Chapter 1

LOGICAL THEORIES FOR FRAGMENTS OF ELEMENTARY GEOMETRY

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Abstract We survey models and theories of geometric structures of parallelism, orthogonality, incidence, betweenness and order, thus gradually building towards full *elementary geometry* of Euclidean spaces, in Tarski's sense. Besides the geometric aspects of such structures we look at their logical (first-order and modal) theories and discuss logical issues such as: expressiveness and definability, axiomatizations and representation results, completeness and decidability, and interpretations between structures and theories.

1. Introduction and historical overview

In ancient Babylon and Egypt geometry was just a set of empirical observations and practical skills and methods for measuring land and designing irrigation systems, although it already had a degree of sophistication (e.g., Pythagoras' theorem was already known and the triangle with sides 3-4-5 was used in practice for producing right angles). In ancient Greek times it evolved into a *liberal art*.

We dare claim that Geometry only became a science with Euclid's epic work "Elements" written over 2300 years ago, which was not only the first truly scientific treatment of geometry, but also the first systematic application of the *axiomatic method* in mathematics. Geometry remained a central subject in mathematics throughout the centuries, but when in the first half of the 17th century Descartes introduced coordinate systems, and with them the *analytic method* in geometry, it gradually began to lose its prime position in mathematics and became part of algebra and calculus, and later – of topology. The modern view, going back to the famous Klein's *Erlangen program*, defined geometry as a study *not of figures, but of transformations,* and classified the different geometric structures and their theories in terms of the groups of transformations which preserve them. This view placed it firmly on algebraic foundations to an extent that some mathematicians consider it as an 'applied group theory'.

On the other hand, the discovery of non-Euclidean geometries by Bolyai, Lobachevsky and Gauss in the early 19th century (see e.g. Coxeter, 1969; Eves, 1972; Meserve, 1983), which showed *inter alia* the independence of Euclid's 'Fifth Postulate' from the other axioms of Euclidean geometry, was an impressive demonstration of the strength and importance of the formal logical approach in mathematics, which also reinforced the importance of geometry to the foundations of mathematics. Euclid's Fifth Postulate claims that, given a line and a point not incident with it in a plane, there exists a unique line in that plane passing through the given point and parallel to the given line. Depending on the acceptance or otherwise of that postulate, several natural lines of development of geometry evolve:

- *affine geometry*, first studied by Euler, which adopts the Fifth Postulate. Thus incidence, parallelism, collinearity, and betweenness, as well as transformations that preserve these relations, play a central role in affine geometry. Such *affine transformations* can be taken, in the spirit of the Erlangen program, as defining the very notion of 'affine'. Affine geometry does not deal with angles, distances, or any other related metric concepts (not even with orthogonality), as these are not invariant under affine transformations. Thus the models of affine geometry, viz. affine spaces, are more general than Euclidean spaces, but have poorer structure.

- *hyperbolic geometry*, introduced by Lobachevsky and Bolyai, and *elliptic* geometry, which adopt the negation of the Fifth Postulate. In hyperbolic geometries, given a line and a point not on that line, there exist infinitely many lines through that point which lie parallel with the original given line (see e.g. Coxeter, 1969; Szczerba and Tarski, 1965; Szczerba and Tarski, 1979 for more details); in elliptic geometry (see Coxeter, 1969; Behnke et al., 1974) there are no parallel lines at all. These will not be discussed in this chapter, but see Hilbert, 1950 for comparison and relationships.

- *absolute geometry*, introduced by J. Bolyai, based only on the first four postulates of Euclid, but independent of the Fifth Postulate. In some extension of absolute geometry the notion of parallelism can be completely rejected, as in *projective geometry* , where every two lines in a projective plane intersect. In others, alternatives of the Fifth Postulate can be adopted, as in the elliptic and hyperbolic geometries. Thus, absolute geometry is a full-fledged system of geometry, involving distances, angles, etc., but based on a weaker axiomatic basis than Euclidean geometry. It is not a subsystem of affine geometry, though the intersection of the two is still rich enough to develop a meaningful and interesting theory of ordered affine structures (see Coppel, 1998; Coxeter, 1969, Ch. 15; Lenz, 1992; Szczerba, 1972).

A fundamental affine relation, i.e., one invariant under affine transformations, is the relation of *betweenness* on triples of points, which extends basic affine structures by introducing *ordering* between points on a line, but not distances. Much of the theory of betweenness is independent from Euclid's Fifth Postulate, and thus lies in the intersection of absolute and affine geometry (see Coppel, 1998; Coxeter, 1969).

Perhaps the simplest important non-affine relation is that of *orthogonality*. Further adding distances and angles, along with the axioms of the field of reals, extends affine geometry to the classical, Euclidean geometry of the real plane and space.

In the beginning of the 20th century Hilbert, the most influential proponent of the axiomatic method in mathematics, illustrated the power of that method by re-casting Euclid's work into a precise and rigorous modern treatment which eventually put geometry on sound axiomatic foundations (see Hilbert, 1950). It was preceded by axiomatic investigations of the foundations of geometry at the end of the 19th century by

Peano, as well as Pieri, 1908; Veblen, 1904; Veblen, 1914 and Pasch, 1882, who analyzed various axiomatic systems and the mutual relationships between the primitive notions of Euclidean geometry. However, the axiomatic method in geometry only reached its logical maturity with the seminal work of Tarski and his students and followers Szczerba, Szmielew, Schwabhäuser, Scott, Monk, Givant and others (see Schwabhäuser et al., 1983 for comprehensive details) in the 1920-70's. Tarski developed systematically the logical foundations of *elementary geometry*, which is "that part of Euclidean geometry that can be formulated and established without the help of any set-theoretical devices" (see Tarski, 1959). Essentially, that means the first-order theory of Euclidean geometry, developed over a suitably expressive first-order language (see further). In particular:

- Tarski, 1951; Tarski, 1967 demonstrated how the elementary geometry of the real plane can be formally interpreted into the elementary (i.e. first-order) theory of real-closed fields. Furthermore, Tarski showed the completeness and decidability of the theory of real-closed fields by means of quantifier elimination, and consequently obtained a decision procedure for the elementary Euclidean geometry. He then extended these results to Euclidean planes over arbitrary real-closed fields.

- Tarski, 1959 showed that the whole elementary geometry can be developed axiomatically using just two geometric relations, viz. *betweenness* and *equidistance* (used as the only primitives also by Veblen, 1904). He thus obtained an explicit axiomatization of the first-order theory of the Euclidean geometry in terms of these primitives and showed that it is complete and decidable, though not finitely axiomatizable.

- In a similar fashion, Szmielew, 1959 studied the first-order theory of the metric hyperbolic geometry, obtained by negating Euclid's axiom in Tarski's first-order axiomatization of the Euclidean geometry.

- Szczerba and Tarski, 1965; Szczerba and Tarski, 1979 studied and characterized the first-order theories of the fragments of the Euclidean, hyperbolic and absolute geometries based on betweenness alone, for which they established explicit axiomatizations.

- Beth and Tarski, 1956; Tarski, 1956 studied the problem of which geometric relations are sufficient to be adopted as primitive notions in terms of which the whole Euclidean geometry can be developed.

- Szmielew, 1983 developed the theory of point-based collinearity structures and showed how to build up the Euclidean geometry from that theory, while Schwabhäuser and Szczerba, 1975 studied line-based structures for elementary geometry.

While the post-Tarski period in the logical foundations of geometry is less active and spectacular, still there are several research lines which deserve discussion. Besides the notable works of Tarski's students mentioned above, they include:

- study of classical and constructive axiomatizations of fragments of projective, affine, absolute, Euclidean, elliptic, hyperbolic, etc. geometries, with emphasis on simplicity and minimality, in Pambuccian, 1989; von Plato, 1995; Lombard and Vesley, 1998; Pambuccian, 2001a; Pambuccian, 2001b; Pambuccian, 2006, etc. For a general discussion of the axiomatics of affine and projective geometry, see Bennett, 1995.
- investigation of primitive relations sufficient for the elementary affine, projective, absolute, etc. geometries; the expressiveness of such relations; and axiomatizations in terms of such relations, in Scott, 1956; Pambuccian, 1995; Pambuccian, 2003; Pambuccian, 2004, etc.
- development of practical methods and algorithms for theorem proving in algebra and geometry: quantifier elimination based methods, such as Seidenberg's implementation of Tarski's method, Seidenberg, 1954, the method of cylindrical algebraic decompositions (see e.g. Caviness and Johnson, 1998; Buchberger et al., 1988), and the more recent and efficient Heintz et al., 1990; Renegar, 1992; Basu et al., 1996; Basu, 1999; Gröbner basis method (Buchberger, 1985), the characteristic set method (Chou and Gao, 1990), and others. For more details and references see Sec. 9.2.

Most of the studies and results mentioned above apply to geometric structures of which the logical languages are rich enough to express properties of ordering and metric. However, there are various weaker, yet natural and important, geometric structures such as *parallelism*, *orthogonality*, *incidence* and *collinearity structures*, which involve points and lines in a real or abstract geometric space. The elementary theories of these latter structures are considerably less studied, mainly from the perspective of discrete and combinatorial geometry. We will discuss these structures and their theories in some detail here, as they play an important role in various models of qualitative spatial reasoning.

While affine and absolute spaces are too general to allow the development of a full-fledged elementary geometry in them, they are still amenable to algebraic treatment by means of *coordinatization*, which enables the study of affine and projective spaces by studying algebraic structures called *ternary rings* (see e.g. Blumenthal, 1961; Heyting, 1963; Szmielew, 1983; Hughes and Piper, 1973; Mihalek, 1972). Since the coordinatization is a first-order interpretation, it is instrumental for the algebraic investigation and characterization of the logical theories of affine spaces, and can be used to establish various logical properties, such as independence results, representation theorems, (lack of) finite model property, decidability and complexity results of these theories.

In this chapter we survey and discuss from a *logical perspective* structures and theories of parallelism, orthogonality, incidence and order, gradually building the full *elementary geometry* of Euclidean spaces, in Tarski's sense. Besides traditional geometric properties and constructions, we discuss various logical issues such as: *definability of relations and properties*, *expressiveness of concepts*, *axiomatic theories and their models*, *representation results and completeness*, *finite model property*, *decidability*, *categoricity* and other model-theoretic properties.

The chapter consists of two parts. In the first part we discuss classical, first-order theories of geometric structures, starting with very weak structures of parallelism (Sec. 3), orthogonality (Sec. 4), incidence (Sec. 5) and collinearity, for which we show how to develop some geometric concepts, such as independence, basis, planarity and dimension. In Sec. 6 and Sec. 7 we outline coordinatization of projective and affine planes as a general method of interpreting them into algebraic structures called planar ternary rings, and discuss the relationship between geometric and algebraic properties, and generally between the logical theories of planes and the associated coordinate rings. We also discuss collineations and general affine transformations, and the associated (invariant) affine concepts and properties. In Sec. 8 we then add betweenness and order in affine planes, discuss definability in these planes, and the relationship of these planes with ordered coordinate rings, as well as the results from Szczerba and Tarski, 1965; Szczerba and Tarski, 1979 on axiomatic theories of betweenness. Eventually we consider some rich languages, i.e. languages containing primitive notions in terms of which the whole elementary geometry can be developed, and present Tarski's axiomatization of the Euclidean geometry in terms of betweenness and equidistance. The first part of the chapter, dealing with elementary theories of geometry, ends with a brief discussion in Sec. 9.2 of the development of decision methods for elementary geometry since Tarski's seminal decidability results, and automated reasoning for elementary geometry.

The second part of the chapter is devoted to *modal logics* arising from classical, mainly two-dimensional geometrical structures. After a short general discussion of spatial modal logics in Sec. 10, we consider modal logics of several sorts: *point-based* (Sec. 11), *line-based* (Sec. 12), and *point-line based logics* (Sec. 14) with incidence relations between the sorts, defining affine or projective incidence structures. In Sec. 13 we show how two-sorted relational structures based on points and lines can be replaced by one-sorted relational structures containing the same geometrical information, and how modal logic can be developed on such structures. In Sec. 14 we discuss point-line spatial logics and show how modal languages can be interpreted on two-sorted relational structures.

2. Preliminaries

2.1 Some terminology and notation

The following notions will be introduced more than once in this chapter. Here we only fix the notation and terminology used further (unless otherwise specified) for the convenience of the reader.

We deal with two basic geometric objects, **points** and **lines**.

Points. Points are usually considered primitive concepts, but as we shall see, they can also be defined in terms of co-punctual lines. Specific points will be denoted as A, B, C etc. and typical point variables will be X, Y, Z, etc. Basic relations on points are **collinearity**, denoted as $\mathbf{Col}(XYZ)$, meaning that the points X, Y and Z lie on a common line (sometimes generalized to n points); **betweenness**, denoted as $\mathbf{B}(XYZ)$, meaning that Y lies on the line segment joining X and Z (with possibly Y coinciding with X or Z); and **equidistance**, denoted $XY \equiv ZU$ and meaning that the line segment formed by X and Y has the same length as the line segment formed by Z and U. Using any of these, one can define the **triangle** relation, which holds when three points are non-collinear.

Lines. Lines can be introduced as primitive concepts, or defined in terms of pairs of different points, or as equivalence classes of points in collinearity structures. Specific lines will be denoted as a, b, c, etc. Typical line variables will be x, y, z etc. Basic relations on lines are:

Incidence denoted as x **Inc** y, meaning that the lines x and y share a common point, and may even coincide. Incidence may be generalized to **co-punctuality** (**concurrence**), denoted $\mathbf{Cop}(x_1 \dots x_n)$, meaning that the lines $x_1, \dots, x_n$ have exactly one point in common.

Intersection of two lines, denoted x **Int** y and meaning that x and y are incident *but different*.

Co-planarity of lines, denoted $\mathbf{Pl}(xy)$ for two lines and generalized to $\mathbf{Pl}(x_1 \dots x_n)$ for n lines, meaning that the lines lie in the same plane.

Strict (irreflexive) parallelism, denoted $x \parallel y$, meaning that the lines x and y are parallel and different, and **weak (reflexive) parallelism**, denoted $x \shortparallel y$, meaning that x and y are parallel or coincide.

Orthogonality, denoted $x \perp y$, meaning that the lines x and y are orthogonal, but not necessarily intersecting, and **perpendicularity**, denoted $x \dot{\perp} y$, meaning that the lines x and y are orthogonal and coplanar (and hence intersecting).

Skewness, denoted $x \bowtie y$, meaning that x and y are not co-planar.

Lines can also be defined as sets of points collinear with a pair of points: given two distinct points P and Q, the **line determined by** P **and** Q, denoted $\mathbf{l}(P, Q)$, is defined as the set of all points X such that $\mathbf{Col}(PQX)$ holds.

Given a point X and a line y, the claim that X is **incident** with y will be denoted as $X\mathbf{I}y$ or simply as $X \in y$, while, assuming Euclid's parallel postulate, the unique line parallel with y and containing X will be denoted $\mathbf{p}(X, y)$. The unique line incident with two distinct points X and Y will be denoted as $\mathbf{l}(X, Y)$ or simply as XY, while the line segment between X and Y will be denoted $|XY|$ and the length of that segment as $\|XY\|$. Given intersecting lines x and y, their point of intersection will be denoted $\mathbf{P}(x, y)$.

For every integer $n \geq 1$, $\mathrm{Diff}_n(X_1 \dots X_n)$ will be the formula stating that $X_1, \dots, X_n$ are distinct, i.e.

$$\mathrm{Diff}_n(X_1 \dots X_n) \ := \bigwedge_{\substack{i \neq j \\ 1 \leq i,j, \leq n}} X_i \neq X_j,$$

and likewise for $\mathrm{Diff}_n(x_1 \dots x_n)$.

2.2 Algebraic background

The terminology on algebraic structures varies considerably in the literature, so we fix ours here. The reader is referred to any standard text in abstract algebra, or to Szmielew, 1983 for more details.

Consider the structure $(G; 0, +)$ where G is a non-empty set, $+$ is a binary operation on G and 0 is some distinguished element in G. Then $(G; 0, +)$ is called an **(additive) loop (with zero** 0**)** if

1. $a + 0 = a = 0 + a$ for all $a \in G$;
2. $a + b = c$ uniquely determines any of a, b, c from the other two.

Likewise we refer to $(G; 1, \cdot)$ as a **multiplicative loop with unit** 1. If the operation $+$ is associative as well, then $(G; 0, +)$ is called a **group**. Note that the second property above guarantees the existence of additive inverses in any group. A group will be called **abelian** when the operation defined in it is commutative.

A structure $(F; 0, +, \cdot)$ is called a **ring** if $(F; 0, +)$ is an abelian group and the multiplication $\cdot$ is both associative and distributive over $+$.

A structure $(G; 0, 1, \cdot)$ (where 0 and 1 are distinct distinguished elements from G) is called a **multiplicative loop with zero** if

1. $(G \backslash \{0\}, 1, \cdot)$ is a multiplicative loop with unit 1;
2. $a \cdot 0 = 0 = 0 \cdot a$ for all $a \in G$.

Here 0 is the zero of $(G; 0, 1, \cdot)$ and 1 is its unit. Again $(G; 0, 1, \cdot)$ will be called a **multiplicative group with zero** when $\cdot$ is associative.

$(F; 0, 1, +, \cdot)$ is called a **double loop** when

1. $(F; 0, +)$ is an additive loop;
2. $(F; 0, 1, \cdot)$ is a multiplicative loop with zero.

A double loop $(F; 0, 1, +, \cdot)$ is called a **left division ring** (respectively, **right division ring**) when $(F; 0, +)$ is an abelian group and $\cdot$ is left-distributive (respectively, right-distributive) over $+$. A **division ring** is a double loop that is both a left and right division ring. A division ring $(F; 0, 1, +, \cdot)$ with associative multiplication $\cdot$ is called a **skew field**, and if $\cdot$ is also commutative then $(F; 0, 1, +, \cdot)$ becomes a **field**. Note that there is some variation in the literature regarding these terms; for example, division rings are called in Szmielew, 1983 quasi-fields. By a classical result of Wedderburn, every finite skew field is a field.

A structure $(G; 0, +, \leq)$ will be called an **ordered loop** if $(G; 0, +)$ is a loop and $\leq$ is a linear ordering on the set G such that

1. $a \leq b \Rightarrow c + a \leq c + b$ (left additive monotony)
2. $a \leq b \Rightarrow a + c \leq b + c$ (right additive monotony)

for all $a, b, c \in G$. If $(G; 0, +)$ is a group then $(G; 0, +, \leq)$ will be called an **ordered group**, etc.

A structure $(F; 0, 1, +, \cdot, \leq)$ will be called an **ordered double loop** if $(F; 0, 1, +, \cdot)$ is a double loop and $\leq$ is a linear ordering on the set F such that both left and right additive monotony holds, and

1. $a \leq b \Rightarrow c \cdot a \leq c \cdot b$ (left multiplicative monotony)

2. $a \leq b \Rightarrow a \cdot c \leq b \cdot c$ (right multiplicative monotony)

for all $a, b, c \in G$ with $c \geq 0$. Likewise if $(F; 0, 1, +, \cdot)$ is instead, say, a left division ring, then $(F; 0, 1, +, \cdot, \leq)$ will be called an **ordered left division ring**, etc. It is easy to see that if $(F; 0, 1, +, \cdot)$ is an ordered double loop then $0 < 1$ and hence F has infinite cardinality since for every $x \in F$, $x < x + 1$.

An ordered structure $(F; 0, 1, +, \cdot, \leq)$ is **Euclidean**, if for every $a \in F$ with $a \geq 0$ there exists $b \in F$ such that $a = b^2$; **real closed**, if it is Euclidean and every polynomial of odd degree over F has a zero in F.

2.3 Logical background

In the treatment of some logical issues, we assume that the reader has background on the basic model theory of first-order logic, suitable references on which include Doets, 1996; Enderton, 1972 and the very comprehensive and more advanced Hodges, 1993. Here we only mention a few more specific concepts and results used in the chapter.

Theories. A (first-order) theory is any set of first-order sentences. A theory T is **complete** if every two models of the theory are elementarily equivalent, i.e., satisfy the same first-order sentences. A typical example of a complete theory is the set $\mathrm{TH}(\mathfrak{A})$ of all first-order sentences satisfied in a given structure $\mathfrak{A}$. A theory T is **ω-categorical** (or, countably categorical) if all countable models of T are isomorphic; T is **decidable** if there is an algorithm which can determine if a given sentence is a logical consequence of T. By the Łoś-Vaught Test (see e.g. Doets, 1996) every ω-categorical theory is complete and decidable.

Padoa's method. Let $\mathcal{L}$ be a first-order language over some signature S, let $\mathbf{s}$ be a symbol not in S and T a theory over the signature $S \cup \{\mathbf{s}\}$. If $\mathfrak{A}$ and $\mathfrak{B}$ are models of T with $\mathfrak{A}|S = \mathfrak{B}|S$ but $\mathbf{s}^{\mathfrak{A}} \neq \mathbf{s}^{\mathfrak{B}}$ then $\mathbf{s}$ cannot be defined by T in $\mathcal{L}$. More generally, if $\mathfrak{A}$ is a model of the theory T and if there exists an automorphism of $\mathfrak{A}|S$ that fails to preserve the symbol $\mathbf{s}$, then $\mathbf{s}$ cannot be defined by T in $\mathcal{L}$. For example, consider the structure $(\mathbb{Z}; +)$. The constant $\mathbf{0}$ is explicitly definable using the formula $\varphi_{\mathbf{0}}(x) := \forall y(x + y = y)$. From this we can then explicitly define subtraction using the formula $\chi_{-}(x, y, z) := \exists u \exists v(\varphi_{\mathbf{0}}(u) \wedge y + v = u \wedge x + v = z)$. To show that multiplication $\cdot$ is not definable in $(\mathbb{Z}; +)$, simply note that the automorphism h of $(\mathbb{Z}; +)$ given by $h(x) = -x$ does not preserve multiplication, since in general $-(x \cdot y) \neq (-x) \cdot (-y)$.

Interpretations. Let S be any signature with $S_A, S_B \subseteq S$, and consider the structures $\mathfrak{A} = (A; S_A)$ and $\mathfrak{B} = (B; S_B)$. An n**-dimensional interpretation** of $\mathfrak{B}$ in $\mathfrak{A}$ consists of the following (the vectors $\bar{x}$ will refer to n-tuples of variables):

1. A formula $\varphi(x_1, \ldots, x_n)$ over the signature S_A which defines some relation $D_B \subseteq A^n$ representing the domain of B interpreted in A;
2. A surjective ("decoding") function $f : D_B \to B$, such that:

(i) for every constant symbol $\mathbf{c} \in S_B$, a formula $\varphi_{\mathbf{c}}(\bar{x})$ in $\mathcal{L}_A$ that defines some element $c_B \in D_B$ such that $f(c_B) = \mathbf{c}^{\mathfrak{B}}$;

(ii) for every m-ary relation symbol $\mathbf{r} \in S_B$, a formula $\varphi_{\mathbf{r}}(\bar{x}_1, \ldots, \bar{x}_m)$ in $\mathcal{L}_A$ that defines some relation $r_B \subseteq D_B^m$ such that $f[r_B] = \mathbf{r}^{\mathfrak{B}}$ (likewise for the equality symbol);

(iii) for every m-ary function symbol $\mathbf{g} \in S_B$, a formula $\varphi_{\mathbf{g}}(\bar{x}_1, \ldots, \bar{x}_m, \bar{x}_{m+1})$ in $\mathcal{L}_A$ that defines some function $g_B : D_B^m \to D_B$ such that $f(g_B(\bar{a}_1, \ldots, \bar{a}_m)) = \mathbf{g}^{\mathfrak{B}}(f(\bar{a}_1), \ldots, f(\bar{a}_m))$.

A classical example is the 2-dimensional interpretation of the rationals $\mathfrak{Q} = (\mathbb{Q}; +, \cdot)$ in the integers $\mathfrak{Z} = (\mathbb{Z}; +, \cdot)$, as ordinary fractions. Interpretations will be used in Sec. 6, where affine planes will be interpreted in algebraic structures called ternary rings.

3. Structures and theories of parallelism

We begin our study with very weak and simple structures which consist of a set of lines subject only to the relation of parallelism (besides equality). We will provide a definitive axiomatic description of such structures which can be extracted from the real Euclidean space of any dimension. In particular, it will turn out that the relation of line parallelism is too weak to distinguish dimensions greater than $n = 1$.

By a **line parallelism frame**, or simply a **parallelism frame**, we mean any structure of the form $\langle \mathbf{Li}, \parallel \rangle$, where $\parallel$ is a binary relation called **parallelism** over a non-empty set $\mathbf{Li}$ of which the elements are called **lines**. When the relation $\parallel$ holds for two lines x and y, we will use phrases such as x **is parallel to** y, etc.

A **pre-model of parallelism** is a parallelism frame $\langle \mathbf{Li}, \parallel \rangle$ satisfying the following conditions:

$\mathsf{Sym}_{\parallel}$: $\forall x \forall y (x \parallel y \to y \parallel x)$ (symmetry)

$\mathsf{PTran}_{\parallel}$: $\forall x \forall y \forall z (x \parallel y \wedge y \parallel z \to x = z \vee x \parallel z)$ (pseudo-transitivity)

A pre-model of parallelism in which the parallelism relation is reflexive (respectively, irreflexive) will be called a **model of weak parallelism**, (respectively, a **model of strict parallelism**). Thus, models of weak and strict parallelism must satisfy respectively the axioms:

$\mathsf{Ref}_{\parallel}$: $\forall x\,(x \parallel x)$, and $\mathsf{Irr}_{\parallel}$: $\neg\exists x\,(x \parallel x)$.

Hereafter, unless otherwise specified, by parallelism we will mean strict parallelism, and weak parallelism will be denoted by the symbol $\shortparallel$. Clearly, these are definable in terms of one another:

$$x \shortparallel y \;\Leftrightarrow\; x \parallel y \vee x = y, \qquad x \parallel y \;\Leftrightarrow\; x \shortparallel y \wedge x \neq y.$$

Thus models of weak parallelism are simply equivalence relations, while models of strict parallelism are isomorphic to disjoint unions of relational structures of the form $\langle \mathbf{W}, \neq \rangle$, where $\neq$ is the difference relation over some non-empty set $\mathbf{W}$. Given a model of strict parallelism $\langle \mathbf{Li}, \parallel \rangle$ and any line $x \in \mathbf{Li}$, the set of lines $\{x\} \cup \{y \in \mathbf{Li} : y \parallel x\}$ will be called the **parallel class (containing** x**)**. The property that a model of strict parallelism contains infinitely many parallel classes can be modelled using the scheme $\mathsf{Par}_{\parallel}$ consisting of the axioms

$$\exists x_1 \ldots \exists x_k \left(\mathrm{Diff}_k(x_1 \ldots x_k) \wedge \bigwedge_{i \neq j} x_i \nparallel x_j \right)$$

for every natural $k \geq 1$.

In models of strict parallelism, it is possible for a line to be parallel with no other line. A model of parallelism will be called k**-serial** if it satisfies the property

$$\forall x \exists y_1 \ldots \exists y_k \left(\mathrm{Diff}_k(y_1 \ldots y_k) \wedge \wedge_{i=1}^{k} y_i \parallel x \right).$$

A model that is k-serial for every natural $k \geq 1$ will be called **infinitely serial**, and the scheme specifying that a model is infinitely serial, consisting of all the above axioms for $k \geq 1$, will be denoted as $\mathsf{Ser}_{\parallel}$.

A model of strict parallelism is **real** if it consists of (not necessarily *all*) lines in the real plane, with the usual relation of strict parallelism.

Given a line u, let $\mathbf{u}$ denote the parallel class of u. Now, with every such parallel class $\mathbf{u}$ we associate a real number $m_{\mathbf{u}}$ meant to represent the slope of the lines in $\mathbf{u}$ in some arbitrarily fixed orthogonal coordinate system in the real plane, so that the mapping m is to be injective. Then, each line v in the class $\mathbf{u}$ can be mapped to a unique real number b_v, and the line v is identified with the line in the real plane having equation $y = m_{\mathbf{u}} x + b_v$. Thus, we have the following elementary characterization of line parallelism in $\mathbb{R}^n$.

PROPOSITION 1.1 *Every model of strict parallelism of cardinality not greater than the continuum is isomorphic to a real model.*

By taking the mappings m and b to be surjective, after setting aside a parallel class for the vertical lines, we obtain:

PROPOSITION 1.2 *Every model of strict parallelism in which there are continuum many parallel classes, each of them with the cardinality of the continuum, is isomorphic to the model of strict parallelism consisting of all lines in* $\mathbb{R}^2$.

COROLLARY 1.3 *For every natural* $n \geq 2$, *the model of strict parallelism consisting of all lines in* $\mathbb{R}^n$ *is isomorphic to the model of strict parallelism consisting of all lines in* $\mathbb{R}^2$.

PROPOSITION 1.4 *The theory of the class of models of strict parallelism which satisfy the schemes* $\mathsf{Par}_{\parallel}$ *and* $\mathsf{Ser}_{\parallel}$ *is* ω*-categorical, and hence complete and decidable.*

Indeed, let $\mathfrak{A}$ and $\mathfrak{B}$ be two countable models of strict parallelism satisfying the schemes $\mathsf{Par}_{\parallel}$ and $\mathsf{Ser}_{\parallel}$. Then $\mathfrak{A}$ contains countably many parallel classes, each of them of countable cardinality, and likewise for $\mathfrak{B}$. Let φ be any bijection between the parallel classes of $\mathfrak{A}$ and the parallel classes of $\mathfrak{B}$, and for every parallel class $\mathbf{x}$ in $\mathfrak{A}$, let $\psi_{\mathbf{x}}$ be a bijection between the lines in $\mathbf{x}$ and the lines in $\varphi(\mathbf{x})$. Then the line u in $\mathfrak{A}$ lying in the parallel class $\mathbf{x}$ is mapped to the line $\psi_{\mathbf{x}}(u)$ in $\mathfrak{B}$, and this establishes an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

The completeness and decidability now follow by the Łoś-Vaught Test.

Since the theory of strict parallelism is reducible to the first-order theory of equality, from Stockmeyer, 1977 it follows that the theory of strict parallelism is PSPACE-complete.

4. Structures and theories of orthogonality

4.1 Orthogonality frames and dimension

By a **line orthogonality frame**, or simply **orthogonality frame**, we will mean any structure of the form $\langle \mathbf{Li}, \perp \rangle$, where $\mathbf{Li}$ is a set, the elements of which will be called **lines**, and $\perp$ is a binary relation on $\mathbf{Li}$, called the **orthogonality** relation. Lines x and y satisfying $x \perp y$ will be called **orthogonal**. If x and y are both orthogonal as well as incident, then they will be called **perpendicular**, denoted $x \dot{\perp} y$. For $n \geq 1$, dimension can be defined in an orthogonality frame using the conjunction of the sentences $\mathsf{dim}_{\perp}^{(n)}$ and $\mathsf{Dim}_{\perp}^{(n)}$, given as

$\mathsf{dim}_{\perp}^{(n)}: \ \exists x_1 \ldots \exists x_n \left(\bigwedge_{i \neq j} x_i \perp x_j \right);$

$\mathsf{Dim}_{\perp}^{(n)}: \ \neg \exists x_1 \ldots \exists x_n \exists x_{n+1} \left(\bigwedge_{i \neq j} x_i \perp x_j \right).$

Clearly $\mathsf{Dim}_{\perp}^{(n)} = \neg\mathsf{dim}_{\perp}^{(n+1)}$. An orthogonality frame satisfies the property of ***n*-dimensionality** (for $n \geq 1$) if both sentences $\mathsf{dim}_{\perp}^{(n)}$ and $\mathsf{Dim}_{\perp}^{(n)}$ hold in that frame. A frame that satisfies 2-dimensionality will also be called **planar**.

When dealing with orthogonality frames, the expression $x_1 \shortparallel x_2$ will be an abbreviation for the formula

$$\forall y \, (y \perp x_1 \leftrightarrow y \perp x_2) \, . \tag{1.1}$$

In the context of orthogonality frames, by parallelism we will mean weak parallelism, and will say that lines x_1 and x_2 are **parallel** when $x_1 \shortparallel x_2$ in the sense of (1.1). Clearly, the binary relation defined by $\shortparallel$ is an equivalence relation.

From the class of all orthogonality frames, we single out those which satisfy the additional axiom

$\mathsf{Pen}_{\perp}$: $\forall x_1 \forall x_2 \, (x_1 \neq x_2 \rightarrow x_1 \not\shortparallel x_2)$.

Such orthogonality frames will be called **pencils**, by analogy with a pencil being a collection of co-punctual lines. However, our pencils of orthogonality are not pencils in the strict sense. The axiom $\mathsf{Pen}_{\perp}$ simply states that there may be no parallel lines, and this mimics pencil structure. But the axioms do not exclude models where the lines are not all co-punctual. In fact, incidence is not even definable from orthogonality, so that it is futile to try and axiomatize co-punctuality of lines in orthogonality frames. For example, let $R^n_{_M}$ be the set of all lines in the Euclidean space $\mathbb{R}^n$, and define $\mathfrak{R}^n_{_M} := (R^n_{_M}; \perp)$, where $\perp$ is the Euclidean orthogonality relation. By using the method of Padoa (i.e. finding an automorphism of $\mathfrak{R}^n_{_M}$ which fails to preserve incidence) we can obtain the following.

Proposition 1.5 *For $n \geq 3$, the relation of line incidence is not definable in $\mathfrak{R}^n_{_M}$.*

Orthogonality pencils can be obtained from orthogonality frames by factoring over parallel classes: if $\mathfrak{A}$ is any orthogonality frame, then $\mathfrak{A}/_{\shortparallel}$, the quotient structure of $\mathfrak{A}$ induced by the parallelism relation defined by (1.1), is an orthogonality pencil; parallelism reduces to equality in orthogonality pencils.

We will call an n-dimensional orthogonality frame **real** if it can be isomorphically embedded in $\mathbb{R}^n$ with the Euclidean orthogonality relation, where two lines are orthogonal when the dot product of their direction vectors is 0.

4.2 Planar orthogonality frames

DEFINITION 1.6 *A* ***2-dimensional model of orthogonality****, or simply a* ***planar*** *model of orthogonality, is an orthogonality frame* $\langle \mathbf{Li}, \perp \rangle$ *with* **Li** *non-empty and subject to the following axioms:*

$\mathsf{Irr}_\perp$: $\neg\exists x\,(x \perp x)$

$\mathsf{Sym}_\perp$: $\forall x \forall y\,(x \perp y \rightarrow y \perp x)$

$\mathsf{prd}_\perp^{(2)}$: $\forall x \exists y\,(y \perp x)$

$\mathsf{Prd}_\perp^{(2)}$: $\forall x \forall y_1 \forall y_2 \left(\bigwedge_{i=1,2} y_i \perp x \rightarrow y_1 \shortparallel y_2 \right)$

The axioms $\mathsf{Irr}_\perp$ and $\mathsf{Sym}_\perp$ specify respectively the irreflexivity and symmetry of $\perp$, while $\mathsf{prd}_\perp^{(2)}$ and $\mathsf{Prd}_\perp^{(2)}$ combined state that, up to parallelism, every line has a unique line orthogonal to it. It can easily be verified that the axioms $\mathsf{dim}_\perp^{(2)}$ and $\mathsf{Dim}_\perp^{(2)}$ hold in these structures. It is useful to note that in Goldblatt, 1987 they study orthogonality structures which admit **self-orthogonal** lines (lines which lie orthogonal to themselves) as well as **singular** lines (lines which lie orthogonal to all lines in the structure).

We will also make use of the following axiom schemes:

- $\mathsf{Inf}_\infty := \{\lambda_k\}_{k \in \mathbb{N}}$, stating the existence of infinitely many lines, where $\lambda_k := \exists x_1 \ldots \exists x_k\,(\wedge_{i \neq j} x_i \neq x_j)$;
- Ser_∞, stating that every parallel class has infinite cardinality, consisting of the axioms $\forall x \exists y_1 \ldots \exists y_k \left(\mathrm{Diff}_k(y_1 \ldots y_k) \wedge \wedge_{i=1}^{k} y_i \shortparallel x\right)$ for every $k \in \mathbb{N}$;
- Par_∞, stating that there are infinitely many parallel classes, consisting of the axioms $\exists x_1 \ldots \exists x_k\,(\wedge_{i \neq j} x_i \not\shortparallel x_j)$ for every $k \in \mathbb{N}$.

Using a similar approach as with the proof of Proposition 1.1, we can map lines in an abstract planar orthogonality model to lines in the real plane, to obtain the following.

PROPOSITION 1.7 (REPRESENTATION THEOREM) *Every planar model of line orthogonality with cardinality at most the continuum is isomorphic to a real planar model of line orthogonality.*

By bijectively associating pairs of mutually orthogonal parallel classes in any two countable planar orthogonality models, we furthermore obtain the following.

PROPOSITION 1.8 *The theory of the class of planar orthogonality models satisfying the schemes* Ser_∞ *and* Par_∞ *is* ω*-categorical.*

COROLLARY 1.9 *The theory of the class of planar orthogonality models satisfying the schemes* Ser_∞ *and* Par_∞ *is complete and decidable.*

4.3 Orthogonality frames in higher dimensions

Given an n-dimensional orthogonality frame, a set of lines $x_1, \dots, x_k$ in that frame (with $n \geq 1$ and $k \leq n$) will be called **independent** when the formula

$$\mathrm{LI}_k^{(n)} : \neg\exists z_1 \dots \exists z_{n-k+1} \left(\wedge_{i \neq j} z_i \perp z_j \wedge \wedge_{i,j} z_i \perp x_j\right) \tag{1.2}$$

is satisfied. Lines which are not independent will be called **dependent**. The formula (1.2) takes $k \leq n$ arguments. For $k > n$ define the lines $x_1, \dots, x_k$ to be dependent and the formula $\mathrm{LI}_k^{(n)}(x_1, \dots, x_k)$ to be false.

Say that y *lies in the* ***span*** *of* $x_1, \dots, x_k$ when the following formula holds:

$$\mathrm{Span}(x_1, \dots, x_k, y) := \forall z \left(\wedge_{i=1}^{k} z \perp x_i \to z \perp y\right). \tag{1.3}$$

Independence and span of lines is an abstraction of the notion of linear independence and span of vectors in a vector space.

Definition 1.10 *An orthogonality frame* $\langle \mathbf{Li}, \perp \rangle$ *with* $\mathbf{Li}$ *non-empty will be called an* ***n-dimensional model of orthogonality****, where* $n \geq 3$*, if it satisfies the axioms* $\mathsf{Irr}_\perp$*,* $\mathsf{Sym}_\perp$ *and* $\mathsf{Dim}_\perp^{(n)}$*, together with the axioms*

$$\mathsf{prd}_\perp^{(n)} : \forall x_1 \dots \forall x_{n-1} \exists y \left(\bigwedge_{i=1}^{n-1} y \perp x_i\right)$$

$$\mathsf{Prd}_\perp^{(n)} : \forall x_1 \dots \forall x_{n-1} \Big(\mathrm{LI}_{n-1}^{(n)}(x_1, \dots, x_{n-1}) \to \forall y_1 \forall y_2 \Big(\bigwedge_{i,j} y_i \perp x_j \to y_1 \shortparallel y_2\Big)\Big)$$

From the axiom $\mathsf{prd}_\perp^{(n)}$ it is immediate that the axiom $\mathsf{dim}_\perp^{(n)}$ will hold in all n-dimensional orthogonality models, and in the case where $n = 3$ it can also be shown that the axiom $\mathsf{Dim}_\perp^{(3)}$ may be dropped. The axioms $\mathsf{prd}_\perp^{(n)}$ and $\mathsf{Prd}_\perp^{(n)}$ imply that every n-dimensional orthogonality model has the property

$$\forall x_1 \dots \forall x_{n-1} \Big(\mathrm{LI}_{n-1}^{(n)}(x_1, \dots, x_{n-1}) \to \exists y \Big(\bigwedge_{i=1}^{n-1} y \perp x_i \wedge \forall z \Big(\bigwedge_{i=1}^{n-1} z \perp x_i \to z \shortparallel y\Big)\Big)\Big) \tag{1.4}$$

i.e. every $n-1$ independent lines $x_1, \dots, x_{n-1}$ have a unique parallel class, which we shall call the **product** of $x_1, \dots, x_{n-1}$ - denote it as $x_1 \times \dots \times x_{n-1}$ - that lies orthogonal to all of the x_i. Line products are an abstraction of the vector cross product in $\mathbb{R}^3$. In the context of pencils, $x_1 \times \cdots \times x_{n-1}$ will not be a parallel class, but simply a single line. Note that the operation $\times$ is *not* total, but only defined on tuples of independent lines. In the 3-dimensional case, the formula (1.4) reduces to $\forall x_1 \forall x_2 \, (x_1 \nparallel x_2 \to \exists y \, (y \perp x_1, x_2 \wedge \forall z \, (z \perp x_1, x_2 \to z \parallel y)))$, i.e. every two non-parallel lines have a unique parallel class orthogonal to both of them.

The axioms used above for the formalization of orthogonality in higher dimensions illustrate the novel expressive power of orthogonality, but they do not constitute a complete first-order axiomatization of the class of orthogonality structures in dimension $n \geq 3$. To our knowledge, the complete axiomatization of the first-order theory of orthogonality in these dimensions has not been established yet. Unlike the case for planar orthogonality models, it can be shown that the theory of line orthogonality in Euclidean n-space is not countably categorical for $n \geq 3$, and this negative result indicates that the problem of identifying this theory is presumably difficult.

Since orthogonality is a metric notion - arguably the simplest and most intuitive of all metric line notions - it has great expressive power, and one anticipates that its theory will capture a non-trivial and substantial fragment of that of the full Euclidean geometry, as witnessed by the fact that notions like linear independence and span of vectors can be abstracted and expressed in the language of orthogonality.

5. Two-sorted point-line incidence spaces

In this section, we consider point-line incidence structures, described by a two-sorted first-order language with equality, equipped with sorts for points and lines and the intersort relation of incidence.

5.1 Point-line incidence frames

A **point-line incidence frame** is a two-sorted structure $\langle \mathbf{Po}, \mathbf{Li}, \mathbf{I} \rangle$ where $\mathbf{Po}$ and $\mathbf{Li}$ are non-empty sets and $\mathbf{I} \subseteq \mathbf{Po} \times \mathbf{Li}$ is a symmetric **incidence relation** between them. The elements of **Po** are called **points**, and the elements of **Li** are called **lines**. If the relation **I** holds for a point X and a line x then we use expressions like X *is* ***incident*** *with* x, X ***lies*** *on* x, X ***belongs*** *to* x, x ***passes*** *through* X, x ***contains*** X etc. When $X\mathbf{I}z$ and $Y\mathbf{I}z$ we also say that *the line* z ***connects*** *the points* X *and* Y while *the point* X *is in the* ***intersection*** *of the lines* y *and* z will mean that $X\mathbf{I}y$ and $X\mathbf{I}z$.

We say that the lines x and y are **incident**, denoted x **Inc** y, if they are incident with a common point, formally

$$x \mathbf{\ Inc\ } y := \exists Z (Z \mathbf{I} x \wedge Z \mathbf{I} y).$$

Further, we say that the lines x and y are **intersecting**, denoted x **Int** y, if they are incident and different. Formally

$$x \mathbf{\ Int\ } y \ := \ x \neq y \wedge x \mathbf{\ Inc\ } y.$$

Given an arbitrary incidence frame $\langle \mathbf{Po}, \mathbf{Li}, \mathbf{I} \rangle$, we also introduce the relation of **collinearity** of three points

$$\mathbf{Col}(XYZ) := \exists x (X \mathbf{I} x \wedge Y \mathbf{I} x \wedge Z \mathbf{I} x)$$

and that of **co-punctuality** of lines

$$\mathbf{Cop}(x_1 \ldots x_n) := \exists X \left(\bigwedge_{i=1}^{n} X \mathbf{I} x_i \right).$$

Thus incidence of lines is a special case of co-punctuality of lines.

5.2 Linear spaces of incidence

Linear spaces (*not* in sense of vector spaces) are the most general incidence structures which are geometrically meaningful. It is instructive to note that a number of fundamental concepts in vector spaces, such as independence, basis and dimension can be generalized to linear spaces. The following definition reflects Hilbert's axioms for incidence.

DEFINITION 1.11 *(see Karzel et al., 1973) A* ***linear space*** *(aka* ***incidence geometry*** *or* ***incidence basis*** *in Mihalek, 1972) is an incidence frame* $\langle \mathbf{Po}, \mathbf{Li}, \mathbf{I} \rangle$ *in which the following axioms hold:*

LS1 *Every two distinct points are incident with a unique common line.*
LS2 *Every line passes through at least two points.*

Given distinct points X and Y in a linear space, the unique line incident with both of them will be denoted by $\mathbf{l}(X, Y)$. Thus, the expression $\mathbf{l}(X, Y)$ assumes that X and Y are distinct. Furthermore, if two lines x and y in a linear space intersect, then by LS1 they have a unique common point, hereafter denoted as $\mathbf{P}(x, y)$ and called the **intersection** of x and y. We will only use this notation in the case of intersecting lines.

5.3 Linear subspaces, independence, bases and dimension

To begin with, linear spaces can be regarded as two-sorted *algebraic* structures, with two partial operations: one applied to two different points produces the unique line passing through them, and the other applied to two intersecting lines produces their intersection point. Thus, a subspace of a linear space could be defined as a non-empty substructure which is closed under these partial operations. However, this definition also allows subspaces consisting of just one line from the space, and two or more, but not all, points on that line. It is more natural to require that a subspace contains with every line in it all points on that line. Therefore, by a **(linear) subspace** (*linear variety* in Gemignani, 1971) of a linear space $\mathfrak{L}$ we will mean every substructure $\mathfrak{L}' = \langle \mathbf{Po}', \mathbf{Li}', \mathbf{I} \rangle$ of the incidence frame $\mathfrak{L}$, which is itself a linear space, and all points lying on lines in $\mathbf{Li}'$ are in $\mathbf{Po}'$. Note that any non-empty intersection of a family of subspaces (i.e. incidence structure in which the sets of lines and points are the respective non-empty intersections of the families of lines and points of the spaces in the family) is a subspace itself. The subspace $\mathfrak{L}'$ of $\mathfrak{L}$ is **generated by the pair of sets of points and lines** $(\mathbf{P}, \mathbf{L})$ in $\mathfrak{L}$, denoted here $\mathfrak{L}' = \lfloor \mathbf{P}, \mathbf{L} \rfloor$, if it is the smallest subspace (i.e., the intersection of all these) of $\mathfrak{L}$ containing the points (lines). Alternatively, $\mathfrak{L}'$ can be obtained from $(\mathbf{P}, \mathbf{L})$ by a finite number of successive steps of adding the line passing through two given points and adding all points lying on a given line. Clearly, it suffices to generate subspaces starting from sets of points only; then we write simply $\mathfrak{L}' = \lfloor \mathbf{P} \rfloor$.

A set of points $\mathbf{P}$ in a linear space is **independent** if none of the points of $\mathbf{P}$ belongs to the subspace generated by the rest of $\mathbf{P}$. An independent set of points which generates (paired with the empty set of lines) a given subspace is called a **basis** of that subspace. The reader is also referred to Ch. **??** for a discussion of matroids and independent sets.

Note that, unlike vector spaces, not all bases in a linear space need to have the same cardinality; for an example, see e.g. Batten, 1986, Sec. 2.1. However, as shown there:

PROPOSITION 1.12 *All bases of a linear space* $\mathfrak{L} = \langle \mathbf{Po}, \mathbf{Li}, \mathbf{I} \rangle$ *have the same cardinality provided that the space satisfies the following* ***exchange property:*** for any $\mathbf{P} \subset \mathbf{Po}$ and $X, Y \in \mathbf{Po}$, if $X \notin \lfloor \mathbf{P} \rfloor$ and $X \in \lfloor \mathbf{P} \cup \{Y\} \rfloor$ then $Y \in \lfloor \mathbf{P} \cup \{X\} \rfloor$.

A **dimension** of a subspace $\mathfrak{L}'$ of a linear space $\mathfrak{L}$ is the least number n such that $\mathfrak{L}'$ can be generated by a (clearly independent) set of $n+1$

points. Thus, every subspace containing just one line has a dimension 1; a subspace containing three non-collinear points has a dimension at least 2. A subspace with dimension 2 of a linear space $\mathfrak{L}$ is a **(linear) plane** in $\mathfrak{L}$.

Examples of linear spaces include the usual Euclidean plane and 3D-space, as well as the points in any open disc in the Euclidean plane or space, where lines are the intersections of the usual lines in the plane (space) with that disc. Besides, there is a huge variety of finite linear spaces (see e.g. Batten, 1986; Mihalek, 1972). For instance, take the vertices of a tetrahedron as points, and the sides of the tetrahedron as lines, where incidence is standard. Note that this is the simplest example of a linear space of dimension greater than 2.

5.4 Linear transformations and collineations

Given linear spaces $\mathfrak{L} = \langle \mathbf{Po}, \mathbf{Li}, \mathbf{I} \rangle$ and $\mathfrak{L}' = \langle \mathbf{Po}', \mathbf{Li}', \mathbf{I}' \rangle$, a mapping f from $\mathfrak{L}$ to $\mathfrak{L}'$ is a pair of mappings $f_{po} : \mathbf{Po} \longrightarrow \mathbf{Po}'$ and $f_{li} : \mathbf{Li} \longrightarrow \mathbf{Li}'$. Such mapping is a **linear transformation** if it preserves incidence both ways, i.e. $X\mathbf{I}x$ iff $f_{po}(X)\mathbf{I}' f_{li}(x)$. Thus, the action of a linear transformation on a line is determined by its action on any two distinct points of the line, and therefore it suffices to consider linear transformations as mappings on the set of points of a linear space. It is immediate from the definition to see that if $\mathfrak{L}$ has dimension greater than 1 then every linear transformation on $\mathfrak{L}$ is injective on the set of points, and on the set of lines, of $\mathfrak{L}$. Thus the notion of linear transformation is the natural notion of a mapping between linear spaces that preserves their structure.

An **isomorphism** between linear spaces is a bijective (on each of the sets of points and lines) linear transformation. A **collineation** is an automorphism of linear space, i.e. an isomorphism of a linear space onto itself. With ι as the identity and function inverse and composition as basic operations, the set of all collineations of a linear space $\mathfrak{L}$ is a group, called the **group of collineations** $\mathsf{Aut}(\mathfrak{L})$ of $\mathfrak{L}$. Many properties of a linear space can be determined by its group of collineations; for more detail see e.g. Gemignani, 1971; Behnke et al., 1974; Coxeter, 1969; Hughes and Piper, 1973.

5.5 Parallelism and planarity in linear spaces

Given a linear space $\mathfrak{L} = \langle \mathbf{Po}, \mathbf{Li}, \mathbf{I} \rangle$, one way to define parallel lines in it is to take intuition from the Euclidean space, where two lines are parallel if they are co-planar but do not intersect. Thus, we call two lines in $\mathfrak{L}$ **(strictly) quasi-parallel** if they have no common incident point and belong to a subspace of dimension 2. Note that this relation need not be transitive. For technical reasons, however, we consider separately the case where every line in the space is incident with exactly 2 points. We will call such spaces **meagre**. In meagre spaces, by strict quasi-parallelism we will mean simply non-incidence.

An alternative definition is based on another intuition from the 'real Euclidean geometry': two lines are parallel if they are not incident, but the diagonals of every quadrilateral with a pair of opposite sides lying on these lines must intersect. Formally, we define the relation $\parallel$ of **(strictly) parallel** lines in $\mathfrak{L}$ as follows:

$$x \parallel y \ := \ \neg\, x \ \mathbf{Inc}\ y \ \wedge \forall X_1 \forall X_2 \forall Y_1 \forall Y_2 \Big(\big(x = \mathbf{l}(X_1, X_2) \wedge y = \mathbf{l}(Y_1, Y_2)\big) \\ \to \big(\mathbf{l}(X_1, Y_1)\ \mathbf{Int}\ \mathbf{l}(X_2, Y_2) \ \vee\ \mathbf{l}(X_1, Y_2)\ \mathbf{Int}\ \mathbf{l}(X_2, Y_1)\big)\Big).$$

Then we define **weak parallelism**:

$$x \shortparallel y \ := \ x \parallel y \vee x = y.$$

Again, in the special case of meagre linear spaces, by strict parallelism we mean non-incidence.

The relation $\parallel$ is irreflexive and symmetric but not necessarily transitive either. Still, if two lines in a linear space are parallel, then they are quasi-parallel too: take any two pairs of distinct points, one on each of the lines, and take the intersection point of the respective 'diagonals'; that point together with the pair of points on any of the lines generates a subspace of dimension 2 containing both lines. The converse need not hold, which can be shown by an example from Batten, 1986, Sec. 2.1 of a linear space of dimension 2 which contains a set of 4 independent points, mentioned in Sec. 5.3.

Given a linear space $\mathfrak{L} = \langle \mathbf{Po}, \mathbf{Li}, \mathbf{I} \rangle$, two lines $x, y \in \mathbf{Li}$ are called **co-planar**, denoted $\mathbf{Pl}(xy)$, if they are incident or parallel:

$$\mathbf{Pl}(xy) \ := \ x\ \mathbf{Inc}\ y \vee x \parallel y.$$

A linear space is **planar** if every two lines in it are co-planar.

Thus, by convention, every meagre linear space is planar.

It is easy to see that a non-meagre linear space $\mathfrak{L}$ is planar iff it satisfies the following property:

$$\forall X_1 \forall X_2 \forall X_3 \forall X_4 \Big(\mathbf{l}(X_1, X_2)\ \mathbf{Inc}\ \mathbf{l}(X_3, X_4) \ \vee \\ \mathbf{l}(X_1, X_3)\ \mathbf{Inc}\ \mathbf{l}(X_2, X_4) \ \vee\ \mathbf{l}(X_1, X_4)\ \mathbf{Inc}\ \mathbf{l}(X_2, X_3)\Big), \qquad (1.5)$$

saying that the diagonals of every quadrilateral must intersect.

We now have two different notions of a 'plane' in a linear space, one based on dimension, the other on planarity. Note, that the tetrahedron is a meagre (and hence planar) space of dimension 3. However, if a non-meagre linear space $\mathfrak{L}$ is planar, then it has dimension 2. Indeed, let A, B, C be any three non-collinear points in it. Then for any point D in the space, at least one of the pairs of lines $(\mathbf{l}(A,B), \mathbf{l}(C,D))$, $(\mathbf{l}(A,C), \mathbf{l}(B,D))$ and $(\mathbf{l}(A,D), \mathbf{l}(B,C))$ are incident, say $\mathbf{l}(A,B)\ \mathbf{Inc}\ \mathbf{l}(C,D)$, and let $X = \mathbf{P}(\mathbf{l}(A,B), \mathbf{l}(C,D))$. Then X belongs to the line $\mathbf{l}(A,B)$ and D belongs to the line $\mathbf{l}(C,X)$.

The converse of the claim above need not hold, again by the example from Batten, 1986 mentioned above; note that any independent set of 4 points in a non-meagre space violates the planarity condition above.

5.6 Projective spaces and planes

DEFINITION 1.13 *(see also Lenz, 1954) A* ***projective space*** *is linear space satisfying the following additional axioms:*

PS1 *If A, B, C are distinct points and a line l intersects AB and AC in two distinct points, then it intersects BC as well.*
PS2 *There are at least four points, no three of which are collinear.*

Projective spaces satisfy the exchange property (see Batten, 1986, Sec. 3.9), and hence, by Proposition 1.12, every two bases in a projective space have the same cardinality, called the **rank** of the space. The dimension of the space is thus defined as 1 less than its rank.

A **projective plane** is a projective space of rank 3, i.e. dimension 2. Equivalently, a projective plane is a projective space in which every two lines intersect; in particular, projective planes contain no parallel lines.

Conversely, one can re-define projective spaces of higher dimension in terms of the sub-planes that they contain. For instance, the **projective 3D-space** can be defined (see Hartshorne, 1967; Mihalek, 1972) as a projective space with the following additional axioms:

PS3 *There exist at least 4 non-coplanar points.*
PS4 *Every three non-collinear points lie on a unique sub-plane.*
PS5 *Every line meets every sub-plane in at least one point.*
PS6 *Every two sub-planes have at least a common line.*

Since every line in a projective plane intersects all other lines in different points, and every point is line-connected with every other point in the plane, it follows that in a finite projective plane every line is incident with the same number of points, and every point is incident with the same number of lines.

If φ is any statement about a projective plane formulated in terms of 'point', 'line' and 'incidence' then the statement φ^*, formed from φ by interchanging the words 'point' and 'line', is called the **dual statement** (with respect to 'point' and 'line') of φ. A statement φ is **self-dual** if $\varphi = \varphi^*$. A theorem about projective planes formulated in terms of the notions 'point', 'line' and 'incidence' is a **projective validity** if it is true in the class of all projective planes, i.e. it is derivable from the axioms for projective planes. One reason why projective planes are interesting is the following 'two for the price of one' result (see e.g., Mihalek, 1972; Hughes and Piper, 1973; Batten, 1986):

THEOREM 1.14 *(Duality Principle for Projective Planes) Let φ be a projective validity. Then the dual φ^* of φ is also a projective validity.*

To prove the duality principle it suffice to note that the duals of the axioms for projective planes provide an equivalent axiomatization, and hence the 'dual' of every proof in the (first-order) theory of projective planes is a proof in that theory, too. For instance, it follows from the duality principle that in every projective plane there are at least four lines, no three of them incident with the same point. The reader is also referred to the self-dual axiomatizations of Esser, 1951; Esser, 1973; Kordos, 1982; Menger, 1948; Menger, 1950.

To illustrate the power of the duality principle, consider the following combinatorial example. Suppose we have some projective plane with the property that every line contains n points. Now fix any line l and any point P not on l. Then every point X on l determines a line PX and since l contains n points then there must be n distinct lines of the form PX. Furthermore, note every point in the plane must lie on exactly one of these lines PX. If all these lines were disjoint (as sets of points) then there would be n^2 points in total, but since the point P is counted n times then the total number of points in the plane is $n^2-(n-1)=n^2-n+1$. Thus, we have just shown that the following is a projective validity:

- If every line is incident with n points, then there are $n^2 - n + 1$ points in the entire plane.

By the duality principle, we can conclude the dual of this result:

- If every point is incident with n lines, then there are $n^2 - n + 1$ lines in the entire plane.

Finally, note that axiom PS2 implies that every line in a projective plane is incident with at least 3 points, and therefore, by the result above, the least projective plane, known as **Fano plane**, given on Fig. 1.1, has 7 points and 7 lines.

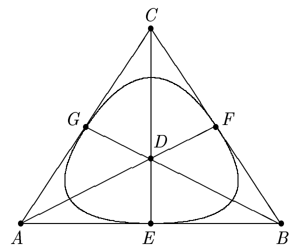

Figure 1.1.

5.7 Affine spaces and planes

DEFINITION 1.15 *(see also Lenz, 1954; Lenz, 1989) An* ***affine structure*** *is a linear space* $\mathfrak{L} = \langle \mathbf{Po}, \mathbf{Li}, \mathbf{I} \rangle$ *in which the relation of weak parallelism* $\shortparallel$ *is an equivalence relation.*

An ***affine space*** *is an affine structure with at least 3 non-collinear points, in which the following axiom (Euclid's Fifth postulate) holds:*

V : *Given a line* x *and a point* X *not on* x*, there is a unique line through* X *that is (strictly) parallel to* x*, denoted by* $\mathbf{p}(X, x)$.

A slightly more general, but essentially equivalent, definition is:

V' : Given a line x and a point X, there is a unique line through X that is weakly parallel to x, denoted by $\mathbf{p}(X, x)$.

An **(affine) subspace** of an affine space $\mathfrak{L}$ is every linear subspace of $\mathfrak{L}$ satisfying V itself, i.e. closed under the operation $\mathbf{p}$.

Examples of affine spaces include the usual Euclidean plane and 3D-space, but not the open disc in the Euclidean plane or space, as the axiom V fails there. An example of a finite affine space is the tetrahedron. For other examples, see e.g. Batten, 1986 and Coxeter, 1969.

Note that, usually the literature on affine and projective geometry deals only with *affine planes*, and only occasionally introduces higher-dimensional affine spaces. Thus, our definition is somewhat more general, as it is not based on coordinatization of the space, neither on the earlier defined notion of affine plane. A small price to pay for that generality was the adjustment of the definitions of parallelism and planarity in the special case of meagre spaces.

Planes and planarity can be re-defined in non-meagre affine spaces using the following observations. Any 'triangle' (three non-collinear points) together with the three lines determined by these points, must define a plane. By the Fifth Postulate, every line in that plane must be incident with at least two of these three lines, i.e. every line is determined by a pair of different points, each incident with some of these lines. Furthermore, every point in that plane belongs to at least one line constructed in this way, e.g. any line determined by that point and a vertex of the original triangle, which intersects the side opposite to that vertex (there will be at least one, by planarity). Turning these observation around, we obtain the following definition: a **plane in an affine space** $\mathfrak{L}$ is an incidence structure $\mathfrak{S}$ constructed as follows: take three non-collinear points P_1, P_2, P_3 in $\mathfrak{L}$ and the lines in $\mathfrak{L}$ determined by them, say x_1, x_2, x_3, where $x_i = \mathbf{l}(P_j, P_k)$, for i, j, k pairwise different. Let $\mathbf{P}$ be the set of points in $\mathfrak{L}$ incident with at least one of the lines x_1, x_2, x_3. Then the lines in $\mathfrak{S}$ are exactly those lines in $\mathfrak{L}$ incident with at least two different points from $\mathbf{P}$, and the points in $\mathfrak{S}$ are those points in $\mathfrak{L}$ incident with at least one of these lines. It is not difficult to see that every such plane is an affine subspace of $\mathfrak{L}$, which will be called the **(affine) plane in the space** $\mathfrak{L}$ generated by the points P_1, P_2, P_3. In particular, every plane in an affine space is closed under the *affine operations* of taking lines through two points, intersections of lines, and construction of lines passing through a given point and parallel to a given line, based on the axiom V.

Now, an affine space is called **planar**, or an **affine plane**, if it coincides with some plane in it. It is easy to show (see e.g. Batten, 1986) that every affine space satisfies the exchange property, and hence all bases in an affine space have the same cardinality. Rank and dimension are introduced as in projective spaces. As for the tetrahedron, despite being of dimension 3, we have a good excuse (to become clear further) to consider it an affine plane as well, and that is the main reason to adjust the definition of parallelism and planarity for meagre spaces. In fact (see Batten, 1986, Sec. 4.1), it is the *only* affine plane of dimension more than 2. Moreover, as shown there, every line in a finite affine space is incident with the same number of points, and vice versa.

Thus, if an affine space is planar then the relation of non-incidence of lines is pseudo-transitive, hence it is a relation of a strict parallelism. To summarize:

PROPOSITION 1.16 *A non-meagre affine space $\mathfrak{L}$ is an affine plane iff it has a dimension 2 iff the relations of non-incidence and strict parallelism between lines in $\mathfrak{L}$ coincide.*

5.8 Relationship between affine and projective planes

Affine planes do not have the duality property. For example, the dual of $\mathsf{LS1}$ does not hold as it violates Euclid's Parallel Postulate V. Still, there is an intimate relationship between affine planes and projective planes, given by the following two theorems, the proofs of which can be found e.g. in Gemignani, 1971; Hughes and Piper, 1973; Mihalek, 1972.

THEOREM 1.17 *Let* $\mathfrak{P} = \langle \mathbf{Po}, \mathbf{Li}, \mathbf{I} \rangle$ *be a projective plane and let* l^* *be a line in* $\mathbf{Li}$*. Define* $\mathbf{Po}^- := \{X \in \mathbf{Po} : X \notin l^*\}$*,* $\mathbf{Li}^- := \mathbf{Li} \backslash \{l^*\}$ *and* $\mathbf{I}^- := \mathbf{I}|_{\mathbf{Po}^- \times \mathbf{Li}^-}$*. Then the structure* $\mathfrak{P}^- = \left\langle \mathbf{Po}^-, \mathbf{Li}^-, \mathbf{I}^- \right\rangle$ *is an affine plane, called the* ***(deletion) affine subgeometry of*** $\mathfrak{P}$ ***induced by*** l^**.*

THEOREM 1.18 *Let* $\mathfrak{A} = \langle \mathbf{Po}, \mathbf{Li}, \mathbf{I} \rangle$ *be any affine plane and let* l^* *be any set, disjoint with* $\mathbf{Po}$ *and* $\mathbf{Li}$*, and of the same cardinality as the number of parallel classes in* $\mathfrak{A}$*. To every parallel class* $[l]_{\shortparallel}$ *in* $\mathfrak{A}$*, assign some distinct element* $P_{[l]_{\shortparallel}} \in l^*$ *to* $[l]_{\shortparallel}$*. Define*

1. $\mathbf{Po}^+ := \mathbf{Po} \cup l^*$*;*
2. $\mathbf{Li}^+ := \mathbf{Li} \cup \{l^*\}$*;*
3. $\mathbf{I}^+ := \mathbf{I} \cup \{(P_{[l]_{\shortparallel}}, l) : l \neq l^*\} \cup \{(P, l^*) : P \in l^*\}$.

Then the structure $\mathfrak{A}^+ = \left\langle \mathbf{Po}^+, \mathbf{Li}^+, \mathbf{I}^+ \right\rangle$*, which we will call the* ***projective extension*** *of* $\mathfrak{A}$*, is a projective plane.*

Thus, affine and projective planes are separated by a single line, the so-called “line at infinity”. The tetrahedron and Fano plane (Fig. 1.1) illustrate the latter two results. It is because of the Fano plane that we insisted the tetrahedron, being its deletion subgeometry, should be an affine plane.

Note that the constructions between affine and projective planes described above are mutually inverse, up to isomorphism. These constructions can be described in logical terms, relating the first-order theories of the affine and projective planes. On the one hand, every affine plane is first-order interpretable into its projective extension in an obvious way; on the other hand, the first-order theory of a projective plane can be reduced to the first-order theory of its affine subgeometry. Consequently, given a class of projective planes, its elementary theory is decidable iff the elementary theory of the class of respective affine subgeometries is decidable, too. Likewise, the elementary theory a class of affine planes is decidable iff the elementary theory of the class of respective projective extensions is decidable, too.

6. Coordinatization

In this section we give an overview on the coordinatization and subsequent algebraization of affine planes. We will introduce a special class of algebraic structures called "ternary rings", the elements of which can serve as coordinates of points in the plane. It will turn out that affine planes and ternary rings are inter-definable in the sense that from every affine plane one can extract a ternary ring while every ternary ring gives rise to an affine plane. In fact, these constructions are essentially *logical (first-order) interpretations*, which thus relate their first-order theories. In particular, we will see that natural and important geometric properties of affine planes, viz. Desargues' and Pappus' properties, correspond to natural algebraic properties in these ternary rings. Furthermore, we will demonstrate the interaction between special dilations of affine planes and the properties of Desargues and Pappus, and will discuss the logical consequences of the coordinatization. In particular, we will extract the axiomatizations and (un)decidability of the first-order theories of some important affine planes and classes of planes from their associated coordinate rings.

The method of coordinatization applies likewise to projective planes, and most of the results obtained below have their close projective analogues. Since both constructions are very similar, we will only present here coordinatization of affine planes. For a more detailed account of coordinatization of affine and projective planes and the relationships (with proofs) between geometric and algebraic properties, the reader is referred to Blumenthal, 1961; Artin, 1957; Heyting, 1963; Szmielew, 1983; Mihalek, 1972; Hughes and Piper, 1973, etc.

6.1 Coordinate systems in affine planes

We adopt weak parallelism in the discussion for the rest of this section.

Let any affine plane $\mathfrak{A} = \langle \mathbf{Po}; \mathbf{Li}; \mathbf{I} \rangle$ be given. The following procedure assigns coordinates to the plane.

- Take any triplet of non-collinear points O, X and Y. The point O will be called the **origin** and the triplet OXY will be called the **coordinate system**.

- Let I be the point of intersection of the line in the parallel class of OX containing Y, with the line in the parallel class of OY containing X. The point I will be called the **unit point** while the lines OX, OY and OI will be called respectively the **x-axis**, **y-axis** and **unit line**.

- Let Γ be any abstract set, containing elements 0 and 1, of the same cardinality as the number of points on the unit line. In fact, since all lines in an affine plane contain the same number of points, Γ

can have the cardinality of any line in the plane. We call the set Γ the **coordinate set**.

- Now let γ be any bijection between points on the unit line and Γ and such that $\gamma(O) = 0$ and $\gamma(I) = 1$.

Points in the plane are assigned coordinates consisting of ordered pairs from Γ^2 in the following manner:

- If P is a point on the unit line and $\gamma(P) = p$ then the coordinates of P are (p, p).
- Let P be any point not on the unit line. Suppose the line in the parallel class of the y-axis, containing P, intersects the unit line in the point with coordinates (a, a), and suppose that the line in the parallel class of the x-axis, containing P, intersects the unit line in the point with coordinates (b, b). Then the coordinates of P are (a, b) (refer to Fig. 1.2).

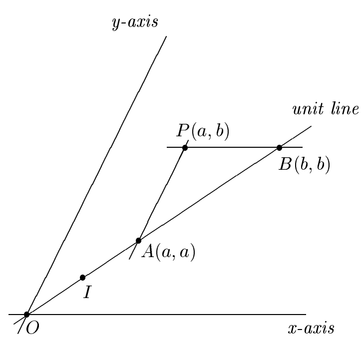

Figure 1.2.

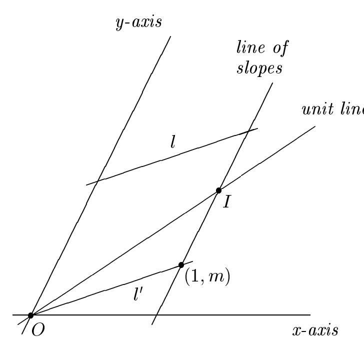

Figure 1.3.

The point P with coordinates (x, y) will be denoted as $P(x, y)$, and points will be identified with their coordinates ($\mathbf{P}(x, y)$ may also refer to the point of intersection of lines x and y, but the context should make it clear what the intended meaning is). For example, the points O, X, Y and I can be given simply as $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. The value x is called the **abscissa** of P and y is called the **ordinate** of P. Any line not in the parallel class of the y-axis will intersect the y-axis in some point $(0, c)$. This value c is called the ***y*-intercept** of the line.

Next we define the **slope** of a line, using what will be called the **line of slopes**, which is that line in the parallel class of the y-axis intersecting the unit point I. Every point on the line of slopes will have coordinates $(1, m)$ for some $m \in \Gamma$. There are two types of line to consider:

1. If l is a line parallel to the y-axis, its slope is left undefined.

2. Consider any line l not in the parallel class of the y-axis. There will be a unique line l' parallel to l and incident with O. This line l' will intersect the line of slopes in some point $(1, m)$. The slope of l is defined as the value m (refer to Fig. 1.3).

Thus the slope of the x-axis is 0 and the slope of the unit line is 1.

The **equation** of a line is any equation formulated in terms of variables x and y such that all and only those points (x, y) belonging to the line satisfy the equation. For example, the line parallel to the x-axis with y-intercept b has equation $y = b$, and the line parallel to the y-axis intersecting the x-axis in the point $(a, 0)$ has equation $x = a$. The unit line has equation $y = x$. But we need further algebraic machinery to describe the equations of lines other than these trivial examples. This is provided by the operation $T : \Gamma^3 \rightarrow \Gamma$, defined as follows. Let the triple of values $(m, a, c) \in \Gamma^3$ be given. To compute the value of $T(m, a, c)$ consider the line l of slope m and y-intercept c. Then l intersects the line parallel to the y-axis, and containing the point $(a, 0)$, in the point $(a, T(m, a, c))$ (refer to Fig. 1.4).

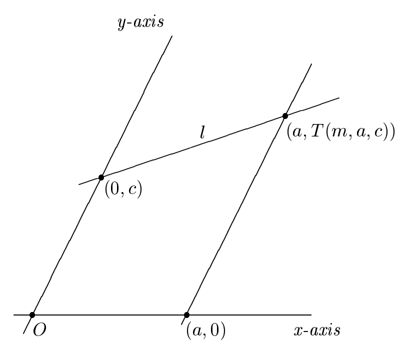

Figure 1.4.

T enables us to obtain an equation for any line in the plane. If the line has undefined slope, it has an equation $x = a$ for some $a \in \Gamma$, while the line with slope m and y-intercept c has equation $y = T(m, x, c)$. The above definition of T uses a geometric construction, but T can also be described purely algebraically using the structure of ternary ring.

6.2 Ternary rings and coordinate systems

Definition 1.19 *A* ***ternary ring*** *(also known as a Hall planar ternary ring in reference to Marshall Hall Jr.) is an algebraic structure* $\mathfrak{T} = (F; 0, 1, T)$ *consisting of a set* F *together with distinguished elements* $0, 1 \in F$ *and a ternary operation* T *on* F *subject to the following axioms:*

T_0 : $0 \neq 1$

T_1 : $T(a, 1, 0) = a$

T_2 : $T(1, b, 0) = b$

T_3 : $T(a, 0, c) = c$

T_4 : $T(0, b, c) = c$

T_5 : $T(a, b, x) = d$ *has a solution for* x

T_6 : *If* $T(a, b, c) = T(a, b, c')$ *then* $c = c'$

T_7 : *If* $b \neq b'$ *then the simultaneous equations* $T(x, b, y) = d$ *and* $T(x, b', y) = d'$ *have a solution for* x *and* y.

T_8 : *If* $a \neq a'$ *then* $T(a, x, c) = T(a', x, c')$ *has a solution for* x

T_9 : *For* $b \neq b'$*, if* $T(a, b, c) = T(a', b, c')$ *and* $T(a, b', c) = T(a', b', c')$ *then* $a = a'$ *and* $c = c'$

It is easy to see that the ternary operation T satisfies all the axioms $\mathsf{T}_0 - \mathsf{T}_9$ hence we have the following important result.

Theorem 1.20 *Let the affine plane* $\mathfrak{A}$ *be coordinatized with coordinate system* OXY *and coordinate set* Γ *and let* T *be the resulting ternary operation on* Γ*. Then the structure* $(\Gamma; 0, 1, T)$ *is a ternary ring.*

The ternary ring $(\Gamma; 0, 1, T)$ above will be called a **(coordinate) ring attached to the plane** $\mathfrak{A}$ (by means of the coordinate system OXY), denoted $\mathbf{T}_{OXY}(\mathfrak{A})$. Given the coordinate system OXY, there is only one, up to isomorphism, ternary ring attached to the plane by means of OXY. A ternary ring $\mathfrak{T}$ is said to be **attached** to the affine plane $\mathfrak{A}$ provided there is *some* coordinate system OXY such that $\mathfrak{T} \cong \mathbf{T}_{OXY}(\mathfrak{A})$.

A converse to the last theorem is also true.

Theorem 1.21 *For any ternary ring* $\mathfrak{T} = (F; 0, 1, T)$*, the plane* $\mathbf{A}(\mathfrak{T})$ *with point universe* F^2*, and line universe consisting of all sets of the form* $\{(a, y) : y \in F\}$ *and* $\{(x, T(m, x, c)) : x \in F\}$ *for every* $a, m, c \in F$*, is an affine plane, called* ***the affine plane over the ternary ring*** $\mathfrak{T}$.

The constructions given in the two theorems above are inverse in the following sense. Let an affine plane $\mathfrak{A}$ be given and fix some coordinate system OXY. Then $\mathbf{A}(\mathbf{T}_{OXY}(\mathfrak{A})) \cong \mathfrak{A}$. Let a ternary ring $\mathfrak{T} = (F; 0, 1, T)$ be given. Then $\mathbf{T}_{(0,0)(0,1)(1,0)}(\mathbf{A}(\mathfrak{T})) \cong \mathfrak{T}$.

If two ternary rings are isomorphic then so will be the affine planes over those ternary rings. But surprisingly, there are non-isomorphic ternary rings such that the affine planes over those ternary rings are still isomorphic. In particular, coordinatizing an affine plane with different coordinate systems may sometimes give rise to non-isomorphic ternary rings attached to the same plane, and later on we will give a sufficient condition for uniqueness of the coordinate rings.

Given a ternary ring $(F; 0, 1, T)$, **addition** $+$ and **multiplication** $\cdot$ are defined on F as follows:

$$a + b = T(1, a, b); \quad a \cdot b = T(a, b, 0).$$

The structure $(F; 0, 1, +, \cdot)$ thus formed is actually a double loop. Hence the class of double loops contains the class of ternary rings.

The geometric analogue to addition is the translation of a line in the plane, that of multiplication is the rotation of a line in the plane. To calculate $a+b$ proceed as follows. Take the points $A(a, a)$ and $B(b, b)$ that lie on the unit line. Intersect the line parallel to the x-axis containing B with the y-axis to obtain the point $Q(0, b)$. Then take the line parallel to the unit line containing the point Q and intersect it with the line parallel to the y-axis containing the point A to obtain the point $P(a, c)$. The line parallel to the x-axis containing P intersects the unit line in the point $C(c, c)$. We define $C = A + B$ and $c = a + b$ (refer to Fig. 1.5). To calculate $a \cdot b$ proceed as follows. Take the points $A(a, a)$ and $B(b, b)$ that lie on the unit line. Intersect the line parallel to the x-axis containing A with the line of slopes to obtain the point $Q(1, a)$. Then the line parallel to the y-axis containing the point B will intersect the line OQ in some point P. Intersect the line parallel to the x-axis containing P with the unit line to obtain the point $C(c, c)$. We define $C = A \cdot B$ and $c = a \cdot b$ (refer to Fig. 1.6).

We are now able to give the linear equations of two more classes of lines. The line of slope 1 with y-intercept c will have equation $y = x + c$ while the line of slope m and y-intercept 0 will have equation $y = m \cdot x$.

Call a left division ring **strong** if it satisfies the additional property

$$m_1 \neq m_2 \Rightarrow \forall c_1 \forall c_2 \exists x \left(m_1 \cdot x + c_1 = m_2 \cdot x + c_2 \right),$$

informally lines with different directions intersect. We can define a ternary operation T in the strong left division ring $(F; 0, 1, +, \cdot)$ as $T(a, b, c) := a \cdot b + c$. Then it turns out that $(F; 0, 1, T)$ will be a ternary

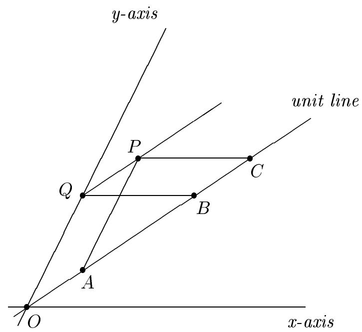

Figure 1.5.

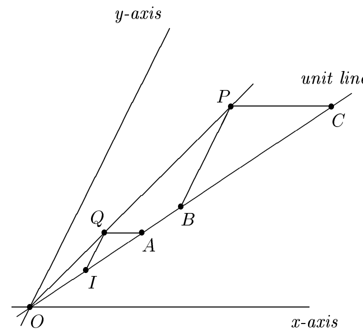

Figure 1.6.

ring. Hence the class of ternary rings contains the class of strong left division rings. It is easy to see that every division ring (with full distributivity) is strong. Let $\mathcal{T}$, $\mathcal{LD}$, $\mathcal{D}$, $\mathcal{SF}$ and $\mathcal{F}$ denote respectively the classes ternary rings, strong left division rings, division rings, skew fields and fields. Then we have the following chain if inclusions:

$$\mathcal{T} \supseteq \mathcal{LD} \supseteq \mathcal{D} \supseteq \mathcal{SF} \supseteq \mathcal{F}.$$

6.3 The properties of Desargues and Pappus

The theorems of Desargues and Pappus, known from Euclidean geometry, turn out to hold in a more general, affine setting. While retaining their elementary nature, these properties of the Euclidean plane will lose their status as theorems when taken in arbitrary affine planes, as they may or may not hold true depending on the affine plane concerned. The Desargues and Pappus properties deal with configurations of six points, in the former case lying in pairs on three lines, and in the latter case lying in triples on two lines. We distinguish cases where the lines (i) are parallel, or (ii) are mutually incident. The specific geometric properties thus described will correspond to specific classes of the algebraic structures described above.

DEFINITION 1.22 *An affine plane satisfies the* ***First Desargues Property*** D_1*, if the following holds (see Fig. 1.7).*

$$\begin{aligned}\mathsf{D}_1: \ \big(\neg\mathbf{Col}(AA'B) \wedge \neg\mathbf{Col}(AA'C) \wedge AA' \shortparallel BB' \wedge AA' \shortparallel CC' \\ \wedge\ AB \shortparallel A'B' \wedge AC \shortparallel A'C'\big) \rightarrow\ BC \shortparallel B'C'\end{aligned}$$

The Euclidean plane satisfies D_1. As an example (see Blumenthal, 1961) of a plane that does not satisfy D_1, consider the real plane $\mathbb{R}^2$

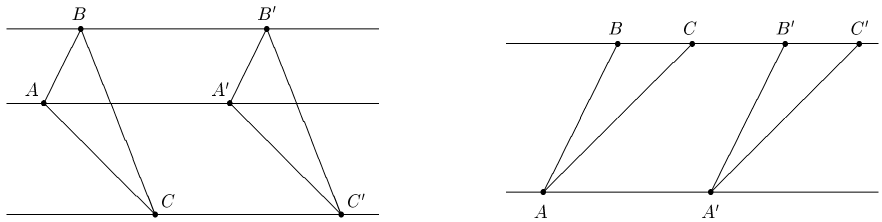

Figure 1.7.

with lines modified as follows. Any line of either undefined slope or non-positive slope will be left unaltered, but any line $y = mx + c$ with strictly positive slope $m > 0$ is changed to the union of the two rays

$$y = mx + c \quad \text{when} \quad y < 0,$$
$$y = \tfrac{1}{2}mx + \tfrac{1}{2}c \quad \text{when} \quad y \geq 0.$$

It is easy to see that this modified plane is affine. Fig. 1.8 gives a configuration of points that falsify D_1.

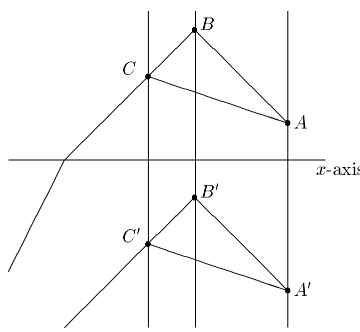

Figure 1.8.

We have the following representation theorem.

THEOREM 1.23 *If an affine plane* $\mathfrak{A}$ *satisfies* D_1 *then* $\mathfrak{A} \cong \mathbf{A}(\mathfrak{F})$ *for some strong left division ring* $\mathfrak{F} \in \mathcal{LD}$*. Conversely, if* $\mathfrak{F} \in \mathcal{LD}$ *is any strong left division ring, then* $\mathbf{A}(\mathfrak{F})$ *satisfies* D_1*.*

The First Desargues Property allows us to give the linear equation for any line with slope m and y-intercept c. Say that a ternary ring $(R; 0, 1, T)$ is **linear** if $T(a, b, c) = a \cdot b + c$ for all $a, b, c \in R$ (where $+$ and $\cdot$ are interpreted in the expanded structure $(R; 0, 1, +, \cdot, T)$).

Theorem 1.24 *An affine plane $\mathfrak{A}$ satisfies the property* D_1 *if and only if every ternary ring* $(\Gamma; 0, 1, T)$ *attached to* $\mathfrak{A}$ *is linear.*

If $\mathfrak{A}$ is an affine plane satisfying D_1, then every line of undefined slope has the form $x = a$ while the line of slope m with y-intercept c has equation $y = m \cdot x + c$. This concludes the task of finding a linear equation for every line of the plane.

Let $\mathbf{P}$ be the quaternary **parallelogram relation** defined by

$$\mathbf{P}(ABCD) \Leftrightarrow AB \parallel CD \wedge AC \parallel BD.$$

The property D_1 guarantees that $\mathbf{P}$ will be transitive in the sense

$$\mathbf{P}(ABCD) \wedge \mathbf{P}(ABEF) \Rightarrow \mathbf{P}(CDEF)$$

where the points are so as to exclude obvious degenerate cases.

Definition 1.25 *An affine plane is said to satisfy the* ***Second Desargues Property*** D_2 *if the following holds (see Fig. 1.9).*

$$\begin{aligned} \mathsf{D}_2 :\ \big(&\mathrm{Diff}_7(OABCA'B'C') \wedge \neg\mathbf{Col}(ABC) \wedge \neg\mathbf{Col}(A'B'C') \\ &\wedge \mathbf{Col}(OAA') \wedge \mathbf{Col}(OBB') \wedge \mathbf{Col}(OCC') \wedge BC \parallel B'C' \\ &\wedge AC \parallel A'C' \wedge AC \parallel OB\big) \rightarrow\ AB \parallel A'B' \end{aligned}$$

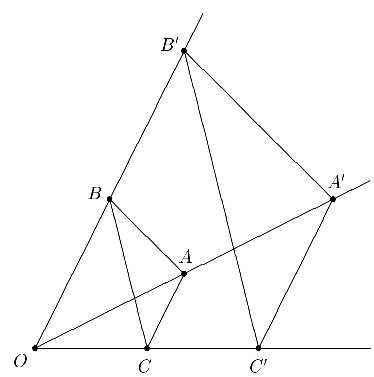

Figure 1.9.

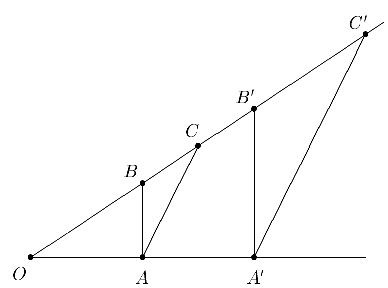

Figure 1.10.

D_2 (also known in the German-language literature as *Trapezdesargues*) endows the attached ternary ring with right distributivity.

Theorem 1.26 *Let an affine plane* $\mathfrak{A}$ *satisfy both the properties* D_1 *and* D_2*. Then every ternary ring attached to* $\mathfrak{A}$ *is a division ring.*

Definition 1.27 *An affine plane is said to satisfy the* ***Third Desargues Property*** D_3 *if the following holds (see Fig. 1.9 and Fig. 1.10):*

$$\begin{aligned}\mathsf{D}_3: \ \big((O \neq A, B, C, A', B', C') \wedge \neg\mathbf{Col}(AA'B) \wedge \neg\mathbf{Col}(AA'C) \\ \wedge\ \mathbf{Col}(OAA') \wedge \mathbf{Col}(OBB') \wedge \mathbf{Col}(OCC') \\ \wedge\ AB \parallel A'B' \wedge AC \parallel A'C'\big) \rightarrow\ BC \parallel B'C'\end{aligned}$$

Clearly $\mathsf{D}_3 \Rightarrow \mathsf{D}_2$ and it can also be shown that $\mathsf{D}_3 \Rightarrow \mathsf{D}_1$.

D_3 endows the ternary ring attached to a plane with associative multiplication. We have the following representation theorem.

Theorem 1.28 *If an affine plane* $\mathfrak{A}$ *satisfies* D_3 *then* $\mathfrak{A} \cong \mathbf{A}(\mathfrak{F})$ *for some skew field* $\mathfrak{F} \in \mathcal{SF}$*. Conversely, if* $\mathfrak{F} \in \mathcal{SF}$ *is any skew field, then* $\mathbf{A}(\mathfrak{F})$ *satisfies* D_3*.*

For any point O, let $\mathbf{T}_O$ be the quaternary **trapezium relation**:

$$\mathbf{T}_O(ABCD) \Leftrightarrow \mathbf{Col}(OAB) \wedge \mathbf{Col}(OCD) \wedge AC \parallel BD.$$

The property D_3 guarantees that $\mathbf{T}_O$ will be transitive in the sense

$$\mathbf{T}_O(ABCD) \wedge \mathbf{T}_O(ABEF) \Rightarrow \mathbf{T}_O(CDEF)$$

where the points are taken so as to exclude obvious degenerate cases.

In particular, note that in any affine plane satisfying the Third Desargues Property the midpoint of a line segment AB, being the intersection point of the diagonals AB and CD of any parallelogram $ACBD$, is definable in terms of A and B.

As shown in Szmielew, 1983, if the plane satisfies D_3, then the coordinate ternary ring attached to the plane is invariant, up to isomorphism, of the coordinate system used; equivalently, every skew field can be restored uniquely from the affine plane over it. Formally:

Theorem 1.29 *(Szmielew, 1983, Sec. 4.6)*
If $\mathfrak{F}, \mathfrak{T}$ *are skew fields such that* $\mathbf{A}(\mathfrak{F}) \cong \mathbf{A}(\mathfrak{T})$ *then* $\mathfrak{F} \cong \mathfrak{T}$*.*

Corollary 1.30 *If* $\mathfrak{A}$ *satisfies* D_3 *and* OXY*,* $O'X'Y'$ *are two coordinate systems in* $\mathfrak{A}$ *then* $\mathbf{T}_{OXY}(\mathfrak{A}) \cong \mathbf{T}_{O'X'Y'}(\mathfrak{A})$*.*

Hereafter, whenever $\mathfrak{A}$ satisfies D_3 we will denote the unique coordinate ring attached to $\mathfrak{A}$ by $\mathbf{T}(\mathfrak{A})$.

Definition 1.31 *An affine plane satisfies the* ***First Pappus Property*** P_1 *if the following holds (see Fig. 1.11).*

$$\begin{aligned}\mathsf{P}_1: \ \big(\mathbf{Col}(ABC) \wedge \mathbf{Col}(A'B'C') \wedge AB \parallel A'B' \\ \wedge\ AB' \parallel A'B \wedge AC' \parallel A'C\big) \rightarrow\ BC' \parallel B'C.\end{aligned}$$

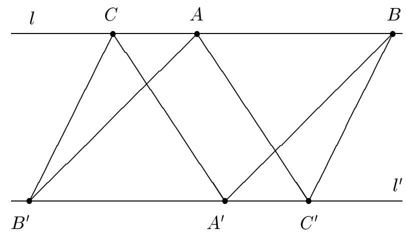

Figure 1.11.

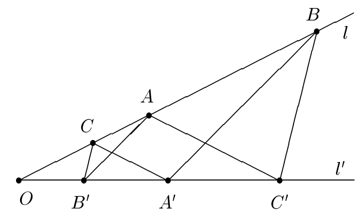

Figure 1.12.

DEFINITION 1.32 *An affine plane satisfies the* ***Second Pappus Property*** P_2 *if the following holds (see Fig. 1.12).*

$$\mathsf{P}_2: \ \big((O \neq A, B, C, A', B', C') \wedge \mathbf{Col}(OABC) \wedge \mathbf{Col}(OA'B'C') \\ \wedge\ AB \neq A'B' \wedge AB' \shortparallel A'B \wedge AC' \shortparallel A'C\big) \rightarrow\ BC' \shortparallel B'C$$

It can be shown that $\mathsf{P}_2 \Rightarrow \mathsf{P}_1$. A famous theorem by Hessenberg establishes the implication $\mathsf{P}_2 \Rightarrow \mathsf{D}_3$. In fact, the following string of implications holds:

$$\mathsf{P}_2 \Rightarrow \mathsf{D}_3 \Rightarrow \mathsf{D}_1 \Rightarrow \mathsf{P}_1.$$

From algebraic considerations one can also derive

$$\mathsf{D}_1 \not\Rightarrow \mathsf{D}_3 \not\Rightarrow \mathsf{P}_2.$$

For instance, the affine plane over the skew field of quaternions satisfies D_3, but not P_2, because of Theorem 1.33 below. It remains an open problem whether P_1 implies (and hence is equivalent to) D_1. See Szmielew, 1983 and Menghini, 1991 for further details.

The property P_2 endows the attached ternary ring with commutative multiplication, and the following representation theorem holds.

THEOREM 1.33 *If an affine plane* $\mathfrak{A}$ *satisfies* P_2 *then* $\mathfrak{A} \cong \mathbf{A}(\mathfrak{F})$ *for some field* $\mathfrak{F}$*. Conversely, if* $\mathfrak{F}$ *is a field, then* $\mathbf{A}(\mathfrak{F})$ *satisfies* P_2*.*

In this section we have followed the terminology from Blumenthal, 1961. Other names for the First and Third Desargues Properties, used in the literature are respectively the Minor, or Weak, and Major, or Strong, Desargues Properties; likewise for the First and Second Pappus Properties (Szmielew, 1983). Hereafter, by 'the Desargues Property' we will mean the Third Desargues Property, and by 'the Pappus Property' we will mean the Second Pappus Property. Accordingly, we will speak about *Desarguesian* and *Pappian* affine planes.

Finally, we note that the Desargues and Pappus properties of affine planes have precise analogues for projective planes, satisfying the same relationships with their algebraic counterparts. In fact, the projective versions of Desargues and Pappus properties are simpler, since they need not take into account the cases of parallel vs intersecting lines. For instance, all affine Desargues' properties turn out to be particular cases in projective extensions of affine planes of the *projective Desargues' property* which simply states that '*If two triangles are perspective from a point (meaning that the three pairs of respective vertices are co-punctual), then they are perspective from a line (meaning that the three intersecting points of the respective pairs opposite sides of these pairs of vertices are collinear).*' Actually, this property holds in a projective plane iff it can be embedded into a projective 3D-space. Likewise, the two affine Pappus properties are combined in one projective Pappus property. For more details, see e.g. Blumenthal, 1961, Hartshorne, 1967, Mihalek, 1972, Blumenthal and Menger, 1970, Hughes and Piper, 1973.

6.4 Analytic geometry and affine transformations of affine planes over a field

Affine planes with the Pappus Property are close enough to the real affine plane that one can introduce not only coordinatization, but even develop analytic geometry of points and lines in them. In fact, for most of what follows it suffices to assume the Desargues Property, i.e. to consider planes $\mathbf{A}(\mathfrak{F})$, where $\mathfrak{F}$ is a skew field, but to avoid having to deal with the non-commutative multiplication, we assume that $\mathfrak{F}$ is a field. Recall from Sec. 6.1 (see also e.g. Gemignani, 1971, Sec. 3 or Blumenthal, 1961, Sec. V.9) that, given a coordinate system OXY in such an affine plane $\mathbf{A}(\mathfrak{F})$, any line l is determined by an equation $y = ax + m$ if not parallel to the line OY, otherwise by $x = c$, where $a, m, c \in \mathfrak{F}$ are fixed parameters. In either case, the line has a *general equation* $ax + by + c = 0$ where at least one of a, b is not 0, and conversely, every such equation represents a line in $\mathbf{A}(\mathfrak{F})$ in the standard analytic geometric sense. Furthermore, a change of the coordinate system to a new one $O'X'Y'$, with coordinate axes $O'X'$ and $O'Y'$ having equations in the old system respectively $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, leads to change of the coordinates (x, y) of a given point in the plane according to the following equations:

$$x' = u(a_1x + b_1y + c_1),\ y' = v(a_2x + b_2y + c_2),$$

where $u = (a_1e_x + b_1e_y + c_1)^{-1}$ and $v = (a_2e_x + b_2e_y + c_2)^{-1}$ where (e_x, e_y) are the coordinates of the new unit point I' in the old coordinate system OXY. The non-parallelism of the new coordinate axes is analytically expressed by the condition $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \neq 0$.

Thus, change of the coordinate system in $\mathbf{A}(\mathfrak{F})$ can be represented (after multiplying out in the equations above) as a transformation of the plane, determined by **affine equations**:

$$\alpha(x) = ax + by + c, \ \alpha(y) = a'x + b'y + c',$$

where $a, b, c, a', b', c' \in \mathfrak{F}$ are such that $ab' - a'b \neq 0$. Such transformation is called an **affine transformation**, or an **affinity**. Conversely, we will see further that every affine transformation can be viewed as a change of the coordinate system.

It is easy to see that every affine transformation is a collineation on the plane. Moreover, using coordinatization, we can characterize explicitly *all* collineations in an affine plane over a field $\mathbf{A}(\mathfrak{F})$. First, recall that collineations of linear spaces preserve incidence, and therefore parallelism, of lines, and collinearity of points. Actually, a bijection α of the points in the plane is a collineation iff it preserves line parallelism, i.e. if α maps the point P to the point P^α then:

$$AB \parallel CD \ \Leftrightarrow \ A^\alpha B^\alpha \parallel C^\alpha D^\alpha.$$

Now, consider a collineation α in $\mathbf{A}(\mathfrak{F})$ and let OXY be any coordinate system in $\mathbf{A}(\mathfrak{F})$. Since α preserves every line (as a set of points), in particular the unit line, it determines a bijection $h : \mathfrak{F} \longrightarrow \mathfrak{F}$ by sending the point from the unit line (x, x) to the point $(h(x), h(x))$. Recall, that addition and multiplication in $\mathfrak{F}$ were geometrically defined in Sec. 6.2 on the unit line by means of the 'affine operations' (Sec. 5.7) of taking the intersection point of two lines, producing the line through two points, and producing the line parallel to a given line through a given point. These constructions are preserved by collineations, and therefore h is an *automorphism* of $\mathfrak{F}$. Now, for any point P with coordinates (x, y) in the system OXY, its image under α is the point P^α with coordinates $(h(x), h(y))$ in the system $O'X'Y'$, because the point P can be obtained (see Sec. 6.1) from the points (x, x) and (y, y) by affine operations.

We can now obtain an explicit algebraic characterization of the collineation α. Suppose the images of O, X, Y under α are $O'(c, c'), X'(a + c, a' + c'), Y'(b + c, b' + c')$. (Note that such $a, b, c, a', b', c' \in \mathfrak{F}$ always exist.) It is easy to check that O', X', Y' are non-collinear iff $ab' \neq a'b$. Now, following the construction on Fig.1.2 (or, as a standard exercise in linear algebra) one can compute the coordinates of $\alpha(P)$ in OXY:

$$\alpha(x) = ah(x) + bh(y) + c, \ \alpha(y) = a'h(x) + b'h(y) + c'.$$

Conversely, it is immediate to check that every mapping defined by such equations in a given coordinate system, where h is an automorphism of $\mathfrak{F}$ and $ab' - a'b \neq 0$, is a collineation.

Thus, we have obtained the following (see Gemignani, 1971, Sec. 3):

THEOREM 1.34 *A mapping α in the plane $\mathbf{A}(\mathfrak{F})$, where $\mathfrak{F}$ is a field, is a collineation, iff it can be defined in some coordinate system by equations*

$$\alpha(x) = ah(x) + bh(y) + c,\ \alpha(y) = a'h(x) + b'h(y) + c',$$

where h is an automorphism of $\mathfrak{F}$ and $a, b, c, a', b', c' \in \mathfrak{F}$ are such that $ab' \neq a'b$.

Therefore, every collineation of $\mathbf{A}(\mathfrak{F})$ is uniquely determined by its action on any three non-collinear points *O, X, Y in the plane, i.e. by their images O', X', Y', and any mapping in $\mathbf{A}(\mathfrak{F})$ that sends the three non-collinear points O, X, Y respectively to three non-collinear points $O'(c, c'), X'(a + c, a' + c'), Y'(b + c, b' + c')$ can be uniquely extended to a collineation of $\mathbf{A}(\mathfrak{F})$ defined by the equations above.*

We now see that affine transformations form a special case of collineations, corresponding to the identity automorphism of $\mathfrak{F}$.

Note that the affinities of a plane form a subgroup of its group of collineations. In the case when $\mathfrak{F}$ is rigid, i.e. has no non-trivial automorphisms, as is the field of reals $\mathbb{R}$, every collineation in $\mathbf{A}(\mathfrak{F})$ is an affinity, but in general this need not be the case, e.g. (see Gemignani, 1971, Sec. 3.2) the complex conjugate mapping $h(z) = \overline{z}$ is an automorphism of the field of complex numbers $\mathbb{C}$, and therefore any collineation of $\mathbf{A}(\mathbb{C})$ associated with h , e.g. $(x, y) \rightarrow (\overline{x}, \overline{y})$, is not an affinity.

A particular case of affine transformations is **dilation** (or **dilatation**). This is a collineation δ which sends every line to a parallel one, i.e.,

$$A^\delta B^\delta \parallel AB.$$

The set of dilations of an affine plane $\mathfrak{A}$ will be denoted as $\mathsf{Dil}(\mathfrak{A})$. It can be easily shown (see Gemignani, 1971; Behnke et al., 1974; Coxeter, 1969; Hughes and Piper, 1973) that every dilation of $\mathbf{A}(\mathfrak{F})$ can be defined in a suitable coordinate system by equations

$$\alpha(x) = ax + c,\ \alpha(y) = ay + c',$$

for some $a, c, c' \in \mathfrak{F}$ such that $a \neq 0$. Therefore, if a dilation is different from the identity dilation ι (for which every point is a fixed point), then it has either no fixed points (if $a = 1$ and $(c, c') \neq (0, 0)$) or exactly one fixed point $C((1-a)^{-1}c, (1-a)^{-1}c')$; accordingly, it will be called respectively a **translation** or **homothety with center** C. The set of all translations of $\mathfrak{A}$ will be denoted $\mathsf{Tr}(\mathfrak{A})$ while the set of homotheties with center C will be denoted $\mathsf{Ht}_\mathrm{C}(\mathfrak{A})$; the set of all homotheties will be denoted $\mathsf{Ht}(\mathfrak{A})$. When the plane is fixed, we will sometimes omit it from these notations. The identity dilation is taken by definition as both a translation as well as a homothety with any point taken as its center.

If α is any non-identity translation and P, Q are any points in the plane then $PP^\alpha \parallel QQ^\alpha$. Hence, all lines PP^α lie in the same 'direction', which will be called the **direction** of the translation α. Given any line l, we define the **set of translations with direction** l:

$$\mathsf{Tr}_l := \{\alpha \in \mathsf{Tr} : \alpha \text{ has direction } l\} \cup \{\iota\},$$

Another important class of affine transformations is the class of **rotations**, given by equations

$$\alpha(x) = ax + by,\ \ \alpha(y) = a'x + b'y,$$

for some $a, b, a', b' \in \mathfrak{F}$ such that $\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} = 1$. Rotations are not dilations, but have a fixed point, viz. $O(0,0)$.

For any line l and any point O in a given affine plane, we have:

$$\mathsf{Tr}_l \leq \mathsf{Tr} \leq \mathsf{Dil} \leq \mathsf{Aut},\quad \mathsf{Ht}_\mathsf{O} \leq \mathsf{Dil} \leq \mathsf{Aut},$$

where $\leq$ means subgroup.

A set of dilations D is **transitive** on a set of points S if, for every $A, B \in S$ the equation $A^\delta = B$ has a solution for δ in D. Given any line l, the set Tr_l will be called transitive if it is transitive on the set of points in the line l. The set Tr will be called transitive if it is transitive on the entire universe of points. The set Ht_O will be called transitive when, for every line l containing O, the set Ht_O is transitive on the set of points $l \setminus \{O\}$. Finally Ht will be called transitive when Ht_O is transitive for every point O.

A set of dilations D is called **commutative** when compositions of dilations in D commute. It can be shown that (i) the set Tr will be commutative if and only if the set Tr_l is commutative for every l, and (ii) if Tr is transitive then it is also commutative.

Transitivity and commutativity of the dilations of a given affine plane are closely related to the Desargues and Pappus properties satisfied in that plane. For a more comprehensive discussion on these relations, the reader is referred to Szmielew, 1983, from where we cite the following.

Theorem 1.35 *An affine plane satisfies*

i) D_1 *iff the set* Tr *is transitive;*

ii) D_3 *iff the set* Ht *is transitive;*

iii) P_2 *iff the set* Ht_O *is transitive and commutative for every point* O*.*

7. On the first-order theories of affine and projective spaces

Here we will discuss some logical results about definability in affine spaces and axiomatization and decidability of the first-order theories of affine and projective spaces, that can be obtained as consequences from the method of coordinatization.

7.1 On affine relations in affine spaces

Two lines x and y in an affine structure are called **crossing** or **skew**, denoted $x \bowtie y$, if they are not incident and not parallel. Thus, each of the relations **Int** , $\shortparallel$ and $\bowtie$ in affine spaces is definable in terms of incidence between a point and a line. Therefore, affine spaces and planes can be defined with these relations taken as primitives, but that would not enhance the expressiveness of the language.

Note that every relation in an affine plane, definable in terms of incidence alone, is preserved under collineations. Therefore, using collineations one can show e.g. that orthogonality of lines in $\mathbb{R}^n$ is not definable in terms of the relation of incidence alone, for any $n \geq 2$. Indeed, the mapping in $\mathbb{R}^n$ that halves the first coordinate of a point is clearly a collineation, but it does not preserve orthogonality.

Further, we can define an **affine relation** in affine planes as one which is preserved under affine transformations. Thus, incidence and parallelism are affine relations, while orthogonality is not.

Note on the other hand, that many not obviously affine concepts can be defined in affine terms, or constructed with purely affine means, in affine planes satisfying special additional properties, e.g. in $\mathbf{A}(\mathbb{R})$. For example, the equidistance relation on strictly parallel line segments, denote it here as $\equiv_1$, is given by the formula

$$X_1X_2 \equiv_1 Y_1Y_2 \;\Leftrightarrow\; X_1 = X_2 \wedge Y_1 = Y_2 \vee \Big(\mathbf{l}(X_1, X_2) \parallel \mathbf{l}(Y_1, Y_2)$$

$$\wedge \Big(\mathbf{l}(X_1, Y_1) \parallel \mathbf{l}(X_2, Y_2) \vee \mathbf{l}(X_1, Y_2) \parallel \mathbf{l}(X_2, Y_1)\Big)\Big)$$

(see Fig. 1.13), while equidistance on arbitrary parallel line segments, denote it here as $\equiv_2$, is given by the formula

$$X_1X_2 \equiv_2 Y_1Y_2 \;\Leftrightarrow\; \exists Z_1 \exists Z_2 \,(X_1X_2 \equiv_1 Z_1Z_2 \wedge Y_1Y_2 \equiv_1 Z_1Z_2)$$

(see Fig. 1.14). In particular, the midpoint operation between two points, $\oplus$, is given by the formula

$$X = Y_1 \oplus Y_2 \;\Leftrightarrow\; XY_1 \equiv_2 XY_2.$$

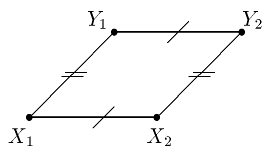

Figure 1.13.

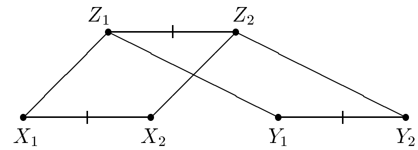

Figure 1.14.

As we will note in Sec. 8, betweenness of points in $\mathbb{R}^n$ for $n \geq 1$ is an affine relation, but is not definable in terms of incidence alone, because it may not be preserved by collineations which are not affinities.

7.2 Coordinatization as logical interpretation

Every coordinatization of an affine plane $\mathfrak{A}$ defines an interpretation $\mathfrak{A}$ in the corresponding ternary ring $\mathfrak{T} = (\Gamma; 0, 1, T)$ attached to it. Indeed, if we treat the plane as a collinearity structure $\langle \mathbf{Po}, \mathbf{Col} \rangle$, then $\mathfrak{A}$ can be 2-dimensionally interpreted in $\mathfrak{T}$ as follows:

i) The domain of $\mathfrak{A}$ interpreted in $\mathfrak{T}$ is given by the formula

$$\psi(x_1, x_2) := (x_1 = x_1).$$

ii) The pair (a, b) in Γ^2 is mapped to the point $P(a, b)$ in $\mathfrak{A}$.

iii) Collinearity of points in $\mathfrak{A}$ is given by the formula

$$\psi_{\mathbf{Col}}(x_1 y_1 x_2 y_2 x_3 y_3) := \wedge_{i,j=1,2,3}\ x_i = x_j \\ \vee\ \exists m \exists c \left(\wedge_{i=1,2,3}\ y_i = T(m, x_i, c)\right).$$

If the plane is regarded as a two-sorted incidence structure $\langle \mathbf{Po}, \mathbf{Li}, \mathbf{I} \rangle$ then $\mathfrak{A}$ can be 4-dimensionally interpreted in $\mathfrak{T}$ as follows:

i) The domain of $\mathfrak{A}$ interpreted in $\mathfrak{T}$ is given by the formula

$$\psi(x_1, x_2, x_3, x_4) := (x_1 = x_1).$$

ii) The quadruple (a, b, c, d) in Γ^4 is mapped to the point $P(a, b)$ if $(a, b) = (c, d)$, and to the line determined by the points $P(a, b)$ and $Q(c, d)$ otherwise.

iii) Incidence of a point and a line in $\mathfrak{A}$ is given by the formula

$$\psi_{\mathbf{I}}(x_1 x_2 x_3 x_4 y_1 y_2 y_3 y_4) := \\ \Big((x_1 = x_3 \wedge x_2 = x_4) \wedge (y_1 \neq y_3 \vee y_2 \neq y_4)\Big) \ \wedge$$

$$\Big(\exists m \exists c\big(x_2 = T(m, x_1, c) \wedge y_2 = T(m, y_1, c) \\ \wedge\, y_4 = T(m, y_3, c)\big) \;\vee\; (x_1 = y_1 \wedge y_1 = y_3)\Big)$$

(informally: $\bar{x}$ is a point, $\bar{y}$ is a line and $\bar{x}$ lies on $\bar{y}$).

Alternatively, lines in planes satisfying D_1 can be interpreted in the coordinate ring as triples of coefficients, by their general equations.

Conversely, any ternary ring can be interpreted in the affine plane over it, by taking the points on the unit line as a domain of the interpretation, and defining addition and multiplication by means of first-order formulae in the language of incidence, constructed following the geometric description of these operations, as described above and illustrated by Fig. 1.5 and Fig. 1.6.

7.3 Decidability and undecidability of affine and projective theories

The interpretations between affine planes and coordinate rings enable effective translation of first-order formulae from one to the other language and transfer of various logical properties between the first order theories of these classes of structures.

THEOREM 1.36 *For every affine plane $\mathfrak{A}$ and every ternary ring $\mathfrak{T}$:*

1. *If the first order theory of $\mathbf{A}(\mathfrak{T})$ is decidable then the first order theory of $\mathfrak{T}$ is decidable, too.*
2. *If the first order theory of $\mathbf{T}_{OXY}(\mathfrak{A})$ is decidable for some coordinate system OXY in $\mathfrak{A}$, then the first order theory of $\mathfrak{A}$, expanded with point-constants for O, X, Y, is decidable, too.*

Given a class of ternary rings $\mathcal{T}$ we denote by $\mathbf{A}(\mathcal{T})$ the class of affine planes over these rings; likewise, given a class of affine planes $\mathcal{A}$ with the Desargues property, we denote by $\mathbf{T}(\mathcal{A})$ the class of ternary rings attached to the planes in $\mathcal{A}$.

THEOREM 1.37 *For every class of ternary rings $\mathcal{T}$ and every class of affine planes $\mathcal{A}$ satisfying the Desargues property the following holds:*

1. *If the first order theory of $\mathbf{A}(\mathcal{T})$ is decidable then the first order theory of $\mathcal{T}$ is decidable, too.*
2. *If the first order theory of $\mathbf{T}(\mathcal{A})$ is decidable, then the first order theory of $\mathcal{A}$ is decidable, too.*

As shown in Tarski, 1949a; Tarski and Mostowski, 1949; Tarski, 1949b and Tarski et al., 1953, the first order theories of all fields, and the field of rationals, are undecidable, while the first order theory of real closed fields, being the same as the first order theory of the field of reals is complete and decidable. Therefore, we obtain the following.

Corollary 1.38

1. *The first order theories of all Pappian affine planes, and of the rational affine plane are undecidable.*

2. *The first order theories of all affine planes over real closed fields, and of the real affine plane are decidable.*

A simple argument shows that if a first-order theory T has an undecidable extension by means of finitely many axioms T', then it is itself undecidable. Indeed, let ϕ be the conjunction of all axioms extending T to T'. Then for any sentence ψ, $T' \vdash \psi$ iff $T \vdash \phi \to \psi$, hence any decision method for T yields a decision method for T'. Thus, we obtain the following results:

Corollary 1.39 *The following first-order theories are undecidable: the theory of all Desarguesian planes; the theory of all affine planes; the theory of all affine spaces; the theory of all linear spaces; the theory of all incidence structures.*

Analogous results hold for first-order theories of projective planes and spaces (see Ziegler, 1982).

7.4 On the axiomatizations of the first-order theories of the real projective and affine planes

The real affine plane $\mathbf{A}(\mathbb{R})$ is simply the Euclidean plane with the standard points, lines and incidence relation. The real projective plane $\mathbf{P}(\mathbb{R})$ can be obtained from $\mathbf{A}(\mathbb{R})$ by the extension construction described earlier, but also e.g. by the well-known *central projection* of the affine plane onto a sphere touching that plane (see Coxeter, 1969).

Here we briefly discuss the questions: *what are the first-order axiomatizations of the real projective and affine planes* $\mathbf{P}(\mathbb{R})$ *and* $\mathbf{A}(\mathbb{R})$ *in the language with incidence?*

The first-order theory of $\mathbb{R}$ has a well-known axiomatization (the theory of real-closed fields, see e.g. Tarski, 1967, Chang and Keisler, 1973). It extends the axioms for fields with the following axiom schemes:

RealFields : *-1 is not a sum of squares*:

$$\forall x_1 \ldots \forall x_n \neg(x_1^2 + \ldots + x_n^2 = -1)$$

for every integer $n > 0$.

RealClosedFields : *Every polynomial of odd degree has a zero*:

$$\forall a_0 \ldots \forall a_n(\neg a_n = 0 \to \exists x(a_0 + a_1 x + \ldots + a_n x^n = 0))$$

for every odd integer $n > 0$.

PythagoreanFields :

$$\forall x \exists y(y^2 = x \vee y^2 = -x).$$

In view of the mutual interpretability between $\mathbb{R}$ and each of $\mathbf{P}(\mathbb{R})$ and $\mathbf{A}(\mathbb{R})$, and the uniqueness of the coordinate field for each of these planes, translating the axioms above to the geometric language should in principle suffice to axiomatize their first-order theories. Still, it is natural to search for *explicit and geometrically meaningful* axiomatizations of the real projective and affine planes, rather than a translation of the axioms of real closed fields to the geometric language.

When betweenness is added to the language, such a complete axiomatization (involving an infinite axiom scheme of continuity) for the real affine plane has been obtained by Szczerba and Tarski, 1965 and Szczerba and Tarski, 1979, and will be presented in Sec. 8. The language with betweenness, however, is substantially more expressive, so the question is: what affine properties of $\mathbb{R}^2$ can be expressed in terms of incidence alone, in projective and affine settings.

We already know that there are rather non-trivial *universal* properties true in $\mathbf{P}(\mathbb{R})$ and $\mathbf{A}(\mathbb{R})$, such as the Pappus property which guarantees that their coordinate ring is a field.

Furthermore, there is a geometrically natural axiom, known as the **Fano** axiom, which is true in $\mathbf{P}(\mathbb{R})$ but does not follow from the Pappus property. In order to state the Fano axiom, we define **complete quadrangle** in a projective plane to be a configuration of 7 points and 6 lines obtained as follows: take 4 points A, B, C, D, no 3 of which are collinear, consider the 6 lines determined by pairs of these points, and add the 3 'diagonal points' of intersection $\mathbf{P}(AB, CD)$, $\mathbf{P}(AC, BD)$ and $\mathbf{P}(AD, BC)$.

Fano : The three diagonal points in any complete quadrangle are never collinear.

The Fano axiom is true in every projective plane over a field of characteristic different from 2 (see e.g. Coxeter, 1969), but it fails in the

Fano plane. An affine version of the Fano axiom can be formulated, too. In the particular case where the denied collinearity is along the infinite line, it claims precisely that the diagonals of every parallelogram in the plane must intersect.

The Pappus property and Fano axiom are the only additional axioms to those for projective planes offered in Coxeter, 1969, Sec. 14.1 for the real projective plane. However, as shown in an exercise following Coxeter, 1969, Sec. 14.1, for every prime p there is a *finite* projective plane $\mathrm{PG}(2,p)$ of p^2+p+1 points and as many lines satisfying all these axioms, so this system is far from complete.

In fact, there are infinitely many other geometric axioms which should be added to the theory of Pappian planes, in order to obtain the complete theory of $\mathbf{A}(\mathbb{R})$, because it follows from results in Szczerba and Tarski, 1979 that the latter theory is not finitely axiomatizable. For further discussion and results on this, see von Plato, 1995; Pambuccian, 2001b. Still, the question of finding an explicit and geometrically natural axiomatizations for $\mathbf{P}(\mathbb{R})$ and $\mathbf{A}(\mathbb{R})$ apparently remains, as far as we know, unclosed. The same questions can be raised about the first-order theories of the n-dimensional real affine spaces $\mathbf{A}(\mathbb{R}^n)$ (with $n \geq 3$) generated over the field of reals.

8. Betweenness structures and ordered affine planes

We now consider the geometric language in which the only primitive relation is the ternary relation $\mathbf{B}$ of betweenness on points. $\mathbf{B}(XYZ)$ means that the points X, Y and Z are collinear and Y lies between X and Z (with possibly Y coinciding with X or Z). The language consisting of the betweenness relation is significantly more expressive than the language with collinearity, and yet, as we will see later, betweenness is very much an affine notion so that it makes sense to add it to the language of affine geometries, as was done by Tarski.

From the results in Sec. 6.4 is easy to see that collineations on the real affine plane preserve ratios of parallel line segments and consequently also betweenness on points, so that axiomatizing the real affine plane using betweenness does not leave the realm of affine geometry.

8.1 Betweenness structures and ordered geometry

A structure $(S;\mathbf{B})$ consisting of a non-empty set S and a ternary relation $\mathbf{B}$ on S will be called a **linear betweenness structure** provided the following axioms are satisfied:

B1 : $\forall X \forall Y \forall Z (\mathbf{B}(XYZ) \vee \mathbf{B}(YZX) \vee \mathbf{B}(ZXY))$ (connectivity)

B2 : $\forall X \forall Y (\mathbf{B}(XYX) \rightarrow X = Y)$

B3 : $\forall X \forall Y \forall Z (\mathbf{B}(XYZ) \rightarrow \mathbf{B}(ZYX))$ (symmetry)

B4 : $\forall U \forall X \forall Y \forall Z (\mathbf{B}(UXY) \wedge \mathbf{B}(UYZ) \rightarrow \mathbf{B}(XYZ))$

(inner transitivity)

B5 : $\forall U \forall X \forall Y \forall Z (X \neq Y \wedge \mathbf{B}(UXY) \wedge \mathbf{B}(XYZ) \rightarrow \mathbf{B}(UYZ))$

(outer transitivity)

The relation $\mathbf{B}$ is called the **betweenness relation** of the linear betweenness structure. In Szmielew, 1983 it is shown that these axioms are independent, although when the cardinality of the set S is different from 4, the axiom B5 becomes redundant.

Linear orderings and linear betweenness structures are closely related as follows. Let a linear ordering $(S; \leq)$ be given. Then the structure $(S; \mathbf{B}_{\leq})$ with $\mathbf{B}_{\leq}$ defined as

$$\mathbf{B}_{\leq}(XYZ) \Leftrightarrow X \leq Y \leq Z \vee Z \leq Y \leq X$$

is a linear betweenness structure. Conversely, let a linear betweenness structure $(S; \mathbf{B})$ be given and take any distinct $A, B, C \in S$ such that $\mathbf{B}(ABC)$. Then the structure $(S; \leq_{\mathbf{B}})$ with $\leq_{\mathbf{B}}$ defined as

$$X \leq_{\mathbf{B}} Y \Leftrightarrow \big(\mathbf{B}(XYB) \wedge \mathbf{B}(XBC)\big) \vee \big(\mathbf{B}(XBC) \wedge \mathbf{B}(ABY)\big) \vee \big(\mathbf{B}(ABY) \wedge \mathbf{B}(BXY)\big)$$

is a linear ordering. The purpose of the parameters A, B, and C is to fix the direction of $\leq_{\mathbf{B}}$, since clearly every linear betweenness structure gives rise to a *pair* of mutually converse linear orderings. In fact, the parameters A, B, C are inessential in the sense that if $A_i, B_i, C_i \in S$ are distinct with $\mathbf{B}(A_i B_i C_i)$ $(i = 1, 2)$ then the orderings $\leq_{\mathbf{B}}$ determined by these two triples of parameters will be identical. Betweenness structures and linear orderings related as above will be called **adjoint**. Thus, every linear betweenness structure has a pair of mutually converse linear orderings adjoint to it, and every linear ordering has a linear betweenness structure adjoint to it.

A linear betweenness structure $(S; \mathbf{B})$ is **dense** if it satisfies the axiom

$$\forall X_1 \forall X_2 (X_1 \neq X_2 \rightarrow \exists Y (\mathbf{B}(X_1 Y X_2) \wedge X_1 \neq Y \wedge X_2 \neq Y)).$$

A linear betweenness structure $(S;\mathbf{B})$ is called **Dedekind complete** if it satisfies the second-order axiom

$$\forall\mathcal{P}_1\forall\mathcal{P}_2\Big(\exists Y\mathbf{B}(Y\mathcal{P}_1\mathcal{P}_2)\rightarrow\exists Z\mathbf{B}(\mathcal{P}_1 Z\mathcal{P}_2)\Big),$$

stating that if all points of the set $\mathcal{P}_1$ precede all points of the set $\mathcal{P}_2$, i.e. if $\mathbf{B}(YX_1X_2)$ for all $X_1\in\mathcal{P}_1$ and $X_2\in\mathcal{P}_2$, then there is a point which separates $\mathcal{P}_1$ and $\mathcal{P}_2$. Accordingly, a linear ordering $(S;\leq)$ is called **Dedekind complete** if the following second-order axiom is satisfied:

$$\forall\mathcal{P}_1\forall\mathcal{P}_2\Big(\bigwedge_{i=1,2}\mathcal{P}_i\neq\emptyset\wedge\mathcal{P}_1\cup\mathcal{P}_2=S\wedge\mathcal{P}_1\cap\mathcal{P}_2=\emptyset\wedge\mathcal{P}_1\leq\mathcal{P}_2 \\ \rightarrow\ \exists X\big(\mathcal{P}_1\leq X\leq\mathcal{P}_2\big)\Big).$$

This axiom states that if $\mathcal{P}_1$ and $\mathcal{P}_2$ are non-empty disjoint sets that cover the entire set S and if all elements in $\mathcal{P}_1$ are dominated by all elements in $\mathcal{P}_2$, then there is a point that separates $\mathcal{P}_1$ and $\mathcal{P}_2$. Szmielew, 1983 shows that if a linear betweenness structure $(S;\mathbf{B})$ is adjoint to a linear ordering $(S;\leq)$ then $(S;\mathbf{B})$ will be Dedekind complete if and only if $(S;\leq)$ is Dedekind complete.

A linear betweenness structure (respectively a linear ordering) is called **continuous** if it is dense and Dedekind complete.

Now, a betweenness relation is defined on a collinearity structure $\langle\mathbf{Po},\mathbf{Col}\rangle$ by defining it on every line in that structure. Thus we deal with a geometric structure $\mathfrak{C}=\langle\mathbf{Po},\mathbf{Col},\mathbf{B}\rangle$ such that

1. $\mathbf{B}$ is a linear ternary relation on $\mathbf{Po}$, i.e. $\mathbf{B}\subseteq\mathbf{Col}$;
2. $(\mathbf{l}(X,Y);\mathbf{B})$ is a linear betweenness structure for every line $\mathbf{l}(X,Y)$ from $\mathbf{Li}(\mathfrak{C})$.

Point collinearity can be defined in terms of betweenness:

$$\mathbf{Col}(X_1X_2X_3):=\bigvee_{\neq(i,j,k)}\mathbf{B}(X_iX_jX_k).$$

Then, the axioms for betweenness in a collinearity structure are adjusted by adding the axiom B6 and replacing the axiom B1 with the axiom B7:

B6 : $\forall X\forall Y\forall Z(\mathbf{B}(XYZ)\rightarrow\mathbf{Col}(XYZ))$ (linearity)

B7 : $\forall X_1\forall X_2\forall X_3(\mathbf{Col}(X_1X_2X_3)\rightarrow\bigvee_{\neq(i,j,k)}\mathbf{B}(X_iX_jX_k))$

(connectivity on lines)

Consequently, betweenness can serve as the only primitive relation in ordered collinearity structures and their axioms can be phrased exclusively in terms of betweenness. A collinearity structure with a betweenness relation imposed on it will be called an **ordered collinearity geometry**. In case the collinearity structure has dimension ≥ 2, and in particular when dealing with the real collinearity plane, it turns out that the axiom B2 becomes redundant.

The betweenness relation has great expressive power; as will be seen in the next section, betweenness was both Veblen's and Tarski's primitive of choice for formalizing affine notions in first-order logic. For example, given points X and Y one can define the following types of line segment: *closed intervals* $[X, Y] := \{Z : \mathbf{B}(XZY)\}$; open intervals $(X, Y) := [X, Y] \setminus \{X, Y\}$; the ray from X away from Y (when $X \neq Y$) $X/Y := \{Z : \mathbf{B}(YXZ)\}$; the line containing X and Y (when $X \neq Y$) $XY := \{Z : \mathbf{B}(ZXY)\} \cup \{Z : \mathbf{B}(XZY)\} \cup \{Z : \mathbf{B}(XYZ)\}$, etc.

8.2 Definability of betweenness and order in affine planes

Note that even if a linear betweenness structure is defined on every line in a collinearity structure, that may not suffice to have a 'global' betweenness relation on the entire structure, satisfying the axioms B2 - B7, because the linear betweenness relations may not be synchronizable across the structure. To guarantee that, we should guarantee that betweenness is preserved under parallel projections between lines. This property is formalized by the following three axioms:

Pasch : (Invariance - see Fig. 1.15.)

$$\Big(\neg\mathbf{Col}(X_1X_2X_3Y_1Y_2Y_3) \wedge \mathbf{Col}(X_1X_2X_3) \wedge \mathbf{Col}(Y_1Y_2Y_3)$$
$$\wedge\ \mathbf{B}(X_1X_2X_3) \wedge \textstyle\bigwedge_{i,j=1,2,3} X_iY_i \parallel X_jY_j\Big) \ \rightarrow\ \mathbf{B}(Y_1Y_2Y_3)$$

oPasch : (Outer invariance - see Fig. 1.16.)

$$\Big(\neg\mathbf{Col}(X_1X_2X_3Y_2Y_3) \wedge \mathbf{Col}(X_1X_2X_3) \wedge \mathbf{Col}(X_1Y_2Y_3)$$
$$\wedge\ \mathbf{B}(X_1X_2X_3) \wedge X_2Y_2 \parallel X_3Y_3\Big) \ \rightarrow\ \mathbf{B}(X_1Y_2Y_3)$$

iPasch : (Inner invariance - see Fig. 1.17.)

$$\Big(\neg\mathbf{Col}(X_1X_2X_3Y_1Y_3) \wedge \mathbf{Col}(X_1X_2X_3) \wedge \mathbf{Col}(Y_1X_2Y_3)$$

$$\wedge\ \mathbf{B}(X_1X_2X_3) \wedge X_1Y_1 \parallel X_3Y_3\Big)\ \rightarrow\ \mathbf{B}(Y_1X_2Y_3)$$

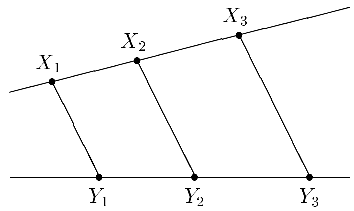

Figure 1.15.

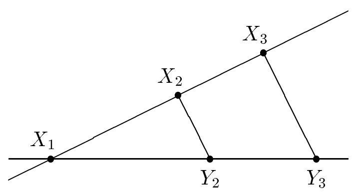

Figure 1.16.

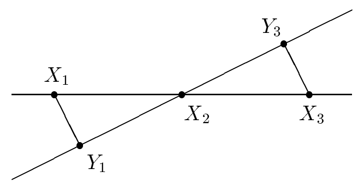

Figure 1.17.

It turns out (see Szmielew, 1983) that in an ordered affine plane, not only are the three Pasch axioms pairwise equivalent, but they are also all equivalent to the Pasch axiom Ax.B5 to be described below.

We now briefly investigate the relationship between ordered affine planes and ordered division rings (see Szmielew, 1983 for details). Let an ordered strong left division ring $\mathfrak{F} = (F; 0, 1, +, \cdot, \leq)$ be given, and let $\mathfrak{F}^- = (F; 0, 1, +, \cdot)$ be the unordered reduct of $\mathfrak{F}$ and $\mathbf{B}$ be the betweenness relation on F adjoint to $\leq$. Then $\mathbf{A}(\mathfrak{F}^-)$ is an affine plane which satisfies the axiom D_1. We can treat $\mathbf{A}(\mathfrak{F}^-)$ as a collinearity structure $\langle F^2, \mathbf{Col}\rangle$. Put $\mathbf{A}(\mathfrak{F}) = \langle F^2, \mathbf{Col}, \mathbf{B}_{\mathfrak{F}}\rangle$, where $\mathbf{B}_{\mathfrak{F}}$ is a ternary relation on the points in $\mathbf{A}(\mathfrak{F}^-)$ defined by stipulating that for any $A = (x_A, y_A), B = (x_B, y_B), C = (x_C, y_C) \in F^2$,

$$\mathbf{B}_{\mathfrak{F}}(ABC) \ \text{ iff } \ \mathbf{Col}(ABC)\ \&\ \mathbf{B}(x_Ax_Bx_C)\ \&\ \mathbf{B}(y_Ay_By_C).$$

On the other hand, let $\mathfrak{A} = \langle \mathbf{Po}, \mathbf{Col}, \mathbf{B}\rangle$ be any ordered affine plane satisfying the axiom D_1 and let $\mathfrak{A}^- = \langle \mathbf{Po}, \mathbf{Col}\rangle$ be the unordered reduct of $\mathfrak{A}$. Fixing some coordinate system OXY in $\mathfrak{A}^-$, the ternary ring $\mathfrak{F}_{OXY}(\mathfrak{A}^-) = (F; 0, 1, +, \cdot)$ attached to $\mathfrak{A}^-$ will be a strong left division ring. Since $\mathbf{B}$ is a betweenness relation in $\mathfrak{A}$ then $\mathbf{B}$ will be a betweenness relation on every line in $\mathfrak{A}$, and hence also on the set F. Thus $(F; \mathbf{B})$ is a linear betweenness structure and it will have a pair of mutually converse linear orderings adjoint to it. Suppose $\leq$ is the linear ordering on F adjoint to $\mathbf{B}$ *and such that* $0 \leq 1$. Put $\mathfrak{F}_{OXY}(\mathfrak{A}) = (F; 0, 1, +, \cdot, \leq)$.

The property that every line in a plane contains at least three points can be axiomatized as follows:

$\mathsf{B8}:\ \forall X \forall Y \exists Z (Z \neq X \wedge Z \neq Y \wedge \mathbf{Col}(XYZ))$

Szmielew, 1983 gives the following representation results.

THEOREM 1.40 *Let* $\mathfrak{A}$ *be an ordered affine plane satisfying* Pasch *and* B8*. If* $\mathfrak{A}$ *also satisfies* D_1 *(respectively,* D_3*;* P_2*) then* $\mathfrak{A} \cong \mathbf{A}(\mathfrak{F})$ *for some ordered strong left division ring (respectively, skew field; field)* $\mathfrak{F}$*.*

THEOREM 1.41 *If* $\mathfrak{F}$ *is an ordered strong left division ring (respectively, skew field; field) then* $\mathbf{A}(\mathfrak{F})$ *is an ordered affine plane satisfying the axioms* Pasch*,* B8*, and* D_1 *(respectively,* D_3*;* P_2*).*

8.3 Axiomatizing betweenness in $\mathbb{R}^2$

Szczerba and Tarski, 1965 and Szczerba and Tarski, 1979 study the affine fragment AE_2, called the **elementary affine Euclidean geometry**, of the Euclidean plane. AE_2 is the elementary geometry formalized in the language with only the betweenness relation $\mathbf{B}$, where a sentence is valid in AE_2 if and only if it is valid in the Euclidean plane E_2. They give a complete axiomatization of AE_2 which will be outlined below (all axioms below are implicitly universally quantified over all occurring free variables).

Ax.B1 : IDENTITY AXIOM

$$\mathbf{B}(XYX) \rightarrow X = Y$$

Ax.B2 : TRANSITIVITY AXIOM

$$Y \neq Z \wedge \mathbf{B}(XYZ) \wedge \mathbf{B}(YZW) \rightarrow \mathbf{B}(XYW)$$

Ax.B3 : CONNECTIVITY AXIOM

$$V \neq W \wedge \mathbf{B}(VWX) \wedge \mathbf{B}(VWY) \rightarrow (\mathbf{B}(VXY) \vee \mathbf{B}(VYX))$$

Ax.B4 : EXTENSION AXIOM

$$\exists X (X \neq Y \wedge \mathbf{B}(XYZ))$$

Ax.B5 : (OUTER FORM OF) PASCH AXIOM

$$\mathbf{B}(XY'Z) \wedge \mathbf{B}(YZ'Y') \rightarrow \exists X' (\mathbf{B}(ZX'Y) \wedge \mathbf{B}(XZ'X'))$$

(given a triangle $YY'Z$, a point X on the extension of the side $Y'Z$ and a point Z' on the inner side (with respect to X) of the triangle, the line XZ' must intersect the triangle in its outer side (with respect to X) $|YZ|$ - see Fig. 1.18.)

Ax.B6 : Desargues axiom

$$\neg\mathbf{Col}(TXY) \wedge \neg\mathbf{Col}(TXZ) \wedge \neg\mathbf{Col}(TYZ) \wedge \mathbf{B}(TXX') \wedge \mathbf{B}(TYY') \wedge \mathbf{B}(TZZ') \wedge \mathbf{B}(YXU) \wedge \mathbf{B}(Y'X'U) \wedge \mathbf{B}(XZW) \wedge \mathbf{B}(X'Z'W) \wedge \mathbf{B}(YZV) \wedge \mathbf{B}(Y'Z'V) \ \rightarrow \ \mathbf{B}(UVW)$$

(triangles perspective from a point are perspective from a line - see Fig. 1.19.)

Ax.B7 : Lower 2-dimensional axiom

$$\exists X \exists Y \exists Z (\neg\mathbf{B}(XYZ) \wedge \neg\mathbf{B}(YZX) \wedge \neg\mathbf{B}(ZXY))$$

Ax.B8 : Upper 2-dimensional axiom (See Fig. 1.20.)

$$\exists V ((\mathbf{B}(YVZ) \wedge \mathbf{Col}(XVW)) \vee (\mathbf{B}(XVZ) \wedge \mathbf{Col}(YVW)) \vee (\mathbf{B}(XVY) \wedge \mathbf{Col}(ZVW)))$$

As.B9 : Elementary continuity axiom schema

$$\forall \overline{W} (\exists U \forall X \forall Y (\varphi(X, \overline{W}) \wedge \psi(Y, \overline{W}) \rightarrow \mathbf{B}(UXY)) \ \rightarrow \ \exists V \forall X \forall Y (\varphi(X, \overline{W}) \wedge \psi(Y, \overline{W}) \rightarrow \mathbf{B}(XVY)))$$

The variables $\overline{W}$ are distinct from U, V, X, Y, and $\varphi(X, \overline{W})$ and $\psi(Y, \overline{W})$ are first-order formulae over $\mathbf{B}$ with free variables only amongst $X, \overline{W}$ in the case of φ, and $Y, \overline{W}$ in the case of ψ. This schema comprises the *parametrically first-order definable instances of the full second order continuity axiom* (see Ax.11 further). Note that Ax.B9 is an infinite schema, and it cannot be replaced by a finite one, as Tarski has shown.

The axiom system given so far is denoted GA_2 and the geometry it describes is called by Szczerba and Tarski the **general affine geometry**. It does not reflect Euclid's parallel postulate at all and it is shown in Szczerba and Tarski, 1979 that:

- GA_2 is incomplete and has continuum many complete extensions.
- In particular, GA_2 is a proper subtheory of the **elementary affine absolute geometry** AA_2 (the affine fragment in the language of

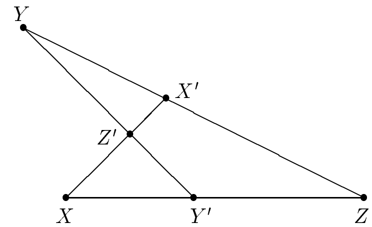

Figure 1.18.

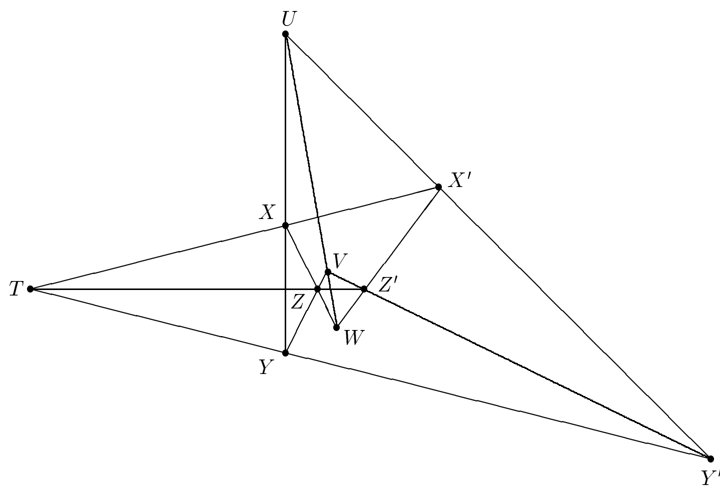

Figure 1.19.

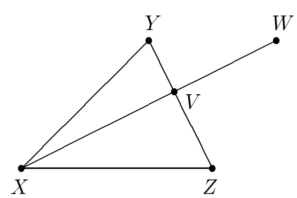

Figure 1.20.

B of the absolute geometry A_2, which is the reduct of the Euclidean geometry obtained by dropping Euclid's parallel postulate).

- However, GA_2 is complete with respect to universal sentences, i.e. if a universal sentence σ is true in *some* model of GA_2 then σ is true in *every* model of GA_2.
- GA_2 is not finitely axiomatizable.

- GA_2 is hereditarily undecidable, meaning that both GA_2, as well as all its subtheories, are undecidable.
- GA_2 is decidable with respect to inductive sentences. Therefore, GA_2 is not an inductive theory.

Here is a form of Euclid's postulate in the language of **B**:

Ax.E : Euclid's axiom

$$Z \neq V \wedge \mathbf{B}(ZVT) \wedge \mathbf{B}(UVW) \;\rightarrow$$
$$\exists X \exists Y (\mathbf{B}(ZUX) \wedge \mathbf{B}(ZWY) \wedge \mathbf{B}(YTX)).$$

The axiom Ax.E says that through any point T in the interior of an angle there is a line intersecting both sides of that angle (see Fig. 1.21).

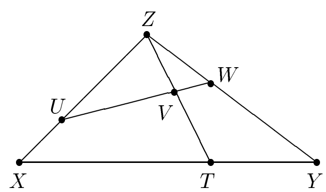

Figure 1.21.

Szczerba and Tarski show that adding that axiom Ax.E to GA_2 renders a complete axiomatization of the elementary affine Euclidean geometry AE_2, and hence also a complete axiomatization of the real affine plane.

Finally, here is a representation result for models of GA_2. Let $\mathfrak{F} = (F; 0, 1, +, \cdot, \leq)$ be any ordered field and define $\oplus$ and $\odot$ as

$$(x_1, y_1) \oplus (x_2, y_2) := (x_1 + x_2, y_1 + y_2), \quad (x_1, y_1) \odot \alpha := (x_1 \cdot \alpha, y_1 \cdot \alpha),$$

for any $(x_1, y_1), (x_2, y_2) \in F^2$ and $\alpha \in F$. We can define a betweenness relation $\mathbf{B}_{\mathfrak{F}}$ in the Cartesian square F^2 over the field $\mathfrak{F}$ as follows: given $a, b, c \in F^2$, we stipulate that

$$\mathbf{B}_{\mathfrak{F}}(abc) \;\Leftrightarrow\; b = [a \odot (1 - \lambda)] \oplus [c \odot \lambda]$$

for some $\lambda \in F$ with $0 \leq \lambda \leq 1$. The structure $\mathbf{A}(\mathfrak{F}) = (F^2; \mathbf{B}_{\mathfrak{F}})$ thus formed will be called the **affine plane over the ordered field** $\mathfrak{F}$. Using the class of all interiors of triangles as a basis, we define a topology on the set F^2. Now let S be any non-empty, convex, open subset of F^2. The structure $\mathbf{A}(\mathfrak{F}; S) = (F^2|_S; \mathbf{B}_{\mathfrak{F}}|_S)$ will be called the S**-restricted affine plane over** $\mathfrak{F}$. A plane over some field will simply be called a **restricted affine plane** if it is an S-restricted affine plane for some S.

THEOREM 1.42 1. *Every model of GA_2 is isomorphic to a restricted affine plane over some real closed ordered field.*

2. *Every restricted affine plane over the ordered field of reals is a model of GA_2.*

3. *If $\mathfrak{F}$ is an ordered real closed field not isomorphic to the field of reals, then there is a restricted affine plane over $\mathfrak{F}$ which is not a model of GA_2.*

9. Rich languages and structures for elementary geometry

We will call a geometric language **rich** if the whole elementary geometry in $\mathbb{R}^n$ is definable in that language. Perhaps the first study on rich primitive notions in elementary geometry is Pieri, 1908, where the Pieri relation Δ is introduced, defined as

$$\Delta(XYZ) := \|XY\| = \|XZ\|\,,$$

meaning that the configuration of points XYZ forms an isosceles triangle with base $|YZ|$ (in the degenerated cases either $Y = Z$ or X is the midpoint of $|YZ|$). Pieri showed that Δ can be used as the *only* primitive relation in $\mathbb{R}^n$ for $n \geq 2$. This result easily implies the richness of many other relations in terms of which Δ is definable, for example the ternary relation of **closer-than**

$$|XY| \leq |XZ|,$$

which states that either the point Y is closer to X than what Z is to X or that Y and Z lie equally far from X. This furthermore implies that the quaternary relation **shorter-than**

$$|XY| \leq |ZU|,$$

which states that the line segment $|XY|$ is shorther than the segment $|ZU|$, or that they are of equal length, is also rich.

Veblen, 1904 considered the two primitive relations of **betweenness** $\mathbf{B}$ and **equidistance** $\equiv$ (or δ), which are the same primitives that Tarski later used. Veblen showed that these primitives are sufficient for the elementary geometry, although he believed to have proved, falsely, that the relation of equidistance is definable in terms of the relation of betweenness (see Tarski and Givant, 1999). In fact, using the coordinatization of the Euclidean plane, and applying Padoa's method, it is easy to see that the equidistance relation $\equiv$ is *not* definable in terms of betweenness, not only in first-order languages, but even in higher-order logic. Indeed, the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $f(x, y) = (x, 2y)$, preserves betweenness, but not equidistance.

On the other hand, Pieri showed that $\mathbf{B}$ is first-order definable in terms of the quaternary closer-than relation $\leq$ defined above:

$$\mathbf{B}(XYZ) \Leftrightarrow \forall U\left((|XU| \leq |XY| \wedge |ZU| \leq |ZY|) \to U = Y\right),$$

meaning that if U and Y are intersection points of spheres with centers X and Z then they must coincide (see Fig. 1.22).

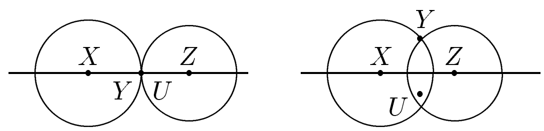

Figure 1.22.

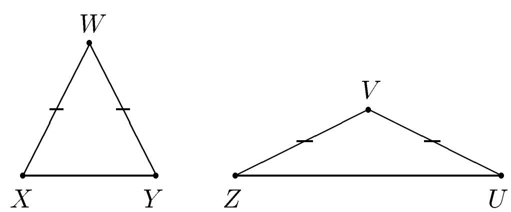

Figure 1.23.

Furthermore, the quaternary shorter-than relation $\leq$ is definable in terms of $\equiv$ as follows:

$$|XY| \leq |ZU| \Leftrightarrow \forall V\left(ZV \equiv UV \to \exists W(XW \equiv YW \wedge YW \equiv UV)\right),$$

meaning that if there is an isosceles triangle with base $|ZU|$ and a given side, then an isosceles triangle with the same side and base $|XY|$ exists too (see Fig. 1.23). It follows that $\equiv$ can be taken as the only primitive for the elementary geometry.

Regarding primitive relations with smaller arities, for every $n \geq 2$, no binary relation can be rich for $\mathbb{R}^n$ (see Beth and Tarski, 1956). Also, as noted earlier, the relation $\mathbf{B}$ alone is not sufficient for the whole elementary geometry in $\mathbb{R}^2$. Still, Beth and Tarski show in Beth and Tarski, 1956 that the ternary relation $\mathbf{E}$, where $\mathbf{E}(XYZ)$ means that the configuration XYZ forms an equilateral triangle (or degeneratively that the points X, Y and Z coincide), is rich for every $\mathbb{R}^n$ with $n \geq 3$, by expressing Pieri's relation in terms of $\mathbf{E}$ in the system $\mathbb{R}^3$. However, they also show that the relation **E** alone is not sufficient for the geometries $\mathbb{R}^n$ with $n = 1$ or $n = 2$. Similar results hold for the ternary relation specifying that three points determine a rectangular isosceles triangle, and the quaternary relation specifying that four points are the vertices of a square. However, it was shown in (see Beth and Tarski, 1956; Scott, 1956) that the completely symmetric ternary relation **R**, where $\mathbf{R}(XYZ)$ means that the points X, Y and Z are distinct and form, in some order, a rectangular triangle, is rich for every $\mathbb{R}^n$ with $n \geq 2$.

Schwabhäuser and Szczerba, 1975 investigate *line* relations which can be taken as primitives for the elementary Euclidean geometry. They establish simple rich systems of such relations for every $\mathbb{R}^n$, $n \geq 2$. For the dimension-free Euclidean geometry they show that the binary relation $\dot{\perp}$ of perpendicularity together with the ternary relation **Cop** of co-punctuality suffice. For dimensions higher than 3 perpendicularity alone suffices, while for $\mathbb{R}^2$ it does not, following Tarski's result mentioned above. For $\mathbb{R}^3$ perpendicularity and the binary relation **Cop** of co-punctuality suffice. Later, Kramer, 1993 proves that $\dot{\perp}$ alone does not suffice as a primitive for $\mathbb{R}^3$, because co-punctuality is not definable there in terms of it. Finally, the question of primitive geometric relations and definability in the case of $\mathbb{R}^1$ turns out to be rather more complicated; the reader is referred to Tarski and Givant, 1999.

9.1 Tarski's system of elementary geometry based on B and δ

In mid 20th century Tarski developed systematically an axiomatic system for the elementary geometry based on the only primitive concept of *point*, and the two primitive relations *betweenness* **B** and *equidistance* $\equiv$ (or δ). Over many years, Tarski and his students refined, simplified, and minimized that system, and a detailed account of that development can be found in Tarski and Givant, 1999, which we follow here for the choice of axioms and notation. Again, all axioms are implicitly universally quantified over all occurring free variables.

Ax.1 : Reflexivity of equidistance.

$$X_1X_2 \equiv X_2X_1$$

Ax.2 : Transitivity of equidistance.

$$(X_1X_2 \equiv Y_1Y_2 \wedge X_1X_2 \equiv Z_1Z_2) \rightarrow Y_1Y_2 \equiv Z_1Z_2$$

Ax.3 : Identity of equidistance.

$$XY \equiv ZZ \rightarrow X = Y$$

$\mathsf{Ax.4}$: EQUAL SEGMENTS CONSTRUCTION. (See Fig. 1.24) There is a segment of length $\|Y_1Y_2\|$ beginning at X_1 in direction of $\overrightarrow{ZX_1}$:

$$\exists X_2(\mathbf{B}(ZX_1X_2) \wedge X_1X_2 \equiv Y_1Y_2)$$

$\mathsf{Ax.5}$: FIVE-SEGMENT AXIOM. (See Fig. 1.25) The corresponding line segments built on two congruent triangles are equal:

$$(X \neq Y \ \wedge \ \mathbf{B}(XYZ) \ \wedge \ \mathbf{B}(X'Y'Z') \ \wedge \ XY \equiv X'Y' \ \wedge \ YZ \equiv Y'Z' \\ \wedge \ XW \equiv X'W' \ \wedge \ YW \equiv Y'W') \ \rightarrow \ ZW \equiv Z'W'$$

$\mathsf{Ax.7}_1$: (OUTER FORM OF) PASCH AXIOM. (See $\mathsf{Ax.B5}$ above.)

$$\mathbf{B}(XY'Z) \wedge \mathbf{B}(YZ'Y') \rightarrow \exists X'(\mathbf{B}(ZX'Y) \wedge \mathbf{B}(XZ'X'))$$

$\mathsf{Ax.8}^{(2)}$: LOWER 2-DIMENSIONAL AXIOM

$$\exists X \exists Y \exists Z(\neg\mathbf{B}(XYZ) \wedge \neg\mathbf{B}(YZX) \wedge \neg\mathbf{B}(ZXY))$$

$\mathsf{Ax.8}^{(n)}$: LOWER n-DIMENSIONAL AXIOM FOR $n \geq 3$

$$\exists U \exists V \exists W \, \exists X_1 \ldots \exists X_{n-1} \Big(\mathrm{Diff}_{n-1}(X_1 \ldots X_{n-1}) \\ \wedge \neg\mathbf{B}(UVW) \wedge \neg\mathbf{B}(VWU) \wedge \neg\mathbf{B}(WUV) \\ \wedge \bigwedge_{i=2}^{n-1} UX_1 \equiv UX_i \ \wedge \ \bigwedge_{i=2}^{n-1} VX_1 \equiv VX_i \ \wedge \ \bigwedge_{i=2}^{n-1} WX_1 \equiv WX_i \Big)$$

The axiom $\mathsf{Ax.8}^{(n)}$ claims that there exist $n-1$ distinct points $X_1, \ldots, X_{n-1}$, and three non-collinear points U, V, W, each of them equidistant from $X_1, \ldots, X_{n-1}$, which implies that the dimension of the space is at least n. Using these axioms, one can express that the dimension of the space is n:

$$\mathsf{Dim}_n := (\mathsf{Ax.8}^{(n)}) \wedge \neg(\mathsf{Ax.8}^{(n+1)}).$$

$\mathsf{Ax.10}_1$: (FORM OF) EUCLID'S AXIOM. (See the axiom $\mathsf{Ax.E}$ above.)

$$Z \neq V \wedge \mathbf{B}(ZVT) \wedge \mathbf{B}(UVW) \ \rightarrow \\ \exists X \exists Y(\mathbf{B}(ZUX) \wedge \mathbf{B}(ZWY) \wedge \mathbf{B}(YTX))$$

$\mathsf{Ax.11}$: SECOND-ORDER CONTINUITY AXIOM. (See also Sec. 8 above.)

$$\exists Y(\mathbf{B}(Y\mathbf{X}_1\mathbf{X}_2)) \rightarrow \exists Z(\mathbf{B}(\mathbf{X}_1 Z \mathbf{X}_2))$$

This axiom says that if all elements of the set $\mathbf{X}_1$ precede all elements of the set $\mathbf{X}_2$ on a line, then there is a point *on* that line which separates $\mathbf{X}_1$ and $\mathbf{X}_2$. This property is not definable in the first-order language of $\mathbf{B}$ and $\equiv$. Its first-order approximation is the corresponding *axiom schema*.

As.11 : CONTINUITY AXIOM SCHEMA. See As.B9 above.

Ax.15 : INNER TRANSITIVITY AXIOM FOR BETWEENNESS.

$$\mathbf{B}(XYW) \wedge \mathbf{B}(YZW) \rightarrow \mathbf{B}(XYZ)$$

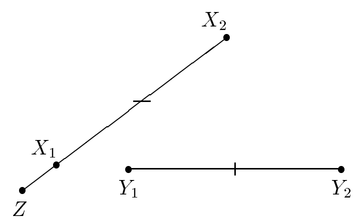

Figure 1.24.

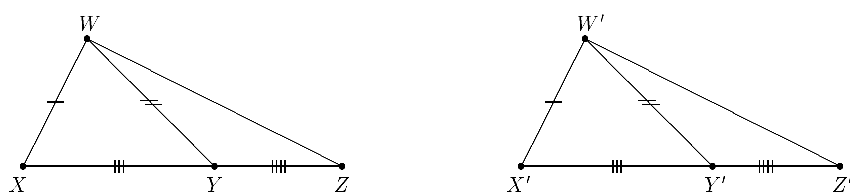

Figure 1.25.

THEOREM 1.43 (TARSKI, 1967) *The set of axioms* Ax.1 – Ax.5, Ax.7_1, Ax.10_1, Ax.15 *plus* Dim_n*, taken with the second order axiom* Ax.11*, characterizes up to isomorphism the full n-dimensional Euclidean geometry $FG^{(n)}$ for every $n \geq 2$. Furthermore, if* As.11 *is taken instead of* Ax.11*, the resulting first-order axiomatic system is a complete axiomatization of the first-order theory of the full n-dimensional Euclidean elementary geometry $EG^{(n)}$ for any $n \geq 2$. If, moreover, the dimension axiom* Dim_n *is omitted, then, according to Scott, 1959, a complete axiomatization of the dimension-free Euclidean geometry is obtained.*

9.2 Decision methods and automated reasoning for elementary geometry

The algebraic approach in geometry goes back at least to Descartes, who introduced the coordinate method in his study of geometry. The first modern development of general algebraic methods used in constructive solutions to classes of geometric problems in affine geometry is due to Hilbert, 1950. However, the first explicit decision method for elementary Euclidean geometry, i.e. a general method for deciding the truth of any first-order sentence in this geometry, was developed in Tarski, 1951. Tarski's decision method is based on a decision procedure for the first-order theory of the field of real numbers, which is also the first-order theory of the class of real-closed fields. For that theory Tarski established *quantifier elimination*, i.e. it was proved that every first-order sentence formulated in the class of real-closed fields is equivalent, over the class of real-closed fields, to a boolean combination of algebraic equations and inequalities. Equations are simply conjunctions of inequalities, and every inequality can be expressed in the form $t \geq 0$ for some term t in the first-order language of rings (i.e. t will be a polynomial). Therefore, eventually, every first-order sentence of that language is equivalent, over the class of real-closed fields, to a boolean combination of formulae of the type $\exists x(x^2 = t)$, where t is a term not containing x. Subsets of $\mathbb{R}^n$ definable by such formulae are called *semi-algebraic sets*. In particular, Tarski's result implies that the parametrically first-order definable relations in $\mathbb{R}^n$ are precisely the semi-algebraic sets of $\mathbb{R}^n$. For a sketch of an algebraic proof of this result based on Sturm's theorem, see e.g. Hodges, 1993.

Tarski's decision procedure is practically inefficient as it has non-elementary complexity. More efficient, elementary decision procedures were developed later, first by Monk, followed by Solovay, Collins, and others.

Currently there are several well-developed and applied automated theorem proving decision methods for the first-order theory of the field of reals and the theory of elementary geometry.

Probably the most popular decision method for the theory of real closed fields, and the first one amenable to practical automation (it has in fact been implemented), is Collins' method of *Cylindrical Algebraic Decompositions* (CAD), based on quantifier elimination (see Caviness and Johnson, 1998; Collins, 1975; Collins, 1998). Given a boolean combination $\mathcal{B}$ of algebraic equations and inequalities, this algorithm computes a so-called *cylindrical algebraic decomposition* of the solution set of $\mathcal{B}$. This cylindrical algebraic decomposition partitions the solution set of $\mathcal{B}$ into a finite disjoint union of spatial regions called *cells*, which have the property that all polynomials occurring in $\mathcal{B}$ preserve sign on each of these cells. Cells in $\mathbb{R}^n$ can be defined inductively on n as follows: a cell in $\mathbb{R}$ is an open interval or a singleton; a cell in $\mathbb{R}^{n+1}$ is either the graph of a continuous function defined over a cell in $\mathbb{R}^n$, or a region bounded between the graphs $f(\mathbf{x})$ and $g(\mathbf{x})$ of two such functions f and g defined on the same cell in $\mathbb{R}^n$, and such that $f < g$ on that cell. In particular, each of these functions can be taken as $-\infty$ or $+\infty$.

Collins' algorithm has a double exponential worst-case time complexity as a function of the number of variables in $\mathcal{B}$. Later, Heintz et al., 1990 and Renegar, 1992 constructed algorithms for quantifier elimination that are double exponential only in the number of quantifier alternations; see also Davenport and Heintz, 1988. Currently the most efficient algorithms for quantifier elimination known to us can be found in Basu et al., 1996 and Basu, 1999; the latter employs *uniform quantifier elimination*.

The *Characteristic Set method*, rooted in work by Ritt and later developed independently by Wu (see Chou, 1984; Chou, 1988; Chou, 1990; Chou and Gao, 1990; Wu, 1984; Wu, 1986), and the *Gröbner Basis method*, developed by Buchberger (see Buchberger, 1985; Buchberger et al., 1988; Chou, 1990), work only on problems that can be formalized by systems of equations, and are only complete for algebraically closed fields. A related alternative method, based on Hilbert's *Nullstellensatz*, has been proposed by Kapur, 1986.

Another method, based on ideas coming from quantifier elimination in linear and quadratic formulae over the reals, has been proposed in Dolzmann et al., 1998. Unlike the Characteristic Set and Gröbner Basis methods, it is also applicable to geometric problems in the Euclidean plane and Euclidean n-space whose complex analogues may fail.

Chou, 1984 shows how the Wu-Ritt method of characteristic sets can be applied to finding locus equations, and in Chou, 1987 it is also shown how this method can be used for the mechanical derivation of formulae in elementary geometry. All of these methods require large computational resources and can easily become unfeasible for more complex formulae. For an overview of automated reasoning in geometry see Chou and Gao, 1990.

This concludes our discussion of first-order theories of geometry.

10. Modal Logic and Spatial Logic

In the remainder of this chapter we will survey some modal logics related to classical geometric structures, and from now on we assume some familiarity with basic modal logic and Kripke semantics.

Modal logics related to spatial structures are also considered in Ch. **??** and Ch. **??**. The former is mainly oriented towards the topological interpretation of modal logic whereas the latter deals with the combination of spatial logics and temporal logics.

Basic modal logic. In order to fix the notations and terminology in basic modal logic we will give a short list of definitions and facts. For all notions mentioned without definitions the reader is invited to consult the book by Blackburn, de Rijke and Venema (see Blackburn et al., 2001) or Hughes and Cresswell, 1996.

Let $\mathcal{C}$ be a class of relational structures of the form $\mathcal{F} = (W, R_1, ..., R_n)$, where W is a nonempty set whose elements are usually called *possible worlds*, and R_1, …, R_n are binary relations on W called *accessibility relations*. In modal logic such relational structures are called Kripke frames. We associate with $\mathcal{C}$ a modal language L which is an extension of the standard language for propositional logic with unary connectives $[R_i]$, $i = 1, \ldots, n$, called modal *box* operators, with the standard definition of a formula, given by the rule:

$$A ::= p \mid \bot \mid \neg A \mid (A \vee B) \mid [R_1]A \mid \ldots \mid [R_n]A.$$

We use the classical abbreviations for 'true' ($\top$), 'false' ($\bot$), conjunction ($\wedge$), implication ($\rightarrow$), and equivalence ($\leftrightarrow$). We also use the dual, *diamond* operators $\langle R_i \rangle$, defined by $\langle R_i \rangle A = \neg[R_i]\neg A$. Standard modal logic has only one box-modality $\Box$ called "necessity", and the corresponding diamond modality $\Diamond$ is called "possibility".

The semantics of L in a given frame $\mathcal{F} = (W, R_1, \ldots, R_n)$ is based on the notion of *valuation* on $\mathcal{F}$, which is a function V assigning to each proposition letter p a subset $V(p)$ of W. Intuitively, we think of $V(p)$ as a set of possible worlds in which p is true. A pair $\mathcal{M} = (\mathcal{F}, V)$ where V is a valuation on $\mathcal{F}$ is called a *Kripke model* based on $\mathcal{F}$. We define the satisfiability relation $\mathcal{M}, w \models A$ — in words, *the formula A is satisfied at the possible world w of the model $\mathcal{M}$* — as in Ch. **??**. In particular we have:

$\mathcal{M}, w \models [R_i]A$ iff $\mathcal{M}, w' \models A$ for all $w' \in W$ such that wR_iw'.

A formula A is true in Kripke model $\mathcal{M}$ iff $\mathcal{M}, w \models A$ for all possible worlds w in $\mathcal{M}$. We say that A is valid in a Kripke frame $\mathcal{F}$ iff A is true in all models defined over $\mathcal{F}$; A is valid in a class $\mathcal{C}$ of Kripke frames iff A is valid in all Kripke frames of $\mathcal{C}$. The set $\mathcal{L}(\mathcal{C})$ of all formulas which are valid in $\mathcal{C}$ is called the logic of $\mathcal{C}$ and the formulas from $\mathcal{L}(\mathcal{C})$ are called the modal laws of $\mathcal{C}$. If $A \in \mathcal{L}(\mathcal{C})$ then we write $\models_{\mathcal{L}(\mathcal{C})} A$.

The above is a semantic definition of a modal logic related to a class $\mathcal{C}$ of frames. Note that one and the same modal logic may be the logic of different classes of frames.

A class $\mathcal{C}$ of frames of the form $(W, R_i, \ldots, R_n)$ is *modally definable* if there exists a formula A such that for every frame $\mathcal{F}$ of the form $(W, R_i, \ldots, R_n)$, $\mathcal{F}$ is in $\mathcal{C}$ iff A is valid in $\mathcal{F}$.

If this is true then we also say that $\mathcal{C}$ is modally definable by A. If the class $\mathcal{C}$ is definable by a first-order condition φ on the relations R_i then we also say that φ is modally definable by A. For instance reflexivity of a relation R is definable by $[R]p \rightarrow p$, symmetry of R is definable by $p \rightarrow [R]\langle R\rangle p$, and transitivity of R is definable by $[R]p \rightarrow [R][R]p$, where p is a propositional variable. So modal definability is in some sense a way to talk about properties of Kripke frames by means of a propositional language. Let us note, that not all first-order properties of Kripke frames are modally definable and that not all modal formulas define a first-order property.

Axiomatically a modal logic is defined as the smallest set of formulas containing a given set of axioms and closed with respect to a given set of inference rules. The elements of a modal logic $\mathcal{L}$ are called *theorems* of $\mathcal{L}$. If A is a theorem of $\mathcal{L}$ then we write $\vdash_{\mathcal{L}} A$. For instance the modal logic K_n of the class of all Kripke frames of the form $(W, R_1, \ldots, R_n)$ has the following axiomatic definition:

Axioms: all substitution instances of classical tautologies, and all formulas of the form $[R_i](A \rightarrow B) \rightarrow ([R_i]A \rightarrow [R_i]B)$ for $i = 1, \ldots, n$.

Inference rules of K_n: Modus ponens "given A and $A \rightarrow B$, derive B" and generalization "given A, derive $[R_i]A$".

Axiomatic definitions of other logics can be obtained by adding to the above axiomatic system additional axioms and possibly additional rules of inference. If the axiomatic system does not contain additional inference rules it is called normal. For instance the logic S4 is an extension of the logic $K_\Box$ with the axioms schemes $\Box A \rightarrow A$ (defining reflexivity of R) and the axiom scheme $\Box A \rightarrow \Box\Box A$ (defining the transitivity of R). The logic S5 is an extension of S4 with the axiom scheme $A \rightarrow \Box\Diamond A$ (defining the symmetry of R). The statement of the equivalence of a given semantic definition of a modal logic with a given axiomatic definition is called completeness theorem with respect to the corresponding class of frames. There are different methods how to prove completeness theorems. One of them is the so called **method of canonical models**. Important theorem related to this method is the famous **Sahlqvist theorem** saying that if the axioms of the logic are of a given specified form then these axioms define first-order conditions and that the logic is canonically complete in the class of frames satisfying these conditions. For instance the axioms of the logics S4 and S5 are od Sahlqvist's type and hence are "canonical", so S4 is complete in the class of all pre-orders and S5 is complete in the class of all equivalence relations. For S5 it is known also that it is complete in the class of all frames with $R = W \times W$, the universal relation in W. Another method for proving completeness theorems is based on the notion of *bounded morphisms*. By means of this method one can prove, for instance, that two different classes of frames, C_1 and C_2 define equal logics $\mathcal{L}(C_1) = \mathcal{L}(C_2)$. Then, if by some method (for instance by the method of canonical models) one can give a complete axiomatization of $\mathcal{L}(C_1)$, then we automatically obtain also a completeness theorem with respect to the class C_2. For the method of canonical models, bounded morphisms, Sahlqvist's theorem and some other methods see Blackburn et al., 2001.

Important topic in Modal Logic are some algorithmic problems and their complexity: the **satisfiability problem** of a given class of frames, the **provability problem** with respect to a given axiomatic system etc.

The above described modal logics contain only unary modal operations. There are also modal logics with binary and in general with n-ary modal operations called polyadic modalities with Kripke semantics using relations with arbitrary finite arity. If, for instance, $A \circ B$ is a binary modality then it can be interpreted in frames with a ternary relation $R \subseteq W^3$ as follows:

$\mathcal{M}, w \models A \circ B$ iff there exist w' and w'' such that $wRw'w''$, $\mathcal{M}, w' \models A$ and $\mathcal{M}, w'' \models B$.

Modal logic and applied modal logic. A major aim of modal logic is to study modal logics of different classes of frames, mainly with respect to *modal definability, axiomatization, decidability, complexity.*

The broad applicability of modal logics rests, inter alia, on the fact that while they are based on propositional languages, every modal formula corresponds in terms of frame validity to a universal monadic second-order formula, and thus can be used to express properties of relational structures. An important discipline of modal logic related to this issue – correspondence theory (see van Benthem, 1984) – is mostly about using modal formulas to define classes of relational structures. Thus, as noted in Blackburn et al., 2001, modal languages are simple yet expressive languages for talking about relational structures.

Another thing making some modal logics applicable is the fact that they represent tractable decidable fragments of first- or second-order logic, which makes them computationally efficient. Namely, the problem of computational efficiency stimulates recently an active research in modal logic in the realm of complexity theory.

The most important thing of the modal approach is that modal logic presents formal methods of reasoning based on modal operators specific for given practical domains. Often the linguistic meaning of these operators, coming from their use in the everyday language, is quite unprecise, therefore giving the exact semantics for the corresponding logic supplies these modalities with exact meaning. The complete axiomatization with respect to a given formal semantics presents a formal system for reasoning in the corresponding semantic domain and the completeness theorem with respect to the given interpretation can be considered as a tool for establishing the adequacy of the proposed semantics.

In summary, applied modal logic is an integral name for modal systems naturally arising from some practical domains. The machinery of applied modal logic contains all tools developed so far in modal logic. Very often the analysis of some new area of application of modal logic needs to invent some new methods, stimulating in this way the general development of the field.

Spatial modal logics. One of the origins of the classical modal logic of *necessity* and *possibility* arises from the analysis of the meaning of these modalities in natural languages. There are many other modes of truth which can be treated as different kinds of modalities: time modalities, space modalities, knowledge modalities, deontic modalities and so on. Examples of time modalities are: *always, sometimes, always in the future, always in the past, at the next moment, tomorrow, since, until*, etc. Examples of space modalities, related to geometrical relations in the space, are: *everywhere, everywhere else, somewhere, somewhere else, near, far, on the left, on the right, on the top, in the middle, between, parallel*, etc. Although geometry, as a mathematical theory of space, is one of the oldest branches of mathematics, and that the theory of time is not even a mathematical discipline, the logic of time is a much better established branch of modal logic than the modal logic of space. One explanation of this fact, as noted in Balbiani, 1998; Venema, 1999, is rooted in the use of temporal logic in computer science, especially in program verification and specification, concurrent programming and databases. Another reason is probably in the simpler mathematical structure of time, very suitable for a modal treatment by Kripke semantics: a set of moments of time together with a precedence relation between moments. In contrast, the structure of space is much more complex. For instance, the structure of the classical Euclidean geometry consists of several sorts of objects as points, lines, planes, with various binary relations between them like for instance collinearity and betweeness in the set of points, parallelism, concurrence and orthogonality in the set of lines and the intersort relations of incidence between different sorts of objects. At first sight, many-sorted mathematical structures are not suitable for modal treating in the above described sense, because the standard Kripke semantics is based on one-sorted structures. But as we shall see later, the modal approach has been extended also to many-sorted geometric structures.

Recently, in connection with new directions in artificial intelligence and information science, such as geometrical information systems and qualitative spatial reasoning Cohn and Hazarika, 2001, application of logic and in particular of modal logic to the theory of space, have become more popular, and this stimulates the development of a new branch of applied modal logic, commonly called *spatial modal logic.*

In 1998, Lemon and Pratt, 1998 produced a criterion by which one can judge the spatial character of a modal logic and observed in the light of their criterion that several of the existent modal logics of space were not spatial at all. According to Lemon and Pratt, a spatial modal logic is one, the models of which are based on mathematical models of space. Obviously, affine geometry and projective geometry and some of their fragments constitute mathematical models of space par excellence. Some of these geometrical structures considered as first-order systems are studied in the first part of this chapter. In this second part we include some modal logics with semantics based on them. In this chapter we will neither consider spatial modal logics based on interpretations of modal languages in topology and metric spaces, nor logics based on the primitive notion of a spatial region and some spatial relations between regions like *contact, part-of, overlap* etc., nor modal logics related to the relativistic interpretation of 4-dimensional space-time. The reader can find information for such logics in other chapters of this book.

11. Point-based spatial logics

In this section we will consider modal logics related to structures, based on a set of points.

The logic of elsewhere and everywhere. One of the first modal logics with explicit spatial interpretation is the logic of "elsewhere" introduced by von Wright, 1979. Under the reading of box given by von Wright, $\Box A$ means "everywhere else it is the case that A". Thus, the modal logic of 'elsewhere' may be formally identified with the validities in the class of all Kripke frames $\mathcal{F} = (W, R)$ in which R is the *difference relation*: $\forall x \forall y(xRy \leftrightarrow x \neq y)$. That is why the box and the diamond of von Wright's logic are usually written $[\neq]$ and $\langle\neq\rangle$. This is not typical spatial relation, because difference can be considered in any set of objects. But what really made von Wright's box popular is the observation that enriching modal languages with $[\neq]$ greatly increases their expressive power, as shown in Goranko, 1990; de Rijke, 1992; Venema, 1993.

The logic of elsewhere can be axiomatized by adding to K the following axioms (von Wright, 1979):

- $A \to [\neq]\langle\neq\rangle A, \quad A \wedge [\neq]A \to [\neq][\neq]A.$

Another example of a simple modal logic with a spatial interpretation is the logic $S5$ considered by Carnap (see Carnap, 1947) as the logic of all structures (W, R) with universal relation R:

$$(U) \qquad \forall x, y: \quad xRy.$$

This gives the following spatial reading of $\Box A$ as "everywhere A".

The condition (U) motivates the box of Carnap's logic to be usually written as $[U]$, and the diamond as $\langle U\rangle$. Note that $[U]A$ is definable in the logic of elsewhere: $[U]A = A \wedge [\neq]A$. Carnap's reading of box has drawn the attention of many logicians, including Goranko and Passy, 1992 and Spaan, 1993, seeing that enriching modal languages with $[U]$ substantially increases their expressive power, too.

Collinearity and qualitative distance. Collinearity of points is probably one of the basic ternary relations between points. Stebletsova, 2000 considers the ternary relation of collinearity between points in projective geometry: $Col(X, Y, Z)$ iff X, Y and Z all lie on a single line. She has studied the spatial logic based on Col in any projective geometry of finite dimension $d \geq 2$. The ternary relation Col is used to interpret the binary modality $\circ$ as follows:

$$\mathcal{M}, w \models A \circ B \text{ iff for some } w' \in W \text{ and for some } w'' \in W \text{ with } Col(w, w', w'') \text{ we have } \mathcal{M}, w' \models A \text{ and } \mathcal{M}, w'' \models B.$$

In this setting, the following formulas are valid:

- $A \circ (B \circ C) \to (A \circ B) \circ C$,
- $A \circ B \to B \circ A$,
- $A \to A \circ A$, and
- $\langle U\rangle A \wedge \langle U\rangle B \to \langle U\rangle(A \circ B)$

where $\langle U\rangle$ is the existential modality between points defined by $\langle U\rangle A = \top \circ A$. Given a finite dimension $d \geq 2$, validity in the class of all projective geometries of dimension d can be axiomatized with a Gabbay-type inference rule, see Gabbay, 1981, but it is not known whether such rules can be replaced by a finite set of additional axioms.

The modal logic of collinearity in projective geometry is rather expressive. For instance, for all finite dimensions $d \geq 2$, there exists a formula in the basic modal language defined above that characterizes exactly those projective spaces of dimension d satisfying the property of Pappus, see Sec. 6.3. Using the fact that Pappus' theorem holds in any finite projective geometry of finite dimension $d \geq 3$, Stebletsova, 2000 has shown that this logic lacks the finite modal property: there exist satisfiable formulas that cannot be satisfied in finite models. What is more, for any finite dimension $d \geq 3$, the satisfiability problem in the class of all projective geometries of dimension d is undecidable. See Stebletsova, 2000 for further details.

Another interesting ternary spatial relation $N(x, y, z)$ of qualitative distance between points is considered in the paper of van Benthem, 1983, with the following intuitive reading: "y is nearer to x than z". Its most obvious properties in the real plane, may be formulated as follows:

Transitivity $\forall x \forall y \forall z \forall t (N(x, y, z) \wedge N(x, z, t) \rightarrow N(x, y, t))$,

Irreflexivity $\forall x \forall y \neg N(x, y, y)$,

Almost-connectedness $\forall x \forall y \forall z \forall t (N(x, y, z) \rightarrow N(x, y, t) \vee N(x, t, z))$,

Selfishness $\forall x \forall y (x \neq y \rightarrow N(x, x, y))$,

Triangle inequality $\forall x \forall y \forall z (N(x, y, z) \wedge N(z, x, y) \rightarrow N(y, x, z))$.

It is known, that N can serve as the basis of elementary plane Euclidean geometry (see Tarski, 1956). Nevertheless, no complete modal spatial logic has been developed so far with Kripke semantics based on that relation (see Aiello and van Benthem, 2002 for further discussion).

12. Line-based spatial logics

In this section we examine some spatial logics based on lines and some standard relations between lines: parallelism, orthogonality and intersection of lines.

The logic of parallelism. Recall, that $\parallel$ denotes the relation of strict parallelism . Parallelism frames are structures in the form $(\mathbf{Li}, \parallel)$, where **Li** is a non-empty set whose elements are called lines and $\parallel$ is the relation of strict parallelism between lines. That relation satisfies the following first order conditions:

- $\forall x : x \not\parallel x$ – *no line is parallel to itself*,
- $\forall x, y : x \parallel y$ implies $y \parallel x$ – *the relation* $\parallel$ *is symmetric*,
- $\forall x, y, z : x \parallel y$ and $y \parallel z$ and $x \neq z$ implies $x \parallel z$ – *the relation* $\parallel$ *is "pseudo- transitive"*.

Frames satisfying all these conditions are called strict models of parallelism and their class is denoted by $\mathcal{C}_{SMP}$. The frames satisfying the second and the third axiom are called pre-models of parallelism and their class is denoted by $\mathcal{C}_{PreMP}$.

Balbiani and Goranko, 2002 consider the modal logic of strict parallelism, where $[\|]A$ means *"A is true at all parallel lines"*. The semantics, based on parallelism frames $(\mathbf{Li}, \|)$, is in the expected way:

$\mathcal{M}, w \models [\|]A$ iff for all $w' \in \mathbf{Li}$ such that $w \parallel w'$, $\mathcal{M}, w' \models A$.

Obviously, the following formulas modally define the class $\mathcal{C}_{PreMP}$:

$$A \to [\|]\langle\|\rangle A, \quad A \wedge [\|]A \to [\|][\|]A.$$

We denote by PAR, the axiom system obtained by adding these formulas to the minimal modal logic K. Let us note that these axioms are of Sahlqvist type and just modally define the class of frames $\mathcal{C}_{PreMP}$. Then, by the Sahlqvist theorem PAR is sound and complete with respect to $\mathcal{C}_{PreMP}$.

Let us remark that PAR and the logic of elsewhere are the same. Hence, repeating the completeness proof of the logic of elsewhere given in Segerberg, 1981, one can moreover show that PAR is also complete with respect to the strict models of parallelism $\mathcal{C}_{SMP}$. The difficulty in working with strict models of parallelism is that there is no formula corresponding to the irreflexivity of the relation $\|$. This lack of expressive power of the modal language helps to show that the satisfiability problem $SAT(\mathcal{C}_{SMP})$ is NP-complete (Demri, 1996).

Adding to PAR the following Sahlqvist formulas for all $n \geq 0$ we obtain the axiom system PAR^E:

(φ_0) $\langle\|\rangle\top$,

(φ_n) $\langle\|\rangle([\|]A_1) \wedge \ldots \wedge \langle\|\rangle([\|]A_n) \to \langle\|\rangle(A_1 \wedge \ldots \wedge A_n)$,

The formula φ_n corresponds to the first-order property on parallelism frames:

(Φ_n) $x \parallel y_1 \wedge \ldots \wedge x \parallel y_n \to (\exists z)(x \parallel z \wedge y_1 \parallel z \wedge \ldots \wedge y_n \parallel z)$.

Note that φ_n is derivable from φ_{n+1}.

Since the model $\mathcal{F}_{n+2} = (\{1, \ldots, n+2\}, \neq)$ of strict parallelism consisting of exactly $n+2$ parallel lines validates φ_n but does not validate φ_{n+1}, we infer that PAR^E is not finitely axiomatizable. Nevertheless, since φ_n is a Sahlqvist formula for all $n \geq 0$, PAR^E is sound and complete with respect to the class $\mathcal{C}^{\infty}_{PreMP}$ of all pre-models of parallelism satisfying (Φ_n), for all $n = 0, 1, \ldots$. Repeating the line of reasoning suggested by Segerberg, 1981 within the context of the logic of elsewhere, one can show that PAR^E is also complete with respect to $\mathcal{C}^{\infty}_{SMP}$ consisting of all strict models of parallelism satisfying the conditions (Φ_n) for all $n = 0, 1, \ldots$.

More interesting completeness result for the logic PAR^E is that it is sound and complete both in the standard parallelism frames in real plane $\mathbb{P}^2$ and in real 3-dimensional space $\mathbb{P}^3$ with the usual relation of strict parallelism. This shows that the language of strict parallelism is not expressive enough to distinguish the standard parallelism frames in $\mathcal{C}^{\infty}_{PreMP}$. Despite this obvious lack of expressive power of our modal language, the advantage of our modal approach is that the decision problem $SAT(\mathcal{C}^{\infty}_{SMP})$ for satisfiability in $\mathcal{C}^{\infty}_{SMP}$ is also NP-complete (see Balbiani and Goranko, 2002 for the details).

The logic of orthogonality. The relation of orthogonality $\perp$ is another typical binary relation between lines. We interpret $[\perp]A$ as *"A is true at all orthogonal lines of the current line"*. Let us note that in every orthogonality frame $\mathcal{F} = (\mathbf{Li}, \perp)$, the binary relation $\|$, defined as follows is an equivalence relation:

$$w \parallel w' \text{ iff for all lines } w'', \; w \perp w'' \text{ iff } w' \perp w''$$

We consider the class $\mathcal{C}_{PQMO}$ of all planar quasi-models of orthogonality $\mathcal{F} = (\mathbf{Li}, \perp)$ where $\perp$ is symmetric and 3-transitive, i.e.:

- $\forall w \forall w' (w \perp w' \rightarrow w' \perp w)$ – symmetry of $\perp$,
- $\forall w \forall w' \forall w'' \forall w''' (w \perp w' \wedge w' \perp w'' \wedge w'' \perp w''' \rightarrow w \perp w''')$ – 3-transitivity of $\perp$.

The class $\mathcal{C}_{PMLO}$ of *planar models of line orthogonality* is the class of all frames $\mathcal{F} = (\mathbf{Li}, \perp)$ where $\perp$ is irreflexive, symmetric and 3-transitive. Let $\mathcal{C}^{\infty}_{PQMO}$ be the class of all planar quasi-models of line orthogonality in which every equivalence class modulo $\|$ is infinite. Similarly, let $\mathcal{C}^{\infty}_{PMLO}$ be the class of all planar models of line orthogonality in which every equivalence class modulo $\|$ is infinite.

The modal logic ORT based on $\mathcal{C}_{PQMO}$ is obtained by adding the following axioms to K:

- $A \rightarrow [\perp]\langle\perp\rangle A, \quad [\perp]A \rightarrow [\perp][\perp][\perp]A.$

These axioms are formulas of Sahlqvist type just defining the properties of symmetry and 3-transitivity of the relation $\perp$, so by the Sahlqvist theorem ORT is complete in $\mathcal{C}_{PQMO}$. Using bounded morphisms one may prove that the classes $\mathcal{C}_{PQMO}$ and $\mathcal{C}_{PMLO}$ define the same logic, which shows that ORT is also complete in the class $\mathcal{C}_{PMLO}$. See Balbiani and Goranko, 2002 for details.

The logic which corresponds to the class of frames $\mathcal{C}^{\infty}_{PQMO}$ is finitely axiomatizable through the axiom system ORT^E obtained by adding to ORT the axiom $\langle\perp\rangle\top$. This axiom system is sound and complete also with respect to validity in the Euclidean orthogonality plane consisting of all lines in the real plane together with the usual orthogonality relation (see Balbiani and Goranko, 2002). So the language of orthogonality is not expressive enough to distinguish the standard orthogonality frame in the class $\mathcal{C}^{\infty}_{PQMO}$.

Let us note that the formula $[\perp]A \rightarrow [\perp][\perp][\perp]A$ is not valid in the Euclidean orthogonality space, consisting of all lines in the real 3-dimensional space together with the usual orthogonality relation. Hence, comparing with the modal logic of parallelism, the modal logic of orthogonality is able to distinguish the Euclidean orthogonality plane and the Euclidean orthogonality 3-dimension space.

A modal logic of parallelism and intersection of lines. Having outlined the modal logics of parallelism and the modal logics of orthogonality, we are now in a position to consider richer geometrical structures. Next, we discuss the line-based modal logic based on the binary relations of parallelism and intersection of lines, with the corresponding modal operators $[\|]$ and $[\times]$. Our aim is to axiomatize the logic of the standard two-dimensional frame consisting of all lines in the real affine plane (called SAP) with the strict parallelism relation $\|$ and the standard relation of intersection: $a \times b$ iff a and b have only one common point. Let us note that the following modal formulas are true in SAP:

- $\varphi \rightarrow [\|]\langle\|\rangle\varphi$,
- $\varphi \wedge [\|]\varphi \rightarrow [\|][\|]\varphi$,
- $\varphi \rightarrow [\times]\langle\times\rangle\varphi$
- $[\times]\varphi \rightarrow [\|][\times]\varphi$,
- $\varphi \wedge [\|]\varphi \wedge [\times]\varphi \rightarrow [\times][\times]\varphi$,
- $\langle\|\rangle\top$,
- $\langle\|\rangle\varphi_1 \wedge \ldots \wedge \langle\|\rangle\varphi_n \rightarrow \langle\|\rangle(\langle\|\rangle\varphi_1 \wedge \ldots \wedge \langle\|\rangle\varphi_n)$, $n = 1, 2, \ldots$,
- $\langle\times\rangle\top$,
- $\langle\times\rangle\varphi_1 \wedge \ldots \wedge \langle\times\rangle\varphi_n \rightarrow \langle\times\rangle(\langle\times\rangle\varphi_1 \wedge \ldots \wedge \langle\times\rangle\varphi_n)$, $n = 1, 2, \ldots$.

We denote by $\mathcal{ML}(SAP)$ the extension of the logic K with these axioms. Since all of them are Sahlqvist formulas, they define the following class $\mathcal{C}(PreSAP)$ of frames (called pre-standard affine planes) in which $\mathcal{ML}(SAP)$ is complete:

- $u \parallel v \rightarrow v \parallel u$,
- $u \parallel v \wedge v \parallel w \wedge u \neq w \rightarrow u \parallel w$,
- $u \times v \rightarrow v \times u$,
- $u \parallel v \wedge v \times w \rightarrow u \times w$,
- $u \times v \wedge v \times w \rightarrow u = w \vee u \parallel w \vee u \times w$,
- $(\forall u \exists v)(u \parallel v)$,
- $u \parallel v_1 \wedge \ldots \wedge u \parallel v_n \rightarrow (\exists w)(u \parallel w \wedge w \parallel v_1 \wedge \ldots \wedge w \parallel v_n)$, $n = 1, 2, \ldots$,
- $(\forall u \exists v)(u \times v)$,
- $u \times v_1 \wedge \ldots \wedge u \times v_n \rightarrow (\exists w)(u \times w \wedge w \times v_1 \wedge \ldots \wedge w \times v_n)$, $n = 1, 2, \ldots$.

Let us call a structure $(\mathbf{Li}, \parallel, \times)$ *general affine plane* if it satisfies all of the above first-order conditions plus the conditions of irreflexivity of the relations $\times$ and $\parallel$ and let us denote the class of all such structures by $\mathcal{C}(GAP)$. Note that the standard affine plane SAP is in this class. Applying the method of bounded morphisms it can be proved that $\mathcal{ML}(SAP)$ is also complete in this class. However this still does not prove that $\mathcal{ML}(SAP)$ is complete with respect to SAP. Applying more complicate techniques from model theory it can be proved that any two frames from $\mathcal{C}(GAP)$ are modally equivalent, i.e. determine equal logics. Since standard affine plane is in $\mathcal{C}(GAP)$, this implies that the logic $\mathcal{ML}(SAP)$ is complete in its standard semantics – SAP.

The classes $\mathcal{C}(PreSAP)$ and $\mathcal{C}(GAP)$ are quite different. Using the selective filtration techniques one can prove that the logic $\mathcal{ML}(SAP)$ has finite model property (fmp) with respect to $\mathcal{C}(PreSAP)$ and hence is decidable, while with respect to $\mathcal{C}(GAP)$ it does not have fmp – all frames from $\mathcal{C}(GAP)$ are infinite. These facts, however, help to prove that satisfiability problem for $\mathcal{C}(GAP)$ is NP-complete.

The completeness theorem of $\mathcal{ML}(SAP)$ implies that the modal language of parallelism and intersection of lines is too weak to distinguish standard 2-dimensional frame from the other frames in the class $\mathcal{C}(GAP)$. But the language can distinguish 2-dimensional standard frame from the 3-dimensional standard frame, the later consisting of all lines in the real 3-dimensional space with the standard relation of strict parallelism and the standard relation of intersection of lines. For instance the following axiom of $\mathcal{ML}(SAP)$ is not true in the 3-dimension space: $[\times]A \to [\|][\times]A$. The reason is that in the 3-dimension space there are lines intersecting one of two parallel lines but not the other.

13. Tip spatial logics

Projective geometry and affine geometry are probably among the most prominent mathematical models of space. They arise from the study of points and lines by means of properties stated in terms of incidence. In this section and in the following one, we will introduce modal logics for incidence between points and lines. There are two different approaches for defining a modal logic of incidence between points and lines. The standard semantics for modal logic assumes Kripke models with only one sort of possible worlds. Therefore, the first approach consists in the replacement of the two-sorted structures based on points and lines by one-sorted structures containing the same geometrical information. The second approach consists in the extension of the modal logic formalism allowing two sorts of formulas, point formulas and line formulas, and two sorts of possible worlds in Kripke models. The remainder of this section describes shortly the first approach (see Balbiani et al., 1997), while the second approach will be considered in Section 14.

Tips. Let $\mathcal{F} = (\mathbf{Po}, \mathbf{Li}, \mathbf{I})$ be a point-line incidence plane, that is:

- $\mathbf{Po}$ is a non-empty set of points with typical elements denoted by X, Y, Z, T, etc, possibly with subscripts,
- $\mathbf{Li}$ is a non-empty set of lines with typical elements denoted by x, y, z, t, etc, possibly with subscripts,
- I is a binary relation of incidence between points and lines.

The relationship XIx will be read "X is incident with x", "X lies in x", "x is incident with X", or "x passes through X". We will always assume that $\mathbf{Po} \cap \mathbf{Li} = \emptyset$, i.e. no point is a line and no line is a point. Hereafter, we will assume in this section that the binary relation I satisfies the following first-order conditions:

- $\forall X \forall Y \exists z (XIz \wedge YIz)$,
- $\forall X \forall Y \forall z \forall t (XIz \wedge YIz \wedge XIt \wedge YIt \to X = Y \vee z = t)$,
- $\forall x \exists Y \exists Z (YIx \wedge ZIx \wedge Y \neq Z)$,
- $\forall X \exists y \exists z (XIy \wedge XIz \wedge y \neq z)$.

The notion of point-line incidence plane can be extended with new first-order conditions in different directions. Two natural extensions are the notion of affine plane and the notion of projective plane. Consider a point-line incidence plane $\mathcal{F} = (\mathbf{Po}, \mathbf{Li}, \mathbf{I})$. Let us define on **Li** the binary relation $\parallel$ in the following way:

- $x \parallel y$ iff for all points Z, if ZIx and ZIy then $x = y$,

A point-line incidence plane $\mathcal{F} = (\mathbf{Po}, \mathbf{Li}, \mathbf{I})$ is called an affine plane if it satisfies the following additional first-order conditions:

- $\forall X \forall y \exists z(XIz \wedge y \parallel z)$, $\quad \forall x \forall y \forall z(x \parallel y \wedge y \parallel z \rightarrow x \parallel z)$.

Obviously, point-line affine planes are Euclidean in the sense that they satisfy the following condition:

- $\forall X \forall y \forall z \forall t(XIz \wedge y \parallel z \wedge XIt \wedge y \parallel t \rightarrow z = t)$.

A point-line incidence plane $\mathcal{F} = (\mathbf{Po}, \mathbf{Li}, \mathbf{I})$ is called a projective plane if it satisfies the following additional first-order conditions:

- $\forall x \forall y \exists Z(ZIx \wedge ZIy)$,
- $\forall x \forall Y \forall Z \exists T(YIx \wedge ZIx \rightarrow TIx \wedge T \neq Y \wedge T \neq Z)$.

It is clear from our definition that if $\mathcal{F} = (\mathbf{Po}, \mathbf{Li}, \mathbf{I})$ is a projective plane then two different points are always incident with exactly one line whereas two different lines have always one point in common.

Traditionally, the Kripke semantics of modal logics is based on one-sorted relational structures. That is why we introduce a new kind of relational structures, called incidence frames, which are one-sorted and which will be used for defining the Kripke semantics of our next spatial logics. Now consider a point-line incidence plane $\mathcal{F} = (\mathbf{Po}, \mathbf{Li}, \mathbf{I})$. We shall say that the pair (X, x) in $\mathbf{Po} \times \mathbf{Li}$ is a **tip** over $\mathcal{F}$ iff XIx. Intuitively, the tip (X, x) can be considered both as the point X and as the line x. Using tips, we can define the following binary relations:

- $(X, x) \equiv_1^{\mathcal{F}} (Y, y)$ iff $X = Y$,
- $(X, x) \equiv_2^{\mathcal{F}} (Y, y)$ iff $x = y$.

In the expression $(X, x) \equiv_1^{\mathcal{F}} (Y, y)$, (X, x) and (Y, y) are considered as the points X and Y and the relation $\equiv_1^{\mathcal{F}}$ can be seen as the equality of points. Similarly, in the expression $(X, x) \equiv_2^{\mathcal{F}} (Y, y)$, (X, x) and (Y, y) are considered as the lines x and y and the relation $\equiv_2^{\mathcal{F}}$ can be seen as the equality of lines.

Using the binary relations $\equiv_1^{\mathcal{F}}$ and $\equiv_2^{\mathcal{F}}$, we can simulate the binary relation of incidence between points and lines and the binary relation of parallelism between lines in $\mathcal{F}$. Let $O^{\mathcal{F}}$ and $\|^{\mathcal{F}}$ be the binary relations between tips defined in the following way:

- $(X,x)O^{\mathcal{F}}(Y,y)$ iff XIy,
- $(X,x) \|^{\mathcal{F}} (Y,y)$ iff $x \parallel y$.

Obviously, the relation of incidence $O^{\mathcal{F}}$ between tips is definable by means of $\equiv_1^{\mathcal{F}}$ and $\equiv_2^{\mathcal{F}}$ as follows: $w_1 O^{\mathcal{F}} w_2$ iff there exists a tip w such that $w_1 \equiv_1^{\mathcal{F}} w$ and $w \equiv_2^{\mathcal{F}} w_2$, hence $O^{\mathcal{F}} = \equiv_1^{\mathcal{F}} \circ \equiv_2^{\mathcal{F}}$ where $\circ$ is the composition of binary relations. Likewise, $w_1 \|^{\mathcal{F}} w_2$ iff for all tips w, if $wO^{\mathcal{F}}w_1$ and $wO^{\mathcal{F}}w_2$ then $w_1 \equiv_2^{\mathcal{F}} w_2$.

Incidence frames. Tips motivate the following definition. Consider a point-line incidence plane $\mathcal{F} = (\mathbf{Po}, \mathbf{Li}, \mathbf{I})$. The incidence frame over $\mathcal{F}$ is the structure $W(\mathcal{F}) = (W^{\mathcal{F}}, \equiv_1^{\mathcal{F}}, \equiv_2^{\mathcal{F}})$ where $W^{\mathcal{F}}$ is the set of all tips over $\mathcal{F}$. It is not too difficult to see that $\equiv_1^{\mathcal{F}}$ and $\equiv_2^{\mathcal{F}}$ are equivalence relations on W satisfying the following additional conditions:

$(I1)$ $\forall w \forall w' (w \equiv_1^{\mathcal{F}} w' \land w \equiv_2^{\mathcal{F}} w' \to w = w')$,

$(I2)$ $\forall w \forall w' \exists w'' (wO^{\mathcal{F}}w'' \land w'O^{\mathcal{F}}w'')$,

$(I3)$ $\forall w \forall w' \forall w'' \forall w''' (wO^{\mathcal{F}}w'' \land w'O^{\mathcal{F}}w'' \land wO^{\mathcal{F}}w''' \land w'O^{\mathcal{F}}w''' \to$
$w \equiv_1^{\mathcal{F}} w' \lor w'' \equiv_2^{\mathcal{F}} w''')$,

$(I4)$ $\forall w \exists w' \exists w'' (w'O^{\mathcal{F}}w \land w''O^{\mathcal{F}}w \land w' \not\equiv_1^{\mathcal{F}} w'')$,

$(I5)$ $\forall w \exists w' \exists w'' (wO^{\mathcal{F}}w' \land wO^{\mathcal{F}}w'' \land w' \not\equiv_2^{\mathcal{F}} w'')$.

Let us remark that $\equiv_1$ and $\equiv_2$ define $=$ in the following way: $w = w'$ iff $w \equiv_1 w'$ and $w \equiv_2 w'$. Moreover, if $\mathcal{F}$ is affine then:

$(A1)$ $\forall w \forall w' \exists w'' (wO^{\mathcal{F}}w'' \land w' \|^{\mathcal{F}} w'')$,

$(A2)$ $\forall w \forall w' \forall w'' (w \|^{\mathcal{F}} w' \land w' \|^{\mathcal{F}} w'' \to w \|^{\mathcal{F}} w'')$.

If $\mathcal{F}$ is projective then:

$(P1)$ $\forall w \forall w' \exists w'' (w''O^{\mathcal{F}}w \land w''O^{\mathcal{F}}w')$,

$(P2)$ $\forall w \forall w' \forall w'' \exists w''' (w'O^{\mathcal{F}}w \land w''O^{\mathcal{F}}w \to$
$w'''O^{\mathcal{F}}w \land w''' \not\equiv_1^{\mathcal{F}} w' \land w''' \not\equiv_1^{\mathcal{F}} w'')$.

These conditions are characteristic in the following sense: if in a set W we have two equivalence relations $\equiv_1$ and $\equiv_2$ satisfying the conditions $(I1)$–$(I5)$ then there exists a point-line incidence plane $\mathcal{F}$ such that the relational structures $(W, \equiv_1, \equiv_2)$ and $W(\mathcal{F}) = (W^{\mathcal{F}}, \equiv_1^{\mathcal{F}}, \equiv_2^{\mathcal{F}})$ are isomorphic. Moreover, if $(W, \equiv_1, \equiv_2)$ satisfies the conditions $(A1)$ and $(A2)$ then the corresponding point-line incidence plane $\mathcal{F} = (\mathbf{Po}, \mathbf{Li}, \mathbf{I})$ is affine, and, if it satisfies the conditions $(P1)$ and $(P2)$ then the corresponding point-line incidence plane is projective. So, in order to define the Kripke semantics of a modal logic of incidence, we can use one-sorted structures of the form $(W, \equiv_1, \equiv_2)$ instead of point-line incidence planes.

Consider a relational structure of the form $(W, \equiv_1, \equiv_2)$ where $\equiv_1$ and $\equiv_2$ are equivalence relations on W. We shall say that $(W, \equiv_1, \equiv_2)$ is an incidence frame if it satisfies the conditions $(I1)$–$(I5)$. Moreover, $(W, \equiv_1, \equiv_2)$ is said to be affine if it satisfies $(A1)$ and $(A2)$, and it is said to be projective if it satisfies $(P1)$ and $(P2)$.

Let us note that the properties of Desargues and Pappus are also expressible in the present language, so it is quite rich.

A modal logic for incidence. Our modal language for incidence frame uses the modal operators $[\equiv_1]$, $[\equiv_2]$, $[\not\equiv_1]$, and $[\not\equiv_2]$. Well-formed formulas are given by the rule:

- $A ::= p \mid \bot \mid \neg A \mid (A \vee B) \mid [\equiv_1]A \mid [\equiv_2]A \mid [\not\equiv_1]A \mid [\not\equiv_2]A.$

Abbreviations: difference – $[\neq]A = [\not\equiv_1]A \wedge [\not\equiv_2]A$, universal modality – $[U]A = A \wedge [\neq]A$, incidence $[O]A = [\equiv_1][\equiv_2]A$, $[O^{-1}]A = [\equiv_2][\equiv_1]A$.

The semantics is based on incidence frames in the expected way. In particular, we have:

- $\mathcal{M}, w \models [\equiv_i]A$ iff for all $w' \in W$ such that $w \equiv_i w'$, $\mathcal{M}, w' \models A$,
- $\mathcal{M}, w \models [\not\equiv_i]A$ iff for all $w' \in W$ such that $w \not\equiv_i w'$, $\mathcal{M}, w' \models A$,

for $i \in \{1, 2\}$.

The following formulas are valid in $\mathcal{C}_{inc}$:

(Ax_1) $A \to [\not\equiv_i]\langle\not\equiv_i\rangle A$, $i \in \{1, 2\}$,

(Ax_2) $A \to [\neq]\langle\neq\rangle A$,

(Ax_3) $A \wedge [\neq]A \to [\neq][\neq]A$,

(Ax_4) $[U]A \to [\equiv_i]A$, $i \in \{1, 2\}$,

(Ax_5) $[\equiv_i]A \wedge [\not\equiv_i]A \to [U]A$, $i \in \{1, 2\}$,

(Ax_6) $\langle\not\equiv_i\rangle A \to [\equiv_i]\langle\neq\rangle A$, $i \in \{1, 2\}$,

(Ax_7) $[\equiv_i]A \to A$, $A \to [\equiv_i]\langle\equiv_i\rangle A$, $[\equiv_i]A \to [\equiv_i][\equiv_i]A$, $i \in \{1, 2\}$,

(Ax_8) $[O][O^{-1}]A \to [U]A$,

(Ax_9) $\langle O\rangle(A \wedge \langle O^{-1}\rangle([\neq]B \wedge C)) \rightarrow ([O](\langle\equiv_2\rangle A \vee [O^{-1}]B) \vee \langle\equiv_1\rangle C)$,

(Ax_{10}) $A \rightarrow \langle O^{-1}\rangle\langle\not\equiv_1\rangle\langle O\rangle A$,

(Ax_{11}) $A \rightarrow \langle O\rangle\langle\not\equiv_2\rangle\langle O^{-1}\rangle A$.

Let MIG (Modal Incidence Geometry) be the axiom system obtained by adding the formulas (Ax_1)–(Ax_{11}) to the minimal normal modal logic in our language. Note that all proper axioms of MIG are Sahlqvist formulas and that the associated first-order properties correspond to conditions defining incidence frames. For example, the formulas (Ax_9), (Ax_{10}), and (Ax_{11}) correspond to the first-order properties $(I3)$, $(I4)$, and $(I5)$ respectively. Nevertheless, MIG is not known to be complete with respect to validity in the class $\mathcal{C}_{inc}$ of all incidence frames. The point is that the interpretation in incidence frames of formulas in the form $[\not\equiv_1]A$ and $[\not\equiv_2]A$ is based on the complements $\not\equiv_1$ and $\not\equiv_2$ of the binary relations $\equiv_1$ and $\equiv_2$. The difficulty with the complementarity relations is that there is no axiom corresponding exactly to the first-order properties saying that:

- $\equiv_i \cap \not\equiv_i = \emptyset$ for $i \in \{1, 2\}$.

We have seen that $[\not\equiv_1]$ and $[\not\equiv_2]$ define $[\neq]$. Moreover, notice that on the class $\mathcal{C}_{inc}$ of all incidence frames the formula $A \wedge [\neq]\neg A$ is satisfied at some tip w in some incidence model $\mathcal{M} = (W, \equiv_1, \equiv_2, V)$ iff w is the only tip in W where A holds. Hence, $A \wedge [\neq]\neg A$ can be considered as a sort of proper name for w. The reader may observe that the first-order properties saying that the binary relations $\equiv_i$ and $\not\equiv_i$ are disjoint are equivalent to the first-order condition of irreflexivity of the binary relation $\not\equiv_1 \cup \not\equiv_2$. Although irreflexivity does not correspond to a modal formula, it can be characterized in some sense by an inference rule. In this connection, see Gabbay, 1981, de Rijke, 1992, and Venema, 1993. This suggests to enrich the axiom system MIG with a special inference rule, the inference rule of irreflexivity:

- "Given $p \wedge [\neq]\neg p \rightarrow A$, prove A",

where p is a proposition letter not occurring in A, thus obtaining the axiom system MIG^+. This inference rule has also an infinitary version:

- "Given $p \wedge [\neq]\neg p \rightarrow A$ for all proposition letters, prove A",

which gives rise to the same set of provable formulas. Soundness of MIG^+ with respect to validity in $\mathcal{C}_{inc}$ is straightforward: we already know that MIG is sound, hence, it is enough to verify that the inference rule of irreflexivity preserves validity in the class $\mathcal{C}_{inc}$. As for the completeness of MIG^+, we build a special model from maximal consistent sets of formulas which are closed under the infinitary version of the rule. Since all proper axioms of MIG^+ are Sahlqvist formulas and our modal language is versatile, the underlying relational structure is an incidence frame. See Blackburn et al., 2001; Venema, 1993 for more details about the importance of inference rules like the inference rule of irreflexivity. We do not know if it is possible to eliminate the inference rule of irreflexivity in our axiom system: the completeness of MIG with respect to validity in the class $\mathcal{C}_{inc}$ is still open. As well, the decidability/complexity issue of validity in $\mathcal{C}_{inc}$ is still unresolved.

Note also that we can obtain a complete axiomatization of the projective incidence frames adding the following axioms to the system MIG:

(**MPG1**) $\langle U\rangle A \to \langle O^{-1}\rangle\langle O\rangle A$,

(**MPG2**) $\langle O\rangle(A \wedge \langle O^{-1}\rangle B) \to \langle\not\equiv\rangle(\langle O\rangle A \wedge \langle\not\equiv\rangle B)$.

Extending the language with the modality $[\|]$ we can axiomatize also the affine incidence frames.

We note that the presented systems have rich expressiveness, containing modalities with the following intuitive readings: $[U]A$ – everywhere; $[\neq]A$ – everywhere else; $[\equiv_1]A$ – in all points; $[\not\equiv_1]A$ – in all other points; $[\equiv_2]A$ – in all lines; $[\not\equiv_2]A$ – in all other lines; $[O]A$ – in all lines through the current point; $[O^{-1}]A$ – in all points on the current line.

14. Point-line spatial logics

Standard modal languages have semantics over one sorted frames. Within the context of dynamic logic, van Benthem, 1994, Marx, 1996, and de Rijke, 1995 were probably the first few to use relational structures made up of several sets of possible worlds together with binary relations between them. One possible application of such languages are many-sorted geometrical structures like incidence geometries based on points and lines and inter-sort relations of incidence between them. In this section we follow Venema, 1999.

Two-sorted modal logic. Consider, for instance, a relational structure of the form $\mathcal{F} = (W_1, W_2, R)$ where W_1 and W_2 are nonempty sets and $R \subseteq W_1 \times W_2$. For the sake of simplicity, we assume that the sets W_1 and W_2 are disjoint. From now on, such structures will be called two-sorted Kripke frames. In $\mathcal{F}$, the binary relation R links elements of W_1 with elements of W_2. If modal languages must be used for talking about relational structures like $\mathcal{F}$, one possibility is to consider a language with two sorts of formulas:

- $A ::= p \mid \bot \mid \neg A \mid (A \vee B) \mid \Box\alpha$ – *formulas of the first sort*,
- $\alpha ::= \pi \mid \bot \mid \neg\alpha \mid (\alpha \vee \beta) \mid \Box A$ – *formulas of the second sort.*

where p and π denote propositional letters of the corresponding sorts. Note that the modality $\Box$ transform the one sort into the other.

A two-sorted Kripke model based on $\mathcal{F}$ is nothing but a structure of the form $\mathcal{M} = (W_1, W_2, R, V)$ where V — the valuation of the model — associates a subset $V(p)$ of W_1 with every propositional letter p of the first type and a subset $V(\pi)$ of W_2 with every propositional letter π of the second type. Elements of W_1 being denoted by upper case letters like X, Y, Z, etc, and elements of W_2 being denoted by lower case letters like x, y, z, etc, formulas like p, $\neg A$, $A \vee B$, and $\Box\alpha$ will be interpreted at elements of W_1, whereas formulas like π, $\neg\alpha$, $\alpha \vee \beta$, and $\Box A$ will be interpreted at elements of W_2 according to the satisfiability relation defined as usual. In particular:

- $\mathcal{M}, X \models \Box\alpha$ iff $(\forall y \in W_2)(XRy$ implies $\mathcal{M}, y \models \alpha)$,
- $\mathcal{M}, x \models \Box A$ iff $(\forall Y \in W_1)(YRx$ implies $\mathcal{M}, Y \models A)$.

The notion of a formula of a given sort to be true (satisfiable) in given model can be define in an obvious way. Although the extension of the standard techniques (canonical model, bisimulation, filtration, etc) and results (completeness, finite frame property, definability, etc) of modal logic to multi-sorted languages like the one we have just described has never been seriously considered, we believe that their extension to multi-sorted modal logics is straightforward. As for the two-sorted modal language considered above, it is a simple matter to check that K_2 – the following axiom system – is sound and complete with respect to validity in the class of all two-sorted Kripke frames:

- Axioms of the first type: all first-type substitution instances of classical tautologies together with all formulas of the form
  $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$ and $A \to \Box\Diamond A$,
- Axioms of the second type: all second-type substitution instances of classical tautologies together with all formulas of the form
  $\Box(A \to B) \to (\Box A \to \Box B)$ and $\alpha \to \Box\Diamond\alpha$,
- Inference rules of the first type: Modus ponens "given A and $A \to B$, prove B" and generalization "given α, prove $\Box\alpha$",
- Inference rules of the second type: Modus ponens "given α and $\alpha \to \beta$, prove β" and generalization "given A, prove $\Box A$".

In the next paragraph we apply these ideas to the cases of plane projective geometry and plane affine geometry.

A two-sorted modal logic for plane projective geometry. The point-line incidence planes defined in Sec. 13 are good candidates to stand for the Kripke semantics of a two-sorted modal language. Let us consider the class $\mathcal{C}_{pg}$ of all projective planes, i.e. two-sorted structures $\mathcal{F} = (\mathbf{Po}, \mathbf{Li}, \mathbf{I})$ where **Po** is a nonempty set of points, **Li** is a nonempty set of lines, and I is a binary relation between points and lines such that:

- $\forall X \forall Y \exists z (XIz \wedge YIz)$,
- $\forall x \forall y \exists Z (ZIx \wedge ZIy)$,
- $\forall X \forall Y \forall z \forall t (XIz \wedge YIz \wedge XIt \wedge YIt \rightarrow X = Y \vee z = t)$.

Of course, we will assume that the sets **Po** and **Li** are disjoint. According to the discussion above, we now turn to the definition of our two-sorted modal languages for talking about projective planes. Let us consider a countable set $\Phi_{\mathbf{Po}}$ of point-type proposition letters, with typical members denoted p, q, r, etc, and a countable set $\Phi_{\mathbf{Li}}$ of line-type proposition letters, with typical members denoted π, ρ, σ, etc. The well-formed formulas are defined by the following rules:

- $A ::= p \mid \bot \mid \neg A \mid (A \vee B) \mid \Box\alpha$ – *point formulas*,
- $\alpha ::= \pi \mid \bot \mid \neg\alpha \mid (\alpha \vee \beta) \mid \Box A$ – *line formulas*.

In some two-sorted model $\mathcal{M} = (\mathbf{Po}, \mathbf{Li}, \mathbf{I}, V)$, the point-formula $\Box\alpha$ is satisfied at a point X iff the line-formula α is satisfied at every line passing through X. Similarly, the line-formula $\Box A$ is satisfied at line x iff the point-formula A is satisfied at every point lying on x.

Seeing that two points are always incident with at least one line and two lines are passing together through at least one point, the reader may easily verify that the point-formula $\Box\Box A$ is satisfied at point X in $\mathcal{M}$ iff the point-formula A is true everywhere in $\mathcal{M}$. Similarly, the line-formula $\Box\Box\alpha$ is satisfied at line x in $\mathcal{M}$ iff the line-formula α is true everywhere in $\mathcal{M}$. Hence, the universal modality $[U]$ for points and the universal modality $[u]$ for lines are definable in the following way: $[U]A = \Box\Box A$ and $[u]\alpha = \Box\Box\alpha$.

The two-sorted modal logic defined by the class $\mathcal{C}_{pg}$ of all projective planes has been studied first by Balbiani, 1998 and Venema, 1999. They proved that the axiom system K_{pg} obtained by adding all instances of the following axioms to K_2 is complete with respect to validity in $\mathcal{C}_{pg}$:

Axioms of the first type:	Axioms of the second type:
$\Box\alpha \to [U]\Diamond\alpha$	$\Box A \to [u]\Diamond A$
$[U]A \to A$	$[u]\alpha \to \alpha$
$[U]A \to [U][U]A$	$[u]\alpha \to [u][u]\alpha$
$A \to [U]\langle U\rangle A$	$\alpha \to [u]\langle u\rangle\alpha$

The proof of the decidability of the set of all formulas of the first type satisfiable in $\mathcal{C}_{pg}$ and the proof of the decidability of the set of all formulas of the second type satisfiable in $\mathcal{C}_{pg}$ can be done using the standard technique of the filtration. As usual, this filtration argument implies that satisfiability of point-formulas and line-formulas within $\mathcal{C}_{pg}$ is in NEXPTIME. What makes interesting our two-sorted modal logic is the following result proved by Venema, 1999: satisfiability of point-formulas and line-formulas within the class $\mathcal{C}_{pg}$ is NEXPTIME-complete.

The expressive power of our two-sorted modal language is weak. For example, neither the difference modality between points nor the difference modality between lines are definable in it. Let us extend our two-sorted language by allowing point-formulas like $[\neq]A$ and line-formulas like $[\neq]\alpha$. In some two-sorted model $\mathcal{M} = (\mathbf{Po}, \mathbf{Li}, \mathbf{I}, V)$, the point-formula $[\neq]A$ will be satisfied at point X iff point-formula A is satisfied at every point different from X whereas the line-formula like $[\neq]\alpha$ will be satisfied at line x iff line-formula α is satisfied at every line different from x. The axiomatisation/completeness and decidability/complexity issues of validity and satisfiability of formulas in the extended two-sorted language with respect to the class $\mathcal{C}_{pg}$ are still open.

A two-sorted modal logic for plane affine geometry. A particular aspect of plane projective geometry is the duality between points and lines. In plane affine geometry, points and lines are no longer interchangeable seeing that, in point-line affine planes, although two different points are always incident with exactly one line, parallel lines have no point in common. This imbalance between points and lines in affine planes is translated into additional difficulties for those who wish to define a two-sorted modal logic for plane affine geometry. The language of this two-sorted modal logic must be able to talk about incidence between points and lines and parallelism between lines. The solution in Balbiani and Goranko, 2002 is to consider the following rules that mutually define the formulas of sort point and the formulas of sort line:

- $A ::= p \mid \bot \mid \neg A \mid (A \vee B) \mid \Box\alpha$,
- $\alpha ::= \pi \mid \bot \mid \neg\alpha \mid (\alpha \vee \beta) \mid \Box A \mid [\|_s]\alpha$.

As for the two-sorted modal logic for projective geometry, point formulas like $\Box\alpha$ are read "α is satisfied at every line incident with the current point" and line formulas like $\Box A$ are read "A is satisfied at every point incident with the current line". The unary modality $[\|_s]$ will be interpreted by the strong, i.e. irreflexive, binary relation of parallelism between lines defined, in any affine plane $\mathcal{F} = (\mathbf{Po}, \mathbf{Li}, \mathbf{I})$, by:

- $x \|_s y$ iff for all points Z, not ZIx or not ZIy.

In this setting, if V is a valuation on $\mathcal{F}$ then the definition of the satisfiability relation in the two-sorted model $\mathcal{M} = (\mathbf{Po}, \mathbf{Li}, \mathbf{I}, V)$ defined by V over $\mathcal{F}$ now contains the following item:

- $\mathcal{M}, x \models [\|_s]\alpha$ iff for all $y \in \mathbf{Li}$ such that $x \|_s y$, $\mathcal{M}, y \models \alpha$.

It is a simple matter to check that, in $\mathcal{M}$, the universal modality $[U]$ for points and the universal modality $[u]$ for lines are definable in the following way: $[U]A = \Box\Box A$ and $[u]\alpha = \Box\Box\alpha \wedge [\|_s]\alpha$. Seeing that $[\|_s]$ corresponds to the strong relation of parallelism, we observe that for all points X in $\mathbf{Po}$ and for all lines x in $\mathbf{Li}$:

- $\mathcal{M}, X \models \Box[\|_s]\Box A$ iff for all $Y \in \mathbf{Po}$ such that $X \neq Y$, $\mathcal{M}, Y \models A$,
- $\mathcal{M}, x \models [\|_s]\Box\Box\alpha$ iff for all $y \in \mathbf{Li}$ such that $x \neq y$, $\mathcal{M}, y \models \alpha$.

Hence, the difference modality $[D]$ for points and the difference modality $[d]$ for lines are definable in the following way: $[D]A = \Box[\|_s]\Box A$ and $[d]\alpha = [\|_s]\Box\Box\alpha$. To illustrate the value of our two-sorted modal language, let us remark that, in the class $\mathcal{C}_{ap}$ of all affine planes, the following formulas are valid:

Formulas of type point:	Formulas of type line:
$\Box\alpha \rightarrow \Diamond\alpha$	$\Box A \rightarrow \Diamond A$
$[U]A \rightarrow [U][U]A$	$[\|_s]\alpha \rightarrow \langle\|_s\rangle\alpha$
$[U]A \rightarrow [D]A$	$\alpha \rightarrow [\|_s]\langle\|_s\rangle\alpha$
$A \wedge [D]A \rightarrow [U]A$	$\alpha \wedge [\|_s]\alpha \rightarrow [\|_s][\|_s]\alpha$
$[U]\Box\alpha \leftrightarrow \Box\alpha \wedge \Box[\|_s]\alpha$	$[u]\alpha \rightarrow [d]\alpha$
$A \wedge \Diamond(\alpha \wedge \Diamond(\neg A \wedge [D]A)) \rightarrow \Box(\Box A \vee \alpha)$	$\alpha \wedge [d]\alpha \rightarrow [u]\alpha$
	$[u]\Box A \leftrightarrow \Box A \wedge [\|_s]\Box A$

These formulas are Sahlqvist formulas. Hence they correspond to first-order conditions on two-sorted structures. For example, the point formula $[U]A \rightarrow [U][U]A$ is related to the property of line-connectedness saying that every two points are incident with a common line whereas the point formula $[U]\Box\alpha \leftrightarrow \Box\alpha \wedge \Box[\|_s]\alpha$ and the line formula $[u]\Box A \leftrightarrow \Box A \wedge [\|_s]\Box A$ are related to the existence part of Euclid's property saying that every point not incident with a given line is incident with at least one line parallel to the given line. As for the line formula $A \wedge \Diamond(\alpha \wedge \Diamond(\neg A \wedge [D]A)) \rightarrow \Box(\Box A \vee \alpha)$, it corresponds to the normality conditions saying that every two distinct points have no more than one common incident line. Whether adding to K_2 all instances of the above formulas yields a complete axiom system for validity in $\mathcal{C}_{ag}$ is still open. However, thanks to the possibility of defining in our language the difference modalities between points or lines, we may axiomatize the validity by means of irreflexivity rules. The axiom system AFF is obtained by adding all instances of the above formulas to the basic logic together with the following special inference rules:

Irreflexivity rule of type point: "given $p \wedge [D]\neg p \rightarrow A$, prove A" where p is a proposition letter of sort point not occurring in A,

Irreflexivity rule of type line: "given $\pi \wedge [d]\neg\pi \rightarrow \alpha$, prove α" where π is a proposition letter of sort line not occurring in α,

The completeness of AFF with respect to $\mathcal{C}_{ap}$ is proved by transferring the analogous techniques based on irreflexivity rules known from the one-sorted case. See Balbiani and Goranko, 2002 for details.

Being able to define the difference modalities between points or lines in our two-sorted modal language, it is expressive enough to allow us to define formulas expressing Desarguesian and Pappian properties and to axiomatize the corresponding logics.

It is still open whether satisfiability of point formulas and line formulas within $\mathcal{C}_{ap}$ is decidable. Nevertheless, following the line of reasoning suggested by Venema, 1999, Balbiani and Goranko, 2002 proved that satisfiability within $\mathcal{C}_{ap}$ is NEXPTIME-hard.

Finally, we note that the two-sorted modal perspective in geometry is discussed further in van Benthem, 1996, where Henkin model for second-order logic are considered as two-sorted geometric structures, and in van Benthem, 1999, where space and time sorts are put together.

Concluding remarks: elementary geometry and spatial reasoning

We end this chapter with two brief remarks.

First, there is an obvious disparity in the influence and utility of modern mathematical logic to algebra and geometry: while the main (and quite deep) applications of logic to algebra are model-theoretic, the immediate rôle of logic in geometry is still mainly confined to axiomatizations of geometric theories and logical independence of geometric concepts and properties. While some recent model-theoretic developments (see Hodges, 1993) have deep applications to geometry, they are still far from being accessible enough to enter the geometer's toolbox. In this chapter we have just hinted that logic can say and do more to geometry than what it has so far.

Second, we admit that the topic of this chapter is not directly related to practical spatial reasoning. Yet, we believe that the issues and results discussed here are relevant to it, because quite often, spatial reasoning ignores many geometric attributes such as distances, angles, precise shapes, etc. Just like topology can sometimes be more appropriate than metric geometry for the reasoning tasks at hand, affine planes (with or without ordering) or even plain linear incidence spaces may turn out to be the right level of abstraction. For instance, this should be the case when a street map is used for orientation and routing in the city, or in designing a method for orientation in a maze. We thus see the practical value of the study of logical theories for geometric structures discussed here in offering a hierarchy of levels of abstraction, and providing logical tools and techniques, to suit the particular needs of the agent for spatial representation and reasoning.

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