Filter and Ultrafilter Extensions of Structures: Universal-algebraic Aspects

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Abstract

Uniform upliftings are defined for the functions and relations of a structure of any (finitary) signature to the sets of filters, resp. ultrafilters over its universe. Thus, general constructions of *filter* and *ultrafilter extensions* of structures are introduced. Subsystems, morphisms, congruences and products are shown to be smoothly uplifted from the structures to their filter and ultrafilter extensions. Given a structure $\mathcal{A}$, its power structure based on the set of non-empty subsets of $\mathcal{A}$ is isomorphic to the substructure of the filter extension of $\mathcal{A}$ based on the set of its principal filters. The power structure of $\mathcal{A}$ and the filter extension of $\mathcal{A}$ are proved to satisfy the same identities. As a consequence, the identities of the filter extension of $\mathcal{A}$ are described in terms of those of $\mathcal{A}$.

1 Introduction

The aim of this article is to introduce a uniform, and natural from universal-algebraic viewpoint, construction of *filter and ultrafilter extensions* for arbitrary structures, These are supposed to extend the basic operations and relations in a structure to the set of filters, resp. ultrafilters, of the structure.

The work originated with the construction of an ultrafilter extension. Thus, the ultrafilter extension of a structure $\mathcal{A}$ (a non-empty set endowed with functions and relations of finite arity) is supposed to be a structure $\mathbf{U}(\mathcal{A})$ of the same signature, based on the set of ultrafilters over the domain of $\mathcal{A}$, with functions and relations defined in an appropriate uniform way, such that:

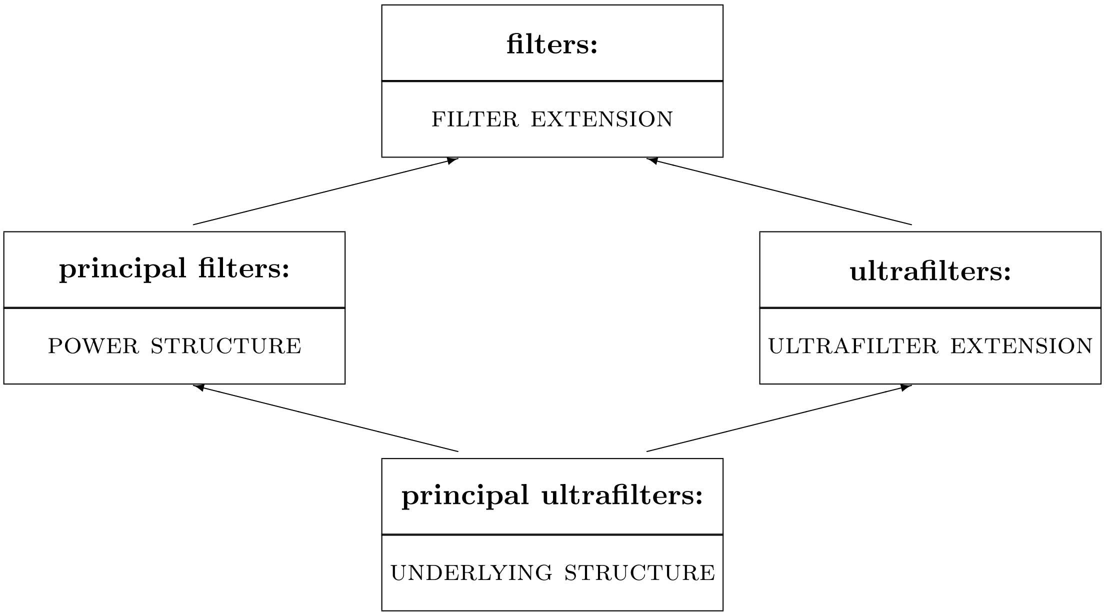
i) the subset of principal ultrafilters in $\mathbf{U}(\mathcal{A})$ is a sub-structure of $\mathbf{U}(\mathcal{A})$ isomorphic to $\mathcal{A}$;

ii) the construction uplifts uniformly the basic algebraic constructs (sub-structures, morphisms, congruences, direct products) from the structures to their ultrafilter extensions.

Ultrafilter extensions or closely related constructions have been introduced for various specific mathematical structures, two important examples being Stone representation for Boolean algebras, later extended by Jónsson and Tarski in [Jónnson & Tarski, 1951] to Boolean algebras with operators, and Wallman's construction in topology (see e.g. [Engelking, 1977]). In particular, ultrafilter extensions of binary relational structures play an important role in model theory of various non-classical logics incl. modal, temporal, dynamic etc. logics where they are in the core of the duality between algebraic and relational semantics (see e.g.

[van Benthem, 1979], cite[Ben 84],[Fin 75],[Gol 76],[Gor 90]). The construction of ultrafilter extensions for arbitrary relational structures introduced here is a smooth generalization of the above mentioned one. The kernel of this construction has been essentially introduced and studied in [Goldblatt, 1989] and particular cases of some results in the present paper have been proved there.

Concerning ultrafilter extensions of functions, there are some technical problems (see section 3.2) the solution of which has suggested the introduction of a *filter extension* of a structure, which in some respect seems now the more natural one. Moreover, the filter extension naturally extends both the ultrafilter extension and another important general algebraic construction, viz. the *power structure* (see [Brink, 1991]) (or *complex structures* in [Grätzer & Whitney, 1984]), which is isomorphic to the substructure of the principal filters of the filter extension of any structure. Thus, eventually the following scheme settled down in the focus of the paper (the arrows indicate isomorphic embeddings):



Here is an outline of the results in the paper. After the preliminary section 2, we introduce in section 3 filter and ultrafilter extensions of relations and functions and prove some basic facts about them. We also discuss the above mentioned problem with ultrafilter extensions of functions. In section 4, we introduce for any structure $\mathcal{A}$ the filter extension $\mathbf{F}(\mathcal{A})$, ultrafilter extension $\mathbf{U}(\mathcal{A})$ and principal_filter extension $\mathbf{F_p}(\mathcal{A})$. $\mathcal{A}$ is shown to be isomorphically embedded in each of these extensions through the substructure based on the set of principal ultrafilters on $\mathcal{A}$. We call these embeddings *canonical.* Also, $\mathbf{F_p}(\mathcal{A})$ is shown to be isomorphic to the power structure of $\mathcal{A}$ based on the set of non-empty subsets of $\mathcal{A}$. In section 5, filter and ultrafilter extensions of mappings between structures are introduced and it is proved that each of the introduced constructions preserves the property of a mapping to be a homomorphism (resp. injective, surjective, strong, bounded homomorphism, isomorphic embedding, isomorphism). Moreover, these extensions commute with the respective canonical embeddings. In particular, the filter (resp. principal_filter, ultrafilter) extension of a substructure of a structure is isomorphic to a substructure of the filter (resp. principal_filter, ultrafilter) extension of the structure. In section 6, filter and ultrafilter extensions of congruences in structures are introduced and it is proved that the filter (resp. principal_filter, ultrafilter) extension of a quotient-structure of a structure over given congruence is isomorphic to the quotient of the filter (resp. principal_filter, ultrafilter) extension of the structure over the respective extension of the congruence. In section 7, filter and ultrafilter extensions of direct products of structures are investigated, and the filter (resp. principal_filter, ultrafilter) extension of a product of structures is proved to be homomorphically mapped onto the product of the respective extensions of the factors. Moreover, for filter and principal_filter extensions, these homomorphisms are shown to be retractions and hence strong homomorphisms. In the last section 8, identities in filter extensions are considered. The main result is that the filter and the principal_filter extension of any structure satisfy the same identities. As corollaries, applying results from [Grätzer & Whitney, 1984] and [Grätzer & Lakser, 1988] about power structures, we give a necessary and sufficient condition a variety of structures to be closed under filter extensions and describe the identities in the filter extension of a structure in terms of those satisfied by the structure.

It should be noted that in this paper we concentrate exclusively on the general universal-algebraic properties of filter and ultrafilter extensions and, in order to keep the text within reasonable bounds without being shallow or sketchy, we do not discuss possible applications to particular mathematical fields; they will be treated elsewhere. Thus, the technical results mentioned above should also be viewed as a collective evidence for the natural algebraic behaviour and robustness of the introduced constructions, and therefore as a (platonic) justification of their study, at least from universal-algebraic perspective.

2 Preliminaries

Since most of the classical texts in universal algebra (e.g. [Cohn, 1965, Grätzer, 1968, Burris & Sankappanavar, 1981]) deal with purely algebraic systems only (i.e. without relations in the signature) we begin with a compendium of definitions of basic universal-algebraic notions and constructions introduced for arbitrary structures in order to fix the notation and terminology, which are close to e.g. [Mal'cev, 1970]. The reader is assumed to have basic knowledge on filters and ultrafilters; either of [Bell & Slomson, 1969] and [Eklof, 1977] would be a more than sufficient reference.

Given a set A, by $\mathbf{P}(A)$ we denote the set of subsets of A and $\mathbf{P}^{+}(A) = \mathbf{P}(A)\backslash\{\emptyset\}$. If n is a positive integer, by $[n]$ we denote the set $\{1, \ldots, n\}$. The principal filter on a set A generated by a subset $X \subseteq A$ is denoted by $\mathbf{f}_A[X]$; the corresponding principal ultrafilter when $X = \{x\}$ is denoted by $\mathbf{u}_A[x]$. When the basic set A is fixed by the context, the index may be omitted. We also denote: the set of filters on A by $\mathbf{F}(A)$; the set of ultrafilters on A by $\mathbf{U}(A)$; the set of principal filters on A by $\mathbf{F_p}(A)$; the set of principal ultrafilters on A by $\mathbf{U_p}(A)$;

A *signature* is an ordered triple $< \mathcal{R}, \mathcal{F}, \mu >$ such that $\mathcal{R}$ and $\mathcal{F}$ are disjoint sets, and μ is a function from $\mathcal{R} \cup \mathcal{F}$ to $\mathcal{N}$. $\mathcal{F}$ is the set of *functional symbols*; $\mathcal{R}$ is the set of *relational symbols;* μ is the *arity* function. If $q \in \mathcal{R}$ (resp. $q \in \mathcal{F}$) and $\mu(q) = n$ then q is called an *n-ary relational symbol* (resp. an *n-ary functional symbol.* When $\mathcal{R} = \emptyset$, σ is a *functional signature;* when $\mathcal{F} = \emptyset$, σ is a *relational* one.

A structure of signature $\sigma =< \mathcal{R}, \mathcal{F}, \mu >$ is every ordered triple $\mathcal{A} =< A, \sigma, \nu >$ where A is a non-empty set called *universe* (or *domain*) of $\mathcal{A}$, denoted also $|\mathcal{A}|$ and ν is a mapping from $\mathcal{R} \cup \mathcal{F}$ to the set of functions and relations in A which associates with every $R \in \mathcal{R}$ an $\mu(R)$-ary relation $\nu^{\mathcal{A}}(R)$ in A and with every $F \in \mathcal{F}$ an $\mu(F)$-ary function $\nu^{\mathcal{A}}(F)$ in A. ν is called an *interpretation* of σ in A. The interpretations under ν of the symbols from the signature are called *basic functions and relations* of the structure. Cardinality of $\mathcal{A}$ is the cardinality of its universe. Henceforth we shall write $q^{\mathcal{A}}$ instead of $\nu_{\mathcal{A}}(q)$.

Note that we only consider *finitary* signatures (with functions and relations of finite arities).

Terms of the signature σ over a fixed, disjoint with $\mathcal{F}$ and $\mathcal{R}$, infinite set of variables $\{v_1, v_2, \ldots\}$ are defined accordingly.

Hereafter an arbitrary signature $\sigma =< \mathcal{R}, \mathcal{F}, \mu >$ is fixed.

2.1 Morphisms.

Let $\mathcal{A} =< A, \sigma, \nu_A >$ and $\mathcal{B} =< B, \sigma, \nu_B >$ be structures of the same signature σ. A mapping $\varphi : A \to B$ is called:

- a *homomorphism* if

  i) for every n-ary $F \in \mathcal{F}$ and $a_1, \ldots, a_n \in A$,

  $$\varphi(F^{\mathcal{A}}(a_1, \ldots, a_n)) = F^{\mathcal{B}}(\varphi(a_1), \ldots, \varphi(a_n));$$

  and

  ii) for every n-ary $R \in \mathcal{R}$ and $a_1, \ldots, a_n \in A$,

  $$R^{\mathcal{A}}(a_1, \ldots, a_n) \text{ implies } R^{\mathcal{B}}(\varphi(a_1), \ldots, \varphi(a_n));$$

- a *strong homomorphism* if it is a homomorphism satisfying the additional condition:

  for every n-ary $R \in \mathcal{R}$ and $b_1, \ldots, b_n$ from the image $\varphi[A]$ of A, if $R^{\mathcal{B}}(b_1, \ldots, b_n)$ then there exist $a_1, \ldots, a_n \in A$ such that $R^{\mathcal{A}}(a_1, \ldots, a_n)$ and $\varphi(a_i) = b_i$, for $i = 1, \ldots, n$;

- a *bounded homomorphism* if it is a homomorphism satisfying a stronger version of the additional condition for a strong homomorphism, viz.:

  for every n-ary $R \in \mathcal{R}$, $a \in A$ and $b_1, \ldots, b_n \in \varphi[A]$ if $R^{\mathcal{B}}(b_1, \ldots, b_n)$ and $b_j = \varphi(a)$ for some $j \in [n]$, then there exist $a_1, \ldots, a_n \in A$, such that $a_j = a, R^{\mathcal{A}}(a_1, \ldots, a_n)$ and $\varphi(a_i) = b_i$, for $i = 1, \ldots, n$.

- an *isomorphic embedding* if it is an injective strong homomorphism.

- an *isomorphism* if it is a bijective strong homomorphism (equivalently, surjective isomorphic embedding).

Note that when φ is an isomorphic embedding, "strong" and "bounded" mean the same, viz.: for every n-ary $R \in \mathcal{R}$ and $a_1, \ldots, a_n \in A$, $R^{\mathcal{A}}(a_1, \ldots, a_n)$ iff $R^{\mathcal{B}}(\varphi(a_1), \ldots, \varphi(a_n))$.

If σ is a functional signature (but not in general) then every homomorphism is strong and hence every bijective homomorphism is an isomorphism.

The notion of bounded homomorphism, although not very popular in the universal algebra is important for applications to computer science, being a special case of a *bisimulation*.

2.2 Substructures.

Let $\mathcal{A} =< A, \sigma, \nu >$ be a structure and $A' \subseteq A$, $A' \neq \emptyset$. A' is *closed in* $\mathcal{A}$ if A' is closed under all basic functions of $\mathcal{A}$ i.e. for every n-ary $F \in \mathcal{F}$ if $a_1, \ldots, a_n \in A'$ then $F^{\mathcal{A}}(a_1, \ldots, a_n) \in$A'.

Let A' be closed in $\mathcal{A}$. Then the structure $\mathcal{A}' =< A', \sigma, \nu' >$, where ν' interprets all symbols from the signature with the restrictions of their interpretations in $\mathcal{A}$, is called a *substructure of* $\mathcal{A}$.

Clearly, if the signature is relational then every non-empty subset of the universe of the structure is a universe of a substructure.

2.3 Congruences and quotient-structures.

Let $\mathcal{A} =< A, \sigma, \nu >$. A binary relation ρ in $|\mathcal{A}|$ is *stable with respect to the n-ary function F in $\mathcal{A}$* if for every two n-tuples $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in $\mathcal{A}$,

$$a_1 \rho b_1, \ldots, a_n \rho b_n \text{ implies } F^{\mathcal{A}}(a_1, \ldots, a_n)\ \rho\ F^{\mathcal{A}}(b_1, \ldots, b_n).$$

Further, ρ is *stable* in $\mathcal{A}$ if it is stable with respect to every basic function of $\mathcal{A}$.

A *congruence* in $\mathcal{A}$ is every stable in $\mathcal{A}$ equivalence relation.

Let θ be a congruence in $\mathcal{A}$. Then the *quotient-structure* $\mathcal{A}/\theta = < A/\theta, \sigma, \nu_\theta >$ *of* $\mathcal{A}$ *modulo the congruence* θ is defined as follows:

i) for every n-ary $F \in \mathcal{F}$ and $a_1/\theta, \ldots, a_n/\theta \in A/\theta$,

$$F^{\mathcal{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = F^{\mathcal{A}}(a_1, \ldots, a_n)/\theta;$$

ii) for every n-ary $R \in \mathcal{R}$ and $a_1/\theta, \ldots, a_n/\theta \in A/\theta$,

$$R^{\mathcal{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) \text{ if } R^{\mathcal{A}}(b_1, \ldots, b_n) \text{ for some } b_1 \in a_1/\theta, \ldots, b_n \in a_n/\theta.$$

It is a routine task to check the correctness of these definitions.

The structure thus defined is called a *quotient-structure* $\mathcal{A}$ *modulo the congruence* θ.

Two facts:

- The canonical mapping $\eta_\theta : \mathcal{A} \to \mathcal{A}/\theta$, given by $\eta_\theta(a) = a/\theta$ is a strong surjective homomorphism.
- (*Homomorphism theorem*) For every homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ the kernel equivalence θ_φ is a congruence in $\mathcal{A}$ called the *kernel congruence of* φ, and the mapping $\tau : \mathcal{A}/\theta_\varphi \to \mathcal{B}$, defined by $\tau(a/\theta_\varphi) = \varphi(a)$ is a bijective homomorphism such that $\tau \circ \eta_{\theta_\varphi} = \varphi$. Moreover, φ is a strong homomorphism then τ is an isomorphism.

Thus, the strongly homomorphic images of a structure $\mathcal{A}$ coincide (up to isomorphism) with the quotient-structures of $\mathcal{A}$, which are in one-to-one correspondence with the congruences in $\mathcal{A}$.

2.4 Cartesian products.

Let $\{\mathcal{A}_i =< A_i, \sigma, \nu_i >\}_{i\in I}$ be a family of structures of σ. The *direct* (or *cartesian*) *product of* that family is the structure

$$\mathcal{A} = \prod_{i\in I}\{\mathcal{A}_i\} = \left\langle \prod_{i\in I}\{A_i\}, \sigma, \nu \right\rangle,$$

where ν interprets σ coordinate-wise:

$$F^{\mathcal{A}}(a_1, \ldots, a_n)(i) = F^{\mathcal{A}_i}(a_1(i), \ldots, a_n(i)) \text{ for every } i \in I;$$

and

$$R^{\mathcal{A}}(a_1, \ldots, a_n) \text{ iff } R^{\mathcal{A}_i}(a_1(i), \ldots, a_n(i)) \text{ for every} i \in I.$$

The coordinate functions $\pi_j : \prod_{i\in I}\{\mathcal{A}_i\} \to \mathcal{A}_j$ defined by $\pi_j(a) = a_j$ are called *projections* of $\prod\{\mathcal{A}_i\}_{i\in I}$. They are homomorphisms but not (in general) strong homomorphisms.

3 Filter and Ultrafilter Extensions of Functions and Relations

3.1 Filter and ultrafilter extensions of relations.

Let A be a non-empty set and $R \subseteq A^{n+1}$ for some $n \in \mathcal{N}$. We define $n+1$ n-ary operations in $\mathbf{P}(A)$ as follows: for each $i \in [n+1]$,

$$< R >_i (X_1, .., X_{i-1}, X_{i+1}, .., X_{n+1}) \ = \tag{1}$$

$\{x \in A \mid$ for all $j \in [n+1] \setminus \{i\}$ there exists $x_j \in X_j$ such that $R(x_1, .., x_{i-1}, x, x_{i+1}, .., x_{n+1})\}$.

Note that $< R >_i$ is a monotone operation on each of its arguments.

We now define $n+1$ $(n+1)$-ary relations in $\mathbf{F}(A)$:

$$R_i(\phi_1, \ldots, \phi_{n+1}) \text{ iff for every } j \in [n+1] \setminus \{i\} \text{ and } X_j \in \phi_j, \tag{2}$$

$$< R >_i (X_1, .., X_{i-1}, X_{i+1}, .., X_{n+1}) \in \phi_i.$$

Finally, we define an $n+1$-ary relation $\mathbf{f}(R)$ in $\mathbf{F}(A)$ as follows:

$$\mathbf{f}(R)(\phi_1, \ldots, \phi_{n+1}) \text{ iff } R_1(\phi_1, \ldots, \phi_{n+1}) \& \ldots \& R_{n+1}(\phi_1, \ldots, \phi_{n+1}),$$

$$\text{i.e., } \mathbf{f}(R) = R_1 \cap \ldots \cap R_{n+1}.$$

Definition 1 $\mathbf{f}(R)$ *is called the* filter extension of the relation R *in* $\mathbf{F}(A)$.

For instance, if R is the equality $=$ in A, then $<=>_1 (X) = <=>_2 (X) = X$, hence $=_1 (\phi_1, \phi_2)$ iff $=_2 (\phi_1, \phi_2)$ iff $\phi_1 = \phi_2$. Thus, $\mathbf{f}(=)$ is the equality in $\mathbf{F}(A)$.

Let us note that if R is a unary relation on A, i.e. a subset of A, then (1) gives $< R >_1 = \{x \in A \mid Rx\} = R$, whence (2) vacuously yields $R_1(\phi)$ iff $R \in \phi$. Thus $\mathbf{f}(R) = \{\phi \mid R \in \phi\}$.

Proposition 2 *1. If $\phi_1, \ldots, \phi_{n+1}$ are principal filters, $\phi_i = \mathbf{f}[X_i]$, then*

$$\mathbf{f}(R)(\phi_1, \ldots, \phi_{n+1}) \text{ iff for any } i \in [n+1], X_i \subseteq < R >_i (X_1, .., X_{i-1}, X_{i+1}, .., X_{n+1}).$$

2. If $\phi_1, \ldots, \phi_{n+1}$ are principal ultrafilters, $\phi_i = \mathbf{u}_A[x_i]$, then

$$\mathbf{f}(R)(\phi_1, \ldots, \phi_{n+1}) \text{ iff } R(x_1, \ldots, x_{n+1}).$$

Proof 1 *Straightforward from the definition.*

Proposition 3 *For any $R \subseteq A^{n+1}$, $\mathbf{f}(R) = R_1 = \ldots = R_{n+1}$ on $\mathbf{U}(A)$.*

Proof 2 *Let $\phi_1, \ldots, \phi_{n+1}$ be ultrafilters, $R_1(\phi_1, \ldots, \phi_{n+1})$. Suppose not $R_i(\phi_1, \ldots, \phi_{n+1})$ for some $i \neq 1$. Without loss of generality we can assume $i = n+1$ for technical ease. Thus, for some $X_1 \in \phi_1, \ldots, X_n \in \phi_n$, $< R >_{n+1} (X_1, .., X_n) \notin \phi_{n+1}$ and hence $(A \setminus < R >_{n+1} (X_1, .., X_n)) \in \phi_{n+1}$. Therefore $Y =< R >_1 (X_2, .., X_n, (A \setminus < R >_{n+1} (X_1, .., X_n)))$ belongs to ϕ_1, hence $X_1 \cap Y \neq \emptyset$. Let $x \in X_1 \cap Y$. Then for some $x_2 \in X_2, \ldots, x_n \in X_n$, and $x_{n+1} \in (A \setminus < R >_{n+1} (X_1, .., X_n))$, $R(x, x_2, \ldots, x_{n+1})$ holds. Therefore, by definition $x_{n+1} \in < R >_{n+1} (X_1, .., X_n)$ — a contradiction.*

Definition 4 $\mathbf{f}(R)$ *restricted on* $\mathbf{U}(A)$*, is called the* ultrafilter extension of the relation R in $\mathbf{U}(A)$*, denoted* $\mathbf{u}(R)$*.*

Thus, $\mathbf{u}(R) = R_1 = \ldots = R_{n+1}$.

3.2 Filter and ultrafilter extensions of functions.

Let $F : A^n \to A$ be a function and $R_F \subseteq A^{n+1}$ be the graph of F. We define an n-ary function $\mathbf{f}(F)$ in $\mathbf{F}(A)$ as follows:

$$\mathbf{f}(F)(\phi_1, \ldots, \phi_n) = \tag{3}$$

$$\{X \subseteq A \mid F[X_1, \ldots, X_n] \subseteq X \text{ for some } X_1 \in \phi_1, \ldots, X_n \in \phi_n\},$$

where

$$F[X_1, \ldots, X_n] = \{F(x_1, \ldots, x_n) \mid x_1 \in X_1, \ldots, x_n \in X_n\}.$$

Note that $F[X_1, \ldots, X_n]$ is monotone on each argument.

Proposition 5 *$\mathbf{f}(F)(\phi_1, \ldots, \phi_n)$ is a filter on A.*

Proof 3 *1. $F[A, \ldots, A] \subseteq A$, hence $A \in \mathbf{f}(F)(\phi_1, \ldots, \phi_n)$.*

2. $\emptyset \notin R_F[\phi_1, \ldots, \phi_n]$ since $F[X_1, \ldots, X_n] = \emptyset$ iff $X_i = \emptyset$ for some $i \in [n]$.

3. If $X \in \mathbf{f}(F)(\phi_1, \ldots, \phi_n)$ and $X \subseteq Y \subseteq A$ then obviously $Y \in \mathbf{f}(F)(\phi_1, \ldots, \phi_n)$.

4. If $X', X'' \in \mathbf{f}(F)(\phi_1, \ldots, \phi_n)$, and $F[X'_1, \ldots, X'_n] \subseteq X'$, $F[X''_1, \ldots, X''_n] \subseteq X''$ for $X'_i, X''_i \in \phi_i$, then $X'_i \cap X''_i \in \phi_i$ for all $i \in [n]$, and $F[X'_1 \cap X''_1, \ldots, X'_n \cap X''_n \subseteq X' \cap X''$ due to monotonicity, hence $X' \cap X'' \in \mathbf{f}(F)(\phi_1, \ldots, \phi_n)$.

Thus, definition (3) is correct.

Definition 6 *$\mathbf{f}(F)$ is called the* filter extension of the function F in $\mathbf{F}(A)$.

Remark 7 *This definition needs some discussion. There are two different rules for evaluation of a composition of functions: from outside inwards (corresponding to the computational rule known as "call by name") and from inside outwards (corresponding to "call by value"). For deterministic (mono-valued) functions these two rules yield the same result. Applied to multi-valued (or partial) functions, however, they act differently. Thus, when one defines a "power function" of a given function, i.e. a function defined on tuples of subsets of the domain, the evaluation from outside inwards yields the following:*

$$F[G_1[X_1, \ldots, X_n], \ldots, G_m[X_1, \ldots, X_n]] =$$
$$\{F(y_1, \ldots, y_m) : y_i \in G_i[X_1, \ldots, X_n], i \in [m]\},$$

while the evaluation from inside outwards yields the rule

$$F[G_1[X_1, \ldots, X_n], \ldots, G_m[X_1, \ldots, X_n]] =$$
$$\{F(G_1(x_1, \ldots, x_n), \ldots, G_m(x_1, \ldots, x_n)) : x_j \in X_j, j \in [n]\}.$$

One can easily see that, in general, these two rules generate different versions of a "power function". In particular, the composition of power functions, according to the latter rule is equal to the power function of the composition while this is not the case with the former rule. Nevertheless, for sound reasons, the former rule is commonly accepted in power structures (see [Brink, 1991, Grätzer & Lakser, 1988]) which explicably causes some peculiarities in their algebraic theory. We too adopt the same rule for computation of a composition. Thus, if $H(x_1, \ldots, x_n) = F(G_1(x_1, \ldots, x_n), \ldots, G_m(x_1, \ldots, x_n))$, then, by certain abuse of notation, $H(X_1, \ldots, X_n)$ will be an abbreviation for $F[G_1[X_1, \ldots, X_n], \ldots, G_m[X_1, \ldots, X_n]]$.

This choice has to be maid once again when we consider filter extension of functions and their composition. For instance, given a binary function F, its filter extension applied to two equal arguments yields

$$\mathbf{f}(F)(\phi,\phi) = \{Y : F[X_1, X_2] \subseteq Y \text{ for some } X_1, X_2 \in \phi\},$$

while the filter extension of the unary function G, defined by $G(x) = F(x,x)$, gives $\mathbf{f}(G)(\phi) = \{Y : F[X,X] \subseteq Y \text{ for some } X \in \phi\}$. Fortunately, both functions turn out to be the same, which can be generalized to the following result.

Proposition 8 *The filter extension of a composition of functions is equal to the composition of their filter extensions.*

Proof 4 *Let F be an m-ary function and $G_1, \ldots, G_m$ be n-ary functions in a set A. Let H be the composition of F with $G_1, \ldots, G_m$. We have to show that, for every $\phi_1, \ldots, \phi_n \in \mathbf{F}(A)$,*

$$\mathbf{f}(H)(\phi_1, \ldots, \phi_n) = \mathbf{f}(F)(f(G_1)(\phi_1, \ldots, \phi_n), \ldots, \mathbf{f}(G_m)(\phi_1, \ldots, \phi_n)).$$

Indeed,
$\mathbf{f}(F)(\mathbf{f}(G_1)(\phi_1, \ldots, \phi_n), \ldots, \mathbf{f}(G_m)(\phi_1, \ldots, \phi_n)) =$
$\{Y \subseteq A \mid F[Y_1, \ldots, Y_m] \subseteq Y \text{ for some } Y_i \in \mathbf{f}(G_i)(\phi_1, \ldots, \phi_n), i \in [m]\} =$
$\{Y \subseteq A \mid F[G_1[X_1^1, .., X_n^1], .., G_m[X_1^m, .., X_n^m]] \subseteq Y \text{ for some } X_j^i \in \phi_j, j \in [n], i \in [m]\} =$
(taking $X_j = X_j^1 \cap \ldots \cap X_j^m, j \in [n]$)
$\{Y \subseteq A \mid F[G_1[X_1, .., X_n], .., G_m[X_1, .., X_n]] \subseteq Y \text{ for some } X_j \in \phi_j, j \in [n]\} =$
$\{Y \subseteq A \mid H[X_1, .., X_n] \subseteq Y \text{ for some} X_j \in \phi_j, j \in [n]\} =$
$\mathbf{f}(H)(\phi_1, \ldots, \phi_n)$.

Proposition 9 *For every n-ary function F in A and filters $\phi_1, \ldots, \phi_n$ on A,*

$$\mathbf{f}(R_F)(\phi_1, \ldots, \phi_n, \mathbf{f}(F)(\phi_1, \ldots, \phi_n)).$$

Proof 5 *First, note that $F[X_1, \ldots, X_n] = < R_F >_{n+1} (X_1, \ldots, X_n)$, and hence*

$$(R_F)_{n+1}(\phi_1, \ldots, \phi_n, \mathbf{f}(F)(\phi_1, \ldots, \phi_n)).$$

Now, let $i \in [n]$ and $X'_j \in \phi_j, j \in [n+1] \setminus \{i\}$, and $X \in \mathbf{f}(F)(\phi_1, \ldots, \phi_n)$. Then $F[X''_1, \ldots, X''_n] \subseteq X$ for some $X''_j \in \phi_j, j \in [n]$. Therefore, $X_j = X'_j \cap X''_j \in \phi_j$ and $F[X_1, .., X_{i-1}, X''_i, X_{i+1}, .., X_n] \subseteq X$. Hence ($X_1, .., X_{i-1}, X_{i+1}, .., X_n$ are non-empty),
$X''_i \subseteq \{x_i \in A \mid F(x_1, .., x_i, .., x_n) = x \text{ for some } x_j \in X_j, j \in [n+1] \setminus \{i\}, \text{ and } x \in X\} \subseteq$
$\{x_i \in A \mid F(x_1, .., x_i, .., x_n) = x \text{ for some } x_j \in X'_j, j \in [n+1] \setminus \{i\}, \text{ and } x \in X\} =$
$< R_F > (X'_1, .., X'_{i-1}, X'_{i+1}, .., X'_n, X)$.

Therefore, the latter set belongs to ϕ_i, hence $(R_F)_i(\phi_1, \ldots, \phi_n, \mathbf{f}(F)(\phi_1, \ldots, \phi_n))$ holds. Thus, finally $\mathbf{f}(R_F)(\phi_1, \ldots, \phi_n, \mathbf{f}(F)(\phi_1, \ldots, \phi_n))$ holds.

So, the filter extension of a function agrees with the filter extension of its graph.

Proposition 10

1. *If F is an n-ary function in A and $\phi_1, \ldots, \phi_n$ are principal filters on A, $\phi_i = \mathbf{f}[X_i]$, then*

$$\mathbf{f}(F)(\phi_1, \ldots, \phi_n) = \{X \subseteq A \mid F[X_1, \ldots, X_n] \subseteq X\} = \mathbf{f}[F[X_1, \ldots, X_n]].$$

2. *If $\phi_1, \ldots, \phi_n$ are principal ultrafilters, $\phi_i = \mathbf{u}_A[x_i]$, then $\mathbf{f}(F)(\phi_1, \ldots, \phi_n)$ is a principal ultrafilter and*

$$\mathbf{f}(F)(\phi_1, \ldots, \phi_n) = \mathbf{u}_A[F(x_1, \ldots, x_n)].$$

Proof 6 *Straightforward from the definition.*

The rest of this section is devoted to the problem of defining an *ultrafilter* extension $\mathbf{u}(F)$ of a function F. As we see from the last proposition, $\mathbf{f}(F)$ works smoothly on the principal ultrafilters. Besides, there is another important case when it is easy to see that $\mathbf{f}(F)$ preserves ultrafilters.

Proposition 11 *If F is a unary function and $u \in \mathbf{U}(A)$ then $\mathbf{f}(F)(u) \in \mathbf{U}(A)$.*

Proof 7 *Let $u \in \mathbf{U}(A)$ and suppose that $X \notin \mathbf{f}(F)(u)$. Then $\{x \in A \mid F(x) \in X\} \notin u$, hence $\{x \in A \mid F(x) \in A \setminus X\} \in u$, so $A \setminus X \in \mathbf{f}(F)(u)$.*

Thus, if A is finite (hence all ultrafilters are principal) or if only unary functions are considered, $\mathbf{f}(F)$ restricted to $\mathbf{U}(A)$ does the job of an ultrafilter extension. In general, however, $\mathbf{f}(F)$ restricted to $\mathbf{U}(A)$, does not work since $\mathbf{f}(F)$ does not preserve free ultrafilters. Here is a simple example. Let $A = \mathcal{N}$, $X = 2\mathcal{N}$, u_1, u_2 be arbitrary free ultrafilters, and

$$F(m, n) = \begin{cases} 0 & \text{if } m \leq n; \\ 1 & \text{if } m > n. \end{cases}$$

Then $F[X_1, X_2] = \{0, 1\}$ for any $X_1 \in u_1, X_2 \in u_2$ since both X_1 and X_2 are infinite. Thus neither X nor $\mathcal{N} \setminus X$ belongs to $\mathbf{f}(F)(u_1, u_2)$.

Still, we should like the notion of ultrafilter extension of a function to agree both with its filter extension and with the ultrafilter extension of its graph. To this end it is natural to require for every ultrafilters $u_1, \ldots, u_n$ that $\mathbf{u}(F)(u_1, \ldots, u_n)$ is an ultrafilter extending the filter $\mathbf{f}(F)(u_1, \ldots, u_n)$. There seem to be two reasonable ways to meet this requirement. One of them is to introduce the ultrafilter extension of F as a *multi-valued* functions:

$$\mathbf{u}[F](u_1, \ldots, u_n) = \{u \in \mathbf{U}(A) \mid \mathbf{f}(F)(u_1, \ldots, u_n) \subseteq u\}.$$

This approach, although natural, would lead to explicable complications in the algebraic theory of the ultrafilter extensions. Still, in areas where multi-valued mappings are of common interest, e.g. in topology, this might be the right approach. The other solution is to introduce a uniform selector function for the set of all ultrafilters extending $\mathbf{f}(F)(u_1, \ldots, u_n)$, which is to select the value of $\mathbf{u}(F)(u_1, \ldots, u_n)$ in a "canonical" way. We propose such a selector defined as follows: for any $u_1, \ldots, u_n \in \mathbf{U}(A)$,

$$\mathbf{u}(F)(u_1, \ldots, u_n) = \{X \subseteq A \mid \ \{x_1 \mid \ldots \{x_n \mid \ F(x_1, \ldots, x_n) \in X\} \ \in \ u_n\} \ldots\} \ \in \ u_1\}. \quad (4)$$

Proposition 12 *If F is an n-ary function in A and $u_1, \ldots, u_n \in \mathbf{U}(A)$, then $\mathbf{u}(F)(u_1, \ldots, u_n)$, defined as above, is an ultrafilter on A which contains $\mathbf{f}(F)(u_1, \ldots, u_n)$.*

Proof 8 *For technical simplicity we shall consider the case $n = 2$. We denote $F_X(x) = \{y \mid F(x, y) \in X\}$ for any $X \subseteq A$ and $x \in A$, and $F_X(u) = (x \mid F_X(x) \in u\}$ for any $u \in \mathbf{U}(A)$.*

We shall prove that

$$\mathbf{u}(F)(u_1, u_2) = \{X \subseteq A \mid F_X(u_2) \in u_1\}$$

is an ultrafilter.

1. *$\emptyset \notin \mathbf{u}(F)(u_1, u_2)$ since $F_\emptyset(x) = \emptyset$ for every $x \in A$ and $\emptyset \notin u_2$.*
2. *Let $X, Y \subseteq A$. For every $x \in A$, $F_{X \cap Y}(x) = F_X(x) \ \cap \ F_Y(x)$, hence $F_{X \cap Y}(u_2) = F_X(u_2) \ \cap \ F_Y(u_2)$, and therefore $F_{X \cap Y}(u_2) \in u_1$ iff $F_X(u_2) \in u_1$ and $F_Y(u_2) \in u_1$, i.e. $X \cap Y \in \mathbf{u}(F)(u_1, u_2)$ iff $X, Y \in \mathbf{u}(F)(u_1, u_2)$.*
3. *For every $X \subseteq A$ and $x \in A$, $F_{A \setminus X}(x) = A \setminus F_X(x)$, hence $F_{A \setminus X}(u_2) = A \setminus F_X(u_2)$, and therefore $F_{A \setminus X}(u_2) \in u_1$ iff not $F_X(u_2) \in u_1$, i.e. $A \setminus X \in \mathbf{u}(F)(u_1, u_2)$ iff not $X \in \mathbf{u}(F)(u_1, u_2)$.*

Thus $\mathbf{u}(F)(u_1, u_2)$ is an ultrafilter.

Now, let $X \in \mathbf{f}(F)(u_1, u_2)$. This mean that for every $x_1 \in X_1$ and $x_2 \in X_2$, $F(x_1, x_2) \in X$. Therefore, for every $x_1 \in X_1$, $X_2 \subseteq F_X(x_1)$, hence $F_X(x_1) \in u_2$. Thus $X_1 \subseteq F_X(u_2)$, hence $F_X(u_2) \in u_1$, so $X \in \mathbf{u}(F)(u_1, u_2)$. Therefore $< R_F >_3 (u_1, u_2) \subseteq \mathbf{u}(F)(u_1, u_2)$, i.e. $\mathbf{u}(F)(u_1, u_2) \in \mathbf{u}[F](u_1, u_2)$.

This proof obviously generalizes for an arbitrary n.

Thus we have a uniform and seemingly natural (at least, as we shall see further, having natural algebraic properties) way of restricting $\mathbf{u}[F]$ to a function $\mathbf{u}(F)$.

Definition 13 $\mathbf{u}(F)$*, defined by (4), is called an* ultrafilter extension of the function F in $\mathbf{U}(A)$.

Still, we must note that the above definition is not completely satisfactory since the result of the construction involved in (4) crucially depends on the order in which we trace the arguments of the function F. For instance, if F is a binary function, we can also introduce

$$\mathbf{u}'(F)(u_1, u_2) = \{X \subseteq A \mid \ \{x_2 \mid \ \{x_1 \mid \ F(x_1, x_2) \in X\} \ \in \ u_1\} \ \in \ u_2\}$$

and similarly prove that $\mathbf{u}'(F)(u_1, u_2) \in \mathbf{u}[F](u_1, u_2)$. Thus we have (at least) two natural rivals for the title "ultrafilter extension" of the function F. It is easy to show that they may be different. For instance, in example 1, one can easily see that $2\mathcal{N} \in \mathbf{u}(F)(u_1, u_2)$, while $2\mathcal{N} \notin \mathbf{u}'(F)(u_1, u_2)$.A bit more complicated example shows that, even if the binary function F is commutative, the two canonical extensions of F can be different.

EXAMPLE 2. Let $A = \mathcal{N}, F(m, n) = m + n$ and

$$X = \bigcup\{[2^n, 2^{n+1}] \mid n \in 2\mathcal{N}\} = \{1, 2, 4, 5, 6, 7, 8, 16, \ldots\}.$$

For any $Y \subseteq \mathcal{N}$ and $n \in \mathcal{N}$, denote $Y - n = \{m \in \mathcal{N} \mid m + n \in Y\}$.

Now, consider the family $\Phi_1 = \{X - n \mid n \in \mathcal{N}\}$. For every $n \in \mathcal{N}$, the set $X \cap (X-1) \cap \ldots \cap (X - n)$ is infinite. Therefore Φ_1 is contained in some free ultrafilter u_1 on $\mathcal{N}$. Likewise, the family $\Phi_2 = \{(\mathcal{N} \setminus X) - n \mid n \in \mathcal{N}\}$ is contained in some free ultrafilter u_2 on $\mathcal{N}$. Now, $\mathbf{u}(F)(u_1, u_2) = \{Z \subseteq \mathcal{N} \mid \{n \mid \{m \mid n + m \in Z\} \in u_2\} \in u_1\} = \{Z \subseteq \mathcal{N} \mid \{n \mid Z - n \in u_2\} \in u_1\}$ contains $\mathcal{N} \setminus X$ and not X, while $\mathbf{u}'(F)(u_1, u_2)$ contains X (and not $\mathcal{N}\setminus$X).

Since F is commutative, $\mathbf{u}'(F)(u_1, u_2) = \mathbf{u}(F)(u_2, u_1)$ and thus, moreover we see that $\mathbf{u}(F)$ is not commutative, i.e. the ultrafilter extension as introduced by (4) does not preserve such a simple identity as commutativity. Another potential flaw of that ultrafilter extension is that it does not enjoy the property of filter extensions, stated in prop. 8, which can be easily exemplified by the functions

$$f(x, y) = \begin{cases} 0 & \text{if } x = y; \\ 1 & \text{otherwise} \end{cases}$$

and $g(x) = f(x, x) = 0$.

Clearly, when an n-ary function F is considered there might be as many as $n!$ different candidates for an ultrafilter extension of F, constructed in this way. We shall call these candidates *canonical extensions of* F.

On the other hand, it is obvious that all canonical extensions of a function, being symmetrically defined, enjoy the same general properties, i.e. each of them would equally well serve our purposes. Thus, the choice of the "proper" extension amongst the canonical ones is solely a matter of some technical convenience. So, we shall conclude this discussion raising the question:

Is there a natural and uniform way to introduce a (unique) ultrafilter extension of a function, "better" than definition (4)?

4 Filter and Ultrafilter Extensions of Structures.

Definition 14 *Let $\mathcal{A} =< A; F_1, \ldots, F_n, \ldots; R_1, \ldots, R_m, \ldots >$ be a structure of an arbitrary signature σ.*

1. Filter extension *of $\mathcal{A}$ is the structure*

$$\mathbf{F}(\mathcal{A}) =< \mathbf{F}(A); \mathbf{f}(F_1), \ldots, \mathbf{f}(F_n), \ldots; \mathbf{f}(R_1), \ldots, \mathbf{f}(R_m), \ldots > .$$

2. Principal_filter extension *of $\mathcal{A}$ is the substructure (due to prop. 2 and 10) $\mathbf{F_p}(\mathcal{A})$ of $\mathbf{F}(\mathcal{A})$, consisting of the principal filters on $\mathcal{A}$.*

3. Ultrafilter extension *of $\mathcal{A}$ is the structure*

$$\mathbf{U}(\mathcal{A}) =< \mathbf{U}(A); \mathbf{u}(F_1), \ldots, \mathbf{u}(F_n), \ldots; \mathbf{u}(R_1), \ldots, \mathbf{u}(R_m), \ldots > .$$

iv) Power structure of $\mathcal{A}$ *(cf. [Grätzer & Whitney, 1984] and [Brink, 1991]) is the structure*

$$\mathbf{P}^+(\mathcal{A}) =< \mathbf{P}^+(A); \mathbf{p}(F_1), \ldots, \mathbf{p}(F_n), \ldots; \mathbf{p}(R_1), \ldots, \mathbf{p}(R_m), \ldots > ,$$

where

$$\mathbf{p}(F)(X_1, \ldots, X_n) = F[X_1, \ldots, X_n]$$

and

$$\mathbf{p}(R)(X_1, \ldots, X_{n+1}) \textit{ iff } X_i \subseteq < R >_i (X_1, .., X_{i-1}, X_{i+1}, .., X_{n+1}) \textit{ for all } i \in [n+1].$$

Thus, $\mathbf{F}(\mathcal{A}), \mathbf{F_p}(\mathcal{A}), \mathbf{P}^+\mathcal{A})$ and $\mathbf{U}(\mathcal{A})$ have the same signature as $\mathcal{A}$. Recall that, although $\mathbf{U}(A) \subseteq F(A), \mathbf{U}(\mathcal{A})$ is not a substructure of $\mathbf{F}(\mathcal{A})$ since the interpretation of functional symbols in these structures do not agree.

The following two propositions follow from prop. 2 and 10.

Proposition 15 *The mapping $\alpha : \mathbf{P}^+(\mathcal{A}) \to \mathbf{F_p}(\mathcal{A})$, defined by $\alpha(X) = \mathbf{f}_{\mathcal{A}}[X]$ is an isomorphism.*

Thus, the power structure of a structure is isomorphically embedded in the filter extension of the structure. When the "full" power structure $\mathbf{P}(\mathcal{A})$ based on the set of *all* subsets of the underlying structure is considered, a similar result holds; see the concluding comments for more details.

Proposition 16

1. *The mapping $\epsilon_{\mathcal{A}} : \mathcal{A} \to \mathbf{U}(\mathcal{A})$, defined by $\epsilon_{\mathcal{A}}(x) = \mathbf{u}_A[x]$ is an isomorphic embedding.*
2. *The same mapping is an isomorphic embedding of $\mathcal{A}$ into $\mathbf{F_p}(\mathcal{A})$ and hence into $\mathbf{F}(\mathcal{A})$.*

We shall call $\epsilon_{\mathcal{A}}$ a *canonical embedding* of $\mathcal{A}$ into $\mathbf{U}(\mathcal{A})$, respectively $\mathbf{F_p}(\mathcal{A})$ and $\mathbf{F}(\mathcal{A})$.

Corollary 17 *If $\mathcal{A}$ is finite then $\mathcal{A} \cong \mathbf{U}(\mathcal{A})$.*

Henceforth an arbitrary signature σ is fixed and, unless otherwise specified, all considered structures will be of signature σ. We shall assume that σ contains an equality symbol $=$ which will always have the standard interpretation.

5 Filter and Ultrafilter Extensions of Homomorphisms and Substructures.

Proposition 18 *Let $\mathcal{A}, \mathcal{B}$ be structures and $\gamma: \mathcal{A} \to \mathcal{B}$ be a mapping. Then:*

1. *For every $\phi \in \mathbf{F}(\mathcal{A})$ the set*

$$\gamma[\phi] = \{Y \subseteq |\mathcal{B}| \mid \gamma[X] \subseteq Y \text{for some } X \in \phi\}$$

   is a filter on $\mathcal{B}$.

2. *If ϕ is a principal filter on $\mathcal{A}, \phi = \mathbf{f}[X]$, then $\gamma[\phi]$ is a principal filter on $\mathcal{B}$ and $\gamma[\phi] = \mathbf{f}[\gamma[X]]$.*

3. *For every $\phi \in \mathbf{U}(\mathcal{A})$, $\gamma[\phi]$ is an ultrafilter on $\mathcal{B}$.*

4. *If ϕ is a principal ultrafilter on $\mathcal{A}, \phi = \mathbf{u}[x]$, then $\gamma[\phi]$ is a principal ultrafilter on $\mathcal{B}$, at that, $\gamma[\phi] = \mathbf{u}[\gamma(x)]$.*

Proof 9

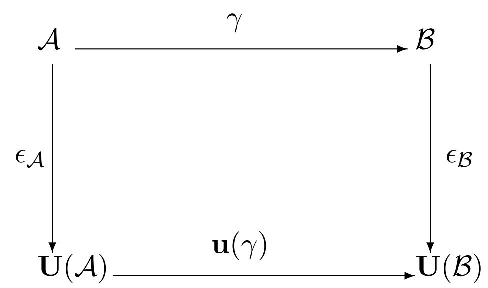
1. *The proof basically repeats that of prop. 5 for the case of unary function.*
2. *is straightforward.*
3. *Add to 1. the argument from the proof of prop. 11.*
4. *follows directly from 2.*

Definition 19 *Let $\gamma: \mathcal{A} \to \mathcal{B}$.*

1. *The mapping $\mathbf{f}(\gamma): \mathbf{F}(\mathcal{A}) \to \mathbf{F}(\mathcal{B})$, defined by $\mathbf{f}(\gamma)(\phi) = \gamma[\phi]$ is called the* filter extension of the mapping γ.
2. *The restriction $\mathbf{f_p}(\gamma): \mathbf{F_p}(\mathcal{A}) \to \mathbf{F_p}(\mathcal{B})$ of $\mathbf{f}(\gamma)$ is called the* principal_filter extension *of γ.*
3. *The restriction $\mathbf{u}(\gamma): \mathbf{U}(\mathcal{A}) \to \mathbf{U}(\mathcal{B})$ of $\mathbf{f}(\gamma)$ is called the* ultrafilter extension of γ.

Theorem 20 *Let $\gamma: \mathcal{A} \to \mathcal{B}$. Then*

1. *The diagram*



is commutative.

2. *If γ is injective then $\mathbf{f}(\gamma), \mathbf{f_p}(\gamma)$, and $\mathbf{u}(\gamma)$ are injective.*

3. *If γ is surjective, then $\mathbf{f}(\gamma), \mathbf{f_p}(\gamma)$, and $\mathbf{u}(\gamma)$ are surjective.*

4. *If γ is a homomorphism then $\mathbf{f}(\gamma), \mathbf{f_p}(\gamma)$, and $\mathbf{u}(\gamma)$ are homomorphisms.*

5. *If γ is an isomorphic embedding (resp. isomorphism) then $\mathbf{f}(\gamma), \mathbf{f_p}(\gamma)$, and $\mathbf{u}(\gamma)$ are isomorphic embeddings (resp. isomorphisms).*

Proof 10 *Let $A = |\mathcal{A}|$ and $B = |\mathcal{B}|$.*

1. *Follows from prop. 18.4.*

2. *Let γ be injective and $\mathbf{f}(\gamma)(\phi_1) = \mathbf{f}(\gamma)(\phi_2) = \psi$ for some $\phi_1, \phi_2 \in \mathbf{F}(A)$. Then for every $X_1 \in \phi_1$, $\gamma[X_1] \in \psi$, hence $\gamma[X_2] \subseteq \gamma[X_1]$ for some $X_2 \in \phi_2$, and therefore $X_2 \subseteq X_1$ (for γ being injective), so $X_1 \in \phi_2$. Thus $\phi_1 \subseteq \phi_2$. Likewise, $\phi_1 \subseteq \phi_2$, hence $\phi_1 = \phi_2$.*

   Now, it immediately follows that $\mathbf{f_p}(\gamma)$ and $\mathbf{u}(\gamma)$ are injective, too.

3. *Let γ be surjective and $\psi \in \mathbf{F}(B)$. We consider the set*

   $$\phi = \{X \subseteq A \mid \gamma^{-1}[Y] \subseteq X \text{ for some } Y \in \psi\}.$$

   ϕ is a filter on A:

   (a) *$\emptyset \notin \phi$ since $\gamma^{-1}[Y] = \emptyset$ iff $Y = \emptyset$, for γ being surjective.*
   (b) *$A = \gamma^{-1}[B]$, hence $A \in \phi$.*
   (c) *Obviously ϕ is closed under supersets.*
   (d) *If $X_1, X_2 \in \phi$ then $\gamma^{-1}[Y_1] \subseteq X_1$ and $\gamma^{-1}[Y_2] \subseteq X_2$ for some $Y_1, Y_2 \in \psi$. Then $Y_1 \cap Y_2 \in \psi$ and $\gamma^{-1}[Y_1 \cap Y_2] \subseteq \gamma^{-1}[Y_1] \cap \gamma^{-1}[Y_2] \subseteq X_1 \cap X_2$, hence $X_1 \cap X_2 \in \phi$.*

   Now it remains to show that $\mathbf{f}(\gamma)(\phi) = \psi$. Indeed, for every $Y \in \psi$, $\gamma^{-1}[Y] \in \phi$ and $\gamma[\gamma^{-1}[Y]] = Y$ (for γ being surjective), hence $Y \in \mathbf{f}(\gamma)(\phi)$, so $\psi \subseteq \mathbf{f}(\gamma)(\phi)$. Vice versa, if $Z \in \mathbf{f}(\gamma)(\phi)$, then $\gamma[X] \subseteq Z$ for some $X \in \phi$, and $\gamma^{-1}[Y] \subseteq X$ for some $Y \in \psi$, hence $Y = \gamma[\gamma^{-1}[Y]] \subseteq \gamma[X] \subseteq Z$, so $Z \in \psi$. Thus $\mathbf{f}(\gamma)(\phi) \subseteq \psi$ and finally, $\mathbf{f}(\gamma)(\phi) = \psi$.

   Therefore $\mathbf{f}(\gamma)$ is surjective.

   If ψ is a principal filter, $\psi = \mathbf{f}[Y]$, then the filter ϕ, constructed above is $\mathbf{f}[\gamma^{-1}[Y]]$. Thus, $\mathbf{f_p}(\gamma)$ is surjective.

   As for $\mathbf{u}(\gamma)$, given an ultrafilter ψ from $\mathbf{U}(\mathcal{B})$, the filter ϕ constructed above is contained in some ultrafilter $u \in \mathbf{U}(\mathcal{A})$. Then $\psi = \mathbf{f}(\gamma)(\phi) \subseteq \mathbf{f}(\gamma)(u) = \mathbf{u}(\gamma)(u)$. Therefore $\psi = \mathbf{u}(\gamma)(u)$ since both sides are ultrafilters. Thus, $\mathbf{u}(\gamma)$ is surjective.

4. Let γ be a homomorphism.

4.1) Let F be an n-ary functional symbol from the signature σ and $\phi_1, \ldots, \phi_n$ be filters on A. Then we must show that

$$\mathbf{f}(\gamma)(\mathbf{f}(F^{\mathcal{A}})(\phi_1, \ldots, \phi_n)) = \mathbf{f}(F^{\mathcal{B}})(\mathbf{f}(\gamma)(\phi_1), \ldots, \mathbf{f}(\gamma)(\phi_n)).$$

The key observation here is that for any subsets $X_1, \ldots, X_n$ of A,

$$\gamma[F^{\mathcal{A}}[X_1, \ldots, X_n]] = F^{\mathcal{B}}[\gamma[X_1], \ldots, \gamma[X_n]],$$

which immediately follows from the fact that γ is a homomorphism.

Now,

$\mathbf{f}(\gamma)(\mathbf{f}(F^{A})(\phi_1, \ldots, \phi_n)) =$

$\{Y \subseteq B \mid \gamma[X] \subseteq Y$ *for some* $X \in \mathbf{f}(F^{\mathcal{A}})(\phi_1, \ldots, \phi_n)\} =$

$\{Y \subseteq B \mid \gamma[F^{\mathcal{A}}[X_1, \ldots, X_n]] \subseteq Y$ *for some* $X_1 \in \phi_1, \ldots, X_n \in \phi_n\} =$

$\{Y \subseteq B \mid F^{\mathcