset) = \{X \subseteq \mathbb{N} \mid X \text{ is cofinite in } \mathbb{N} \text{ and } K \subseteq X\}$.

In the case of single player, $N = \{a\}$, the construction above immediately extends these filters to non-truly playable effectivity functions on $\mathbb{N}$:

- $E_k(\{a\}) = \{X \subseteq \mathbb{N} \mid X \text{ is infinite or } k \in X\}$,
- $E_K(\{a\}) = \{X \subseteq \mathbb{N} \mid X \text{ is infinite or } K \cap X \neq \emptyset\}$,

We know from the proof of Proposition 5 that for all playable E, if the non-monotonic core of $E(\emptyset)$ is nonempty, then it must be complete. The above examples show that this does not have to be the case for other coalitions. For instance, observe that the nonmonotonic core of $E_1(\{a\})$ is non-empty: $E_1^{nc}(\{a\}) = \{\{1\}\}$. Still, the set *Even* of even natural numbers is in $E_1(\{a\})$ but $\{1\} \not\subseteq \mathit{Even}$. Similarly, we can show that even infinite nonmonotonic core does not guarantee its completeness. Indeed, it is easy to see that $E_{\mathit{Even}}^{nc}(\{a\}) = \{\{k\} \mid k \in \mathit{Even}\}$ while we also have $\mathit{Odd} \in E_{\mathit{Even}}(\{a\})$ where *Odd* is the (infinite) set of odd natural numbers. Thus, a playable effectivity function can have its nonmonotonic core for the grand coalition nonempty and consisting entirely of singletons, and yet not be a crown – hence remaining non-truly playable.

We conclude this section with a scenario that motivates our interest in non-truly playable functions on a more practical level.

Example 6 Consider two agents a and b acting in an environment with countably many configurations (labeled by natural numbers). We assume that each agent alone can only prevent a finite number of configurations. Also, there are some uncontrollable configurations – more specifically, the configurations in set $K_a \subseteq \mathbb{N}$ are uncontrollable by a, and those in $K_b \subseteq \mathbb{N}$ cannot be prevented by b. For instance, if $K_a = \{3, 5, 12, 13\}$ then a can play so that none of $1, 2, 4$ is the next state of the environment, but he cannot play to prevent $1, 2, 3$. Finally, the system is nondeterministic with infinite branching, i.e., the grand coalition $\{a, b\}$ cannot narrow down possible next states to a finite set. This scenario can be modeled with an effectivity function similar to the previous examples:

- $E(\{a, b\})$ consists of all infinite subsets of $\mathbb{N}$;
- $E(\{a\})$ consists of all cofinite subsets of $\mathbb{N}$ that subsume K_a;
- $E(\{b\})$ consists of all cofinite subsets of $\mathbb{N}$ that subsume K_b;
- $E(\emptyset)$ consists of all cofinite subsets of $\mathbb{N}$ that subsume $K_a \cup K_b$.

It is easy to check that E is non-truly playable. As a consequence, by Theorem 1, it cannot be implemented as a strategic game with two players. However, there exists a game with *three* players so that $\{a\}$, $\{b\}$, and $\{a, b\}$ have exactly the power assigned to them in the above scenario. We will present the construction of such a game in Section 4.4.

4.4 From Playable to Truly Playable Effectivity Functions

In this section we show that one can reconstruct a non-truly playable effectivity function into a truly playable one with "minimal" modifications. To do so, we interpret choices of the grand coalition containing multiple outcome states as ones that involve inherent nondeterminism. That is, we interpret $\{x_1, x_2, \dots\} \in E(N)$ as a choice where no agent has control over which state out of $x_1, x_2, \dots$ will become the outcome; as a consequence any of these states can possibly be encountered in the next moment. Under such assumption, it is possible to recover true playability (and hence the correspondence to strategic games) by a simple extension of Pauly's procedure. The extension consists in adding an extra player $\mathbf{d}$ (the "decider") who settles the nondeterminism and decides which of $x_1, x_2, \dots$ is going to become the next state.

Proposition 9 *Let $E : 2^N \to 2^{2^S}$ be a playable effectivity function. There exists a truly playable effectivity function $E' : 2^{N \cup \{\mathbf{d}\}} \to 2^{2^S}$ with additional player $\mathbf{d} \notin N$, such that:*

- *$E'(C) = E(C)$ for every $C \subseteq N, C \neq \emptyset$,*
- *$E'(\emptyset) = \{S\}$, and*
- *$E'(N \cup \{\mathbf{d}\}) = 2^S \setminus \{\emptyset\}$.*

Proof Given a playable E, we construct a strategic game whose α-effectivity function satisfies the properties above. Then, existence of a truly playable effectivity function follows immediately. The idea is to take the construction from the proof of Theorem 2.27 in [16] and reassign selection of the outcome state to the additional player $\mathbf{d}$.

Let $h : 2^S \setminus \{\emptyset\} \to S$ be any selector function that selects an arbitrary element from the argument set. In our case, h will designate the "default" outcome for each subset of S. Now, the game $\mathcal{G}$ is constructed as follows:

- $N' = N \cup \{\mathbf{d}\}$;
- The strategies of each player $i \neq \mathbf{d}$ are simply the player's assignments of coalitional choice, i.e., $\Sigma_i = F_i$, as in section 3.2;

- The strategies of $\mathbf{d}$ are state selections: $\Sigma_{\mathbf{d}} = S$;
- The transition function is based on the same partitioning of N as before, that yields $\langle C_1, \ldots, C_k \rangle$. Then, the game proceeds to the state selected by the decider if his choice is consistent with the choices of the others, otherwise it proceeds to the appropriate "default" outcome:

$$o(\sigma_N, s) = \begin{cases} s & \text{if } s \in \bigcap_{i=1}^{k} f(C_l) \\ h(\bigcap_{i=1}^{k} f(C_l)) & \text{else.} \end{cases}$$

Now, it is easy to see that for every $\emptyset \subsetneq C \subsetneq N$ indeed $E^{\alpha}_{\mathcal{G}}(C) = E(C)$ because that was the case in the original construction, and the only difference now is that $\mathbf{d}$ "took over" the selection of a state in $\bigcap_{i=1}^{k} f(C_l)$ from a collective choice of N. For $C = N$, we also have $E^{\alpha}_{\mathcal{G}}(N) = E(N)$ since for every σ_N we get by superadditivity that $\bigcap_{i=1}^{k} f(C_l) \in E(N)$, and every state from the intersection can be potentially selected by $\mathbf{d}$. Moreover, $\{s\} \in E^{\alpha}_{\mathcal{G}}(N \cup \{\mathbf{d}\})$ for every $s \in S$ because $\{s\}$ is enforced by $\sigma_{N\cup\{\mathbf{d}\}} = \langle f_1, \ldots, f_{|N|}, s\rangle$ such that $f_i = S$ for all $i \in N$. Thus, by outcome monotonicity, $E^{\alpha}_{\mathcal{G}}(N \cup \{\mathbf{d}\}) = 2^S \setminus \{\emptyset\}$. Finally, by true playability of $E^{\alpha}_{\mathcal{G}}$, we have $E^{\alpha}_{\mathcal{G}}(\emptyset) = \{\{s \mid \{s\} \in E^{\alpha}_{\mathcal{G}}(N \cup \{\mathbf{d}\})\}\} = \{S\}$. We observe additionally that $E^{\alpha}_{\mathcal{G}}(\mathbf{d}) = \{\{s\} \cup \{h(X) \mid X \in E^{\alpha}_{G}(\mathbf{d}) \text{ and } s \notin X\} \mid s \in S\}$.

5 Strategic Logics and True Playability

In this section, we investigate the impact of true playability on logics of coalitional ability. We begin by indicating that the validities of Coalition Logic do not change if we restrict models to truly playable. As a consequence, CL (and even ATL) cannot distinguish between playable and truly playable models. Then, we discuss two extensions of CL that can discern the two classes of structures.

5.1 Coalition Logic and True Playability

The previous part of the paper analyzed the specific features of coalitional ability in strategic games, providing an alternative representation result to the one originally given in [16]. We can immediately observe that this new relation between effectivity functions and strategic games has no repercussions on the semantics of Coalition Logic and the soundness and completeness results for that logic. The axiomatization of Coalition Logic presented in [16] extends the axiomatization of the classical propositional logic with formulas and rules characterizing playability listed in Proposition 3. In [16] it is proved that this axiomatization is sound and complete with respect to playable coalition models. The following result can be carried over.

Corollary 1 *The axiomatization of playable Coalition Logic from [16] is sound and complete wrt truly playable coalition models (and hence also wrt strategic game models).*

Proof To see this, let us formally define Play to be the class of playable coalitional models, and TrulyPlay as the class of models based on truly playable effectivity functions. Since TrulyPlay $\subset$ Play, every Coalition Logic formula valid in Play is valid in TrulyPlay, too. To see the converse, one can use the finite model property of Coalition Logic with respect to Play and the fact that it coincides with TrulyPlay on finite models.

These results show that Coalition Logic describes strategic interaction at a very abstract level and its expressiveness is insufficient to distinguish playability from true playability. In the next sections we extend the language to make this distinction possible.

Remark 2 The semantics based on effectivity functions can be extended to ATL (see e.g., [7]; also, cf. [16] for the fragment of ATL without "until", called *Extended Coalition Logic*). Again, it can be shown that Play and TrulyPlay determine the same sets of validities for ATL, by checking the soundness of the axiomatization for ATL given in [10] for Play, and using the completeness result for ATL with respect to strategic game models (equivalently, TrulyPlay) proved in the same paper.

5.2 CL with Infinite Disjunctions

One possible extension of CL that can tell apart the classes Play and TrulyPlay involves infinite disjunctions of formulas. The idea is that in truly playable models, every choice of the grand coalition can be narrowed down to a singleton. The infinitary disjunction $\bigvee_{i\in\mathcal{I}}$ for a set of indices $\mathcal{I}$ has the natural interpretation:

$$M, w \models \bigvee_{i\in\mathcal{I}} \phi_i \ \text{ if and only if } \ M, w \models \phi_i \text{ for some } i \in \mathcal{I}.$$

Proposition 10 *For any cardinal number*[7] *κ, let Play_κ (resp. $\mathsf{TrulyPlay}_\kappa$) denote the class of playable (resp. truly playable) coalition models with the domain of outcomes W of cardinality at most κ and let $\{p_\iota\}_{\iota\in\kappa}$ be a set of different propositional letters. Then the following hold:*

1. $\mathsf{Play}_\kappa \not\models [N]\bigvee_{\iota\in\kappa} p_\iota \leftrightarrow \bigvee_{\iota\in\kappa}[N]p_\iota$;
2. $\mathsf{TrulyPlay}_\kappa \models [N]\bigvee_{\iota\in\kappa} p_\iota \leftrightarrow \bigvee_{\iota\in\kappa}[N]p_\iota$.

Proof For (1) simply check the example in Section 3.1 with the set S being κ and every state ι associated with a designating atomic proposition p_ι. Claim (2) follows from Proposition 5.

The above scheme actually characterizes the class of truly playable effectivity frames on domains of bounded cardinality:

Proposition 11 *For every playable effectivity frame F over domain S of cardinality at most κ, we have that F is truly playable if and only if $F \models [N]\bigvee_{\iota\in\kappa} p_\iota \leftrightarrow \bigvee_{\iota\in\kappa}[N]p_\iota$.*

Proof By Proposition 10, if F is truly playable then the formula above is valid in F. Conversely, if $F = (S, E)$ is playable but not truly playable then, for some $w \in S$, $E(w)(N)$ is not a crown, i.e. there is $X \in E(w)(N)$ such that $\{x\} \notin E(w)(N)$ for all $x \in X$. Consider now a model $M = (S, E, V)$ based on F such that V assigns to each $x \in X$ a designating atom that we indicate with p_x. This is possible because there are enough propositional letters available. Then $M \models [N]\bigvee_{\iota\in\kappa} p_\iota$, while $M \not\models \bigvee_{\iota\in\kappa}[N]p_\iota$, hence $M \not\models [N]\bigvee_{\iota\in\kappa} p_\iota \leftrightarrow \bigvee_{\iota\in\kappa}[N]p_\iota$.

[7] We regard cardinals as (special) ordinals in von Neumann sense: any ordinal is the set of all smaller ordinals.

5.3 "Outcome Selector" Modality: Semantic Approach

Adding infinitary operators to a logical language makes its practical applicability unfeasible. This is why we look for other extensions of Coalition Logic that allow to distinguish the two classes of models and/or frames while preserving finiteness of formulae. To this end, we propose another (in fact, very simple) extension of CL, by adding a new modality $\langle\mathcal{O}\rangle$, with a dual $[\mathcal{O}]$, called "*outcome selector*". The informal reading of $\langle\mathcal{O}\rangle\phi$ should be "there is an outcome state, enforceable by the grand coalition and satisfying ϕ". The dual operator is, as usual, defined by $[\mathcal{O}]\phi \equiv \neg\langle\mathcal{O}\rangle\neg\phi$. We will call the extended logic CLO ("Coalition Logic with Outcome selector"), and refer to its language as $\mathcal{L}_{CLO}$.

The semantics of CL modalities remains as before, and we add the following clause to interpret the new modality:

$$M, w \models \langle\mathcal{O}\rangle\phi \quad \text{iff} \quad \exists v.(\{v\} \in E(w)(N) \& M, v \models \phi).$$

Now, the scheme $[N]\phi \leftrightarrow \langle\mathcal{O}\rangle\phi$ can be used to discern between playable and truly playable structures in the following sense.

Proposition 12 *For every playable effectivity frame F, we have that F is truly playable if and only if $F \models [N]\phi \leftrightarrow \langle\mathcal{O}\rangle\phi$.*

Proof Straightforward.

That is, we add the new modality $\langle\mathcal{O}\rangle$ to indicate outcomes *over which the grand coalition N has complete and precise control.* Then, $F \models [N]\phi \leftrightarrow \langle\mathcal{O}\rangle\phi$ requires that every enforceable property can be obtained by a precise choice, and hence the effectivity of N in F must be represented by a crown.

5.4 CL with "Outcome Selector": Axiomatic Approach

In order to characterize true playability axiomatically, we need to redefine the semantics of $\langle\mathcal{O}\rangle$ as a *normal* modality with a standard Kripke semantics. First, we expand coalition models to what we call *extended coalition models* with an additional "outcome enforceability" relation R. Then, we use axioms to impose the right behavior of R.

Definition 10 (Extended coalition frames) An *extended (playable) coalition frame* is a neighbourhood frame $F = (W, E, R)$ where W is a set of outcomes, E a playable effectivity function and R a binary relation on W.

An extended coalition model, pictured in Figure 6, is an extended coalition frame endowed with a valuation function. Given an extended coalition model $M = (W, E, R, V)$, the modality $\langle\mathcal{O}\rangle$ has a standard Kripke semantics with respect to R:

$$M, w \models \langle\mathcal{O}\rangle\phi \text{ if and only if } wRs \text{ and } M, s \models \phi \text{ for some } s \in W.$$

That is, $\langle\mathcal{O}\rangle$ has standard Kripke semantics with respect to the outcome enforceability relation R. Note that extended coalition models do not require any interaction between the effectivity function and the relation R. However, given the intuitive reading of the relation R, the interaction suggests itself, and the following definition accounts for that.

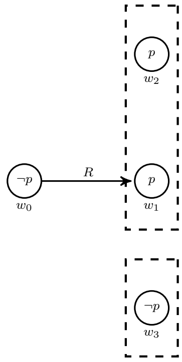

Fig. 6 Extended Coalition Models. The relation R is not dependent on the dynamic effectivity function. There can be outcomes that reachable from w_0 via R but that are not available choices at w_0.

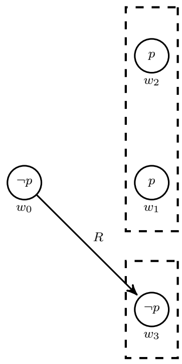

Fig. 7 Standard Coalition Models. The relation R is now dependent on the dynamic effectivity function: each reachable outcome is an available choice and each single outcome choice is also reachable.

Definition 11 (Standard coalition frames) A *standard coalition frame* is an extended coalition frame such that, for all $w, v \in W$, we have wRv if and only if $\{v\} \in E(w)(N)$.

A standard coalition model, illustrated in Figure 7 is a standard coalition frame with a valuation function. Depending on the properties of the underlying effectivity functions we call extended coalition frames and models playable or truly playable.

The following is straightforward:

Proposition 13 *Let M be a standard coalition model, and w a state in it. Then, $M, w \models \langle \mathcal{O} \rangle \phi$ iff there exists v such that $\{v\} \in E(w)(N)$ and $M, v \models \phi$. That is, the new semantics of $\langle \mathcal{O} \rangle$ on standard models coincides with the one introduced in Section 5.3.*

5.5 Characterizing Standard Truly Playable Coalition Frames

We begin by showing that the scheme from Section 5.3 ($[N]\phi \leftrightarrow \langle\mathcal{O}\rangle\phi$) can be used to characterize standard truly playable frames.

Proposition 14 *An extended coalition frame F is standard and truly playable if and only if $F \models [N]p \leftrightarrow \langle\mathcal{O}\rangle p$, where p is any propositional variable.*

Proof Left to right: Assume that F is standard and truly playable. Assume first that $(F, V), w \models [N]p$ for any V and $w \in W$. By definition of E we have that $p^M \in E(w)(N)$. As F is truly playable, E is a crown, therefore there is $v \in p^M$ with $\{v\} \in E(w)(N)$. However F is also standard so wRv. But this means that $(F, V), w \models \langle\mathcal{O}\rangle p$. Conversely, if $(F, V), w \models \langle\mathcal{O}\rangle p$ then wRv for some $v \in p^M$. F being standard we have that $\{v\} \in E(w)(N)$. By outcome monotonicity $p^M \in E(w)(N)$, i.e. $(F, V), w \models [N]p$.

Right to left: Assume that $F \models [N]p \leftrightarrow \langle\mathcal{O}\rangle p$. Let us first prove that F is standard. Suppose wRv for some $w, v \in W$. Let V be a valuation that assigns the proposition p only to v. We have that $M, w \models \langle\mathcal{O}\rangle p$. Then, by the assumptions we also have $M, w \models [N]p$, which means that $\{v\} \in E(w)(N)$. Conversely, suppose now that $\{v\} \in E(w)(N)$. For the same valuation V we must have that $M, w \models [N]p$ and by assumption that $\langle\mathcal{O}\rangle p$, which means that wRv. Thus, F is standard. To prove that F is truly playable, assume that for some $X \subseteq W$, $X \in E(w)(N)$ and let now V be a valuation function such that $p^{F,V} = X$. By definition of E we have that $(F, V), w \models [N]p$, hence by assumption, that $(F, V), w \models \langle\mathcal{O}\rangle p$, which means that wRv for some $v \in p^{(F,V)}$. Then, F being standard, $\{v\} \in E(w)(N)$.

5.6 Standard Truly Playable Models: Axioms

We propose the following axiomatic system TPCL for the class of standard truly playable coalition models TrulyPlay, extending Pauly's axiomatization of CL. The axioms include enough propositional tautologies plus the following schemes, where $C, D \subseteq N$ are any coalitions of agents.

1. $[N]\top$
2. $\neg[C]\bot$
3. $\neg[\emptyset]\phi \to [N]\neg\phi$
4. $[C]\phi \wedge [D]\psi \to [C \cup D](\phi \wedge \psi)$ for any disjoint $C, D \subseteq N$
5. $[\mathcal{O}](\phi \to \psi) \to ([\mathcal{O}]\phi \to [\mathcal{O}]\psi)$.
6. $[N]\phi \leftrightarrow \langle\mathcal{O}\rangle\phi$

The inference rules are: Modus Ponens, plus the Monotonicity rule MON:

$$\frac{\phi \to \psi}{[C]\phi \to [C]\psi}$$

for any coalition C, and the necessitation rule $\text{NEC}_{[\mathcal{O}]}$:

$$\frac{\phi}{[\mathcal{O}]\phi}.$$

We denote derivability of a formula ϕ in TPCL by $\vdash_{TPCL} \phi$.

It is worth pointing out that both Axiom 5 and rule $\text{NEC}_{[\mathcal{O}]}$ are derivable from the rest of TPCL. We have included them in the axiomatic system TPCL only to emphasize the fact that $[\mathcal{O}]$ is a normal modality.

To derive the rule $\text{NEC}_{[\mathcal{O}]}$, let $\vdash_{TPCL} \phi$. Then $\vdash_{TPCL} \top \to \phi$, hence $\vdash_{TPCL} \neg\phi \to \neg\top$. Therefore, $\vdash_{TPCL} [N]\neg\phi \to [N]\neg\top$ by MON, hence $\vdash_{TPCL} \neg[N]\neg\top \to \neg[N]\neg\phi$. Moreover, $\vdash_{TPCL} \neg[N]\bot$ by Axiom 2, and $\vdash_{TPCL} \neg\top \leftrightarrow \bot$. Thus, $\vdash_{TPCL} \neg[N]\neg\top$. Therefore, $\vdash_{TPCL} \neg[N]\neg\phi$ by Modus Ponens. By Axiom 6, it follows that $\vdash_{TPCL} [\mathcal{O}]\phi$.

To derive Axiom 5 from the rest of TPCL, first note that Axiom 6 makes Axiom 5 propositionally equivalent to $\neg[N]\neg(\phi \to \psi) \to (\neg[N]\neg\phi \to \neg[N]\neg\psi)$, which is propositionally equivalent to $(\neg[N]\neg(\phi \to \psi) \wedge [N]\neg\psi) \to [N]\neg\phi$. To derive the latter formula, first note that $\vdash_{TPCL} \neg[N]\neg(\phi \to \psi) \to [\emptyset](\phi \to \psi)$ by Axiom 2. Now, by Axiom 3 we obtain that $\vdash_{TPCL} ([\emptyset](\phi \to \psi) \wedge [N]\neg\psi) \to [N](\neg\psi \wedge (\phi \to \psi))$. Given that $(\neg\psi \wedge (\phi \to \psi)) \to \neg\phi$ is a propositional tautology, we obtain, by using MON, that $\vdash_{TPCL} [N](\neg\psi \wedge (\phi \to \psi)) \to [N]\neg\phi$. Thus, by propositional deduction we obtain in the long run that $\vdash_{TPCL} (\neg[N]\neg(\phi \to \psi) \wedge [N]\neg\psi) \to [N]\neg\phi$, whence the derivation of Axiom 5 follows.

Remark 3 Despite what the above may suggest, we emphasize that Axiom 6 does not define $\langle\mathcal{O}\rangle$ in terms of $[N]$ and does not render it redundant. On the contrary, the two modalities have intrinsically different semantics: the coalitional modality $[N]$ has neighbourhood semantics, while $[\mathcal{O}]$ has normal Kripke semantics (it is in fact the only modality with normal semantics in TPCL). It is only on truly playable frames that their semantics coincide, hence we can use Axiom 6 to rule out frames which are not truly playable. Indeed, as shown in Proposition 14, relating $[N]$ and $[\mathcal{O}]$ by Axiom 6 suffices to enforce the true playability of the underlying frames.

Furthermore, we claim that the outcome modality has an extremely natural semantics, selecting the possible outcomes of the collective actions of the grand coalition. We wonder why such modality has not been introduced during the early studies of coalition logics.

5.7 Soundness and completeness for TPCL

The proof of the following soundness claim is routine and we omit the details.

Proposition 15 *TPCL is sound for the class TrulyPlay: every formula derivable in TPCL is valid in TrulyPlay.*

Now we will establish the following completeness result:

Theorem 2 (Completeness theorem) *Every formula consistent in TPCL is satisfiable in TrulyPlay. Consequently, the logic TPCL is complete for the class TrulyPlay.*

We will prove the completeness using a canonical model construction followed by filtration for monotonic modal logics (for that, we partly reuse constructions from [6] and [16]). This way, we will also obtain finite model property for TPCL. Here we only sketch the standard canonical model construction and refer the reader for further details to [6] and [16].

To construct the canonical model $M^\bullet$, we start with a formula δ which is consistent in TPCL. By a well-known argument, it is contained in some maximal

TPCL-consistent set. We take $W^\bullet$ to be the set of maximally consistent sets, and define for every formula ϕ the *canonical extension* of ϕ, also called *proof set* [6], as $\phi^* = \{s \in W^\bullet \mid \phi \in s\}$.

Definition 12 (Canonical Model) The canonical model for TPCL is $M^\bullet = (W^\bullet, E^\bullet, R^\bullet, V^\bullet)$ where:

$$\begin{array}{lll} w \in V^\bullet(p) & \text{iff} & p \in w \\ X \in E^\bullet(w)(C) & \text{iff} & \exists \psi^* \subseteq X : [C]\psi \in w \qquad\qquad\qquad \text{for } C \neq N \\ X \in E^\bullet(w)(C) & \text{iff} & \forall \psi^* \text{ such that } X \subseteq \psi^* : [N]\psi \in w, \text{ for } C = N \\ wR^\bullet v & \text{iff} & \forall \psi, \text{ if } \psi \in v \text{ then } \langle \mathcal{O} \rangle \psi \in w \end{array}$$

Some remarks:

- That $E^\bullet$ is playable and well-defined is proved in [16].
- The canonical relation for N is defined in [16] in the following equivalent way: $X \in E^\bullet(w)(N)$ if and only if $[\emptyset]\psi \notin w$ for all ψ^* such that $\psi^* \subseteq \overline{X}$. The equivalence follows easily from the fact that $\vdash_{TPCL} [N]\phi \leftrightarrow \neg[\emptyset]\neg\phi$.
- The canonical relation for $\langle \mathcal{O} \rangle$ is defined as a canonical relation for normal modal logics [6].

Proposition 16 (Truth Lemma) *For any $w \in W^\bullet$ we have that $M^\bullet, w \models \phi$ if and only if $\phi \in w$.*

Proof By induction on the length of ϕ: standard for atomic propositions, boolean formulas, and formulas of the form $\langle \mathcal{O} \rangle \psi$; proved in [16] for formulas of the form $[C]\psi$.

The canonical model is an extended coalition model; however, it is neither standard nor truly playable. The reason is that, assuming that for all $\psi \in \mathcal{L}_{CLO}, \psi \in v$ implies that $[N]\psi \in w$ is not sufficient to conclude that $\{v\} \in E^\bullet(w)(N)$ as states are not characterized by unique formulas of the language of CLO. In order to obtain a standard and truly playable model satisfying the given $\mathcal{L}_{CLO}$-consistent formula δ we are going to filter the canonical model with the set $\Sigma(\delta)$, obtained by taking all subformulae of δ and closing under boolean operators. That set is finite *up to propositional equivalence.*

5.8 Filtrations

First, we define a general notion of filtration for extended coalition models and then a special filtration construction that preserves playability. Filtrations of coalition models are introduced, e.g., in [12] for the purpose of axiomatizing Nash-consistent Coalition Logic. Here we only add the filtration for the relation corresponding to the modality $\langle \mathcal{O} \rangle$.

Let $M = (W, E, R, V)$ be an extended coalition model and Σ a subformula-closed set of formulas from $\mathcal{L}_{CLO}$. The equivalence classes induced by Σ on M are defined as follows:

$$v \equiv_\Sigma w \Leftrightarrow \text{ for all } \phi \in \Sigma : M, v \models \phi \text{ if and only if } M, w \models \phi.$$

We denote the equivalence class to which v belongs by $|v|$ and the set $\{|v| \mid v \in X\}$ by $|X|$ for any $v \in W$ and $X \subseteq W$.

Definition 13 (Filtration) Let $M = (W, E, R, V)$ be an extended coalition model and Σ be a subformula closed set of formulas from $\mathcal{L}_{CLO}$. An extended coalition model $M^f_\Sigma = (W^f_\Sigma, E^f_\Sigma, R^f_\Sigma, V^f_\Sigma)$ is a *filtration of* M *through* Σ whenever the following conditions are satisfied:

1. $W^f_\Sigma = |W|$.
2. For all $C \subseteq N$ and $\phi \in \Sigma$, $\phi^M \in E(w)(C)$ implies $\{|v| \mid M, v \models \phi\} \in E^f_\Sigma(|w|)(C)$.
3. For all $C \subseteq N$ and $Y \subseteq |W|$: $Y \in E^f_\Sigma(|w|)(C)$ implies that for all $\phi \in \Sigma$ if $\phi^M \subseteq \{v \mid |v| \in Y\}$ then $\phi^M \in E(w)(C)$.
4. If wRv then $|w|R^f_\Sigma|v|$.
5. If $|w|R^f_\Sigma|v|$ then for all $\langle\mathcal{O}\rangle\phi \in \Sigma$, if $M, v \models \phi$ then $M, w \models \langle\mathcal{O}\rangle\phi$.
6. $V^f_\Sigma(p) = |V(p)|$ for all atoms $p \in \Sigma$.

Intuitively, the first item says that the set of worlds W^f_Σ are given by the equivalence classes of worlds in W that agree in the evaluation of the formulas in Σ. The second item shows how from the effectivity function in the original model is carried over in the filtered model, modulo Σ. The third item instead goes the other way round: if a set Y can be forced by a coalition in the filtered model, then all the sets of the form ϕ^M, such that $\phi \in \Sigma$ and is a subset of the set of members of equivalence classes in Y, can be forced in the original model. The next items describe the two way relation between the original model and the filtered model for the R relation in an analogous way to what done for the coalitional relation. The fourth one says that if two worlds are connected by the R relation in the original model then their counterparts are also connected by the correspondent relation in the filtered model. The fifth one goes the other way round and says that if two worlds are connected by the R^f_Σ relation in the filtered model then their corresponding counterparts in the original model respect the modal valuation of formulas in Σ. The last item constructs the valuation function in the filtered model, which should agree with the one in the original model as for the atoms in Σ. The conditions above are needed to ensure the following Filtration Lemma, as showed in [12] for the neighbourhood relations and e.g. in [6] for the binary relation.

Proposition 17 (Filtration Lemma) *If* $M^f_\Sigma = (W^f_\Sigma, E^f_\Sigma, R^f_\Sigma, V^f_\Sigma)$ *is a filtration of* M *through* Σ *then for all* $\phi \in \Sigma$ *we have that* $M, w \models \phi$ *if and only if* $M^f_\Sigma, |w| \models \phi$.

Definition 14 (Playable Filtration) Let $M = (W, E, R, V)$ be an extended coalition model and $\Sigma(\delta)$ the boolean closure (modulo propositional equivalence) of the set of subformulas of δ, such that $\delta \in \mathcal{L}_{CLO}$. A coalition model $M^F_{\Sigma(\delta)} = (W^F_{\Sigma(\delta)}, E^F_{\Sigma(\delta)}, R^F_{\Sigma(\delta)}, V^F_{\Sigma(\delta)})$ is a *playable filtration* of M through $\Sigma(\delta)$ whenever the following conditions are satisfied:

1. $W^F_{\Sigma(\delta)} = |W|$.
2. For all $C \subsetneq N$ and $Y \subseteq |W|$: $Y \in E^F_{\Sigma(\delta)}(|w|)(C)$ if and only if there exists $\phi \in \Sigma(\delta)$ such that $\phi^M \subseteq \{v \mid |v| \in Y\}$ and $\phi^M \in E(w)(C)$.
3. For all $Y \subseteq |W|$: $Y \in E^F_{\Sigma(\delta)}(|w|)(N)$ if and only if $\overline{Y} \notin E^F_{\Sigma(\delta)}(|w|)(\emptyset)$.
4. $|w|R^F_{\Sigma(\delta)}|v|$ if and only if there exists $w' \in |w|, \exists v' \in |v|$ such that $w'Rv'$.
5. $V^F_{\Sigma(\delta)}(p) = |V(p)|$ for all atoms $p \in \Sigma(\delta)$.

Intuitively the definition constructs a filtration that preserves playability, in the same fashion of Definition 13. The first item constructs the set of states, this time considering the boolean closure of subformulas of δ. The second item deals with the effectivity function of coalitions that do not involve all the players, similarly to what done in item 2 and 3 of Definition 13. The third item specifically deals with the effectivity function of the grand coalition relating it to that of the empty coalition, in a way that preserves N-maximality. The fourth item constructs the filtered relation $R^F_{\Sigma(\delta)}$ making use of the minimal possible filtration and the fifth item deals with the valuation function in the same way of Definition 13.

That $M^F_{\Sigma(\delta)}$ is a filtration in the sense of Definition 13 is proved in [12] for the coalitional modalities. We have added to that a minimal filtration for the modality $\langle \mathcal{O} \rangle$. So $M^F_{\Sigma(\delta)}$ is a filtration in the sense of Definition 13. In [12] it is also shown that playability is preserved by that filtration and that every subset of $W^F_{\Sigma(\delta)}$ is definable by a formula of $\Sigma(\delta)$ as follows. First, for every state $|w| \in |W|$ we define

$$\chi^F_{\Sigma(\delta)}(|w|) := \bigwedge\{\phi \in \Sigma(\delta) \mid M^F_{\Sigma(\delta)}, |w| \models \phi\}.$$

Then, for every $Y \subseteq |W|$ we put

$$\chi^F_{\Sigma(\delta)}(Y) := \bigvee\{\chi^F_{\Sigma(\delta)}(|w|) \mid |w| \in Y\}.$$

It is straightforward to show, using the filtration lemma, that for every $Y \subseteq |W|$:

$$M^F_{\Sigma(\delta)}, |w| \models \chi^F_{\Sigma(\delta)}(Y) \text{ if and only if } |w| \in Y,$$

that is, $\chi^F_{\Sigma(\delta)}(Y)$ indeed characterizes the set y in $M^F_{\Sigma(\delta)}$.

Proposition 18 $M^{\bullet,F}_{\Sigma(\delta)}$ *is standard and truly playable.*

Proof To prove that $M^{\bullet,F}_{\Sigma(\delta)}$ is standard we have to show that for each $w, v \in W$, $|v| R^{\bullet,F}_{\Sigma(\delta)} |w|$ if and only if $\{|v|\} \in E^{\bullet,F}_{\Sigma(\delta)}(|w|)(N)$. From right to left it is straightforward. For the other direction, suppose $|v| R^{\bullet,F}_{\Sigma(\delta)} |w|$. Then $M^{\bullet,F}_{\Sigma(\delta)}, |v| \models \langle \mathcal{O} \rangle \chi^F_{\Sigma(\delta)}(|w|)$ by definition of $R^{\bullet,F}_{\Sigma(\delta)}$ and by the properties of filtrations. By the fact that $R^{\bullet,F}_{\Sigma(\delta)}$ is a minimal filtration we have that $\exists w' \in |w|, \exists v' \in |v|$ such that $v' R^\bullet w'$. By definition of $R^\bullet$ and the Truth Lemma we have that $M^\bullet, v' \models \langle \mathcal{O} \rangle \chi^F_{\Sigma(\delta)}(|w|)$. By the axioms of TPCL and the Truth Lemma we have $M^\bullet, v' \models [N]\chi^F_{\Sigma(\delta)}(|w|)$, hence $M^\bullet, v' \models \neg[\emptyset]\neg\chi^F_{\Sigma(\delta)}(|w|)$. Then $(\neg\chi^F_{\Sigma(\delta)}(|w|))^{M^\bullet} \notin E^\bullet(v')(\emptyset)$ by the definition of $E^\bullet$. But, by Definition 13 $\{(\neg\chi^F_{\Sigma(\delta)}(|w|))^{M^{\bullet,F}_{\Sigma(\delta)}}\} \notin E^{\bullet,F}_{\Sigma(\delta)}(|v|)(\emptyset)$ and in turn $\{(\chi^F_{\Sigma(\delta)}(|w|))^{M^{\bullet,F}_{\Sigma(\delta)}}\} \in E^{\bullet,F}_{\Sigma(\delta)}(|v|)(N)$. Recall now that $(\chi^F_{\Sigma(\delta)}(|w|))^{M^{\bullet,F}_{\Sigma(\delta)}} = |w|$.

Now, to prove that $M^{\bullet,F}_{\Sigma(\delta)}$ is truly playable, assume $Y \in E^{\bullet,F}_{\Sigma(\delta)}(|w|)(N)$. Then, $(\neg\chi^F_{\Sigma(\delta)}(Y))^{M^{\bullet,F}_{\Sigma(\delta)}} \notin E^\bullet(w)(\emptyset)$ by the definition of filtration, which means that for all $\phi \in \Sigma(\delta)$, if $\{v \mid |v| \in (\neg\chi^F_{\Sigma(\delta)}(Y))^{M^{\bullet,F}_{\Sigma(\delta)}}\} \subseteq \phi^M$ then $\phi^M \notin E^\bullet(w)(\emptyset)$. In particular $(\neg\chi^F_{\Sigma(\delta)}(Y))^{M^\bullet} \notin E^\bullet(w)(\emptyset)$. By the definition of $E^\bullet$ we have that $[\emptyset]\neg\chi^F_{\Sigma(\delta)}(Y) \notin w$ and by true playability that $\langle \mathcal{O} \rangle \chi^F_{\Sigma(\delta)}(Y) \in w$. By the definition of canonical relation for $\langle \mathcal{O} \rangle$ we have that there exists v with $w R^\bullet v$ such that $\chi^F_{\Sigma(\delta)}(Y) \in v$. By definition of filtration $|w| R^{\bullet,F}_{\Sigma(\delta)} |v|$ and by the Filtration Lemma $M^{\bullet,F}_{\Sigma(\delta)}, |v| \models \chi^F_{\Sigma(\delta)}(Y)$. Finally, $\{|v|\} \in E^{\bullet,F}_{\Sigma(\delta)}(|w|)(N)$ since $M^{\bullet,F}_{\Sigma(\delta)}$ is standard.

This completes the proof of the Completeness theorem 2.

Corollary 2 (Finite Model Property) *The logic TPCL has the finite model property with respect to the class of models TrulyPlay.*

6 Conclusions

In this paper, we have revisited the correspondence between two classes of abstract game forms: strategic games from non-cooperative game theory on the one hand, and effectivity functions from cooperative game theory on the other. We consider our contribution as threefold. First, we have corrected a well-known and often used result from [16] relating strategic games and playable effectivity functions. We showed that strategic games do not correspond to all playable functions, but to a strict subset of the class, which we call *truly playable effectivity functions*. Second, we have provided several abstract characterizations of truly playable functions. We also showed that the remaining playable effectivity functions (that we call non-truly playable) are induced by non-principal filters, and hence only scenarios with infinitely many possible outcomes can fall in that class.

Third, we have pointed out that Coalition Logic is not expressive enough to characterize true playability. On the other hand, CL can be extended in a simple way to obtain such a characterization. To this aim we have proposed an extension of Coalition Logic with a normal *outcome selector* modality, and called the resulting logic TPCL (Truly Playable Coalition Logic). We established a correspondence between formulas of TPCL and truly playable models (that is, coalitional models truly corresponding to strategic game models). Thus, while bare CL is not sufficient to distinguish between truly playable and playable models, the "outcome selector" modality enhances it with just enough extra expressive power to make that distinction possible.

The importance of our work is mainly theoretical. Essentially, it implies that all the claims that have been proved using Pauly's correspondence between playable effectivity functions and games should be revisited and possibly re-interpreted in the light of the results presented here. Example of such issues, already addressed here, include: axiomatization for Coalition Logic in the class of multi-player game models, axiomatization of ATL in coalitional effectivity models, and the respective finite model properties. In practical terms, this also means that, whenever a decision procedure is built on those theoretical results, the designer should be aware of the correct correspondence between the two classes of game models, which is especially relevant for satisfiability-checking algorithms. Tableaux for extensions of Coalition Logic, like the one for a combination of CL and description logic $\mathcal{ALC}$ from [13], are examples of such procedures.

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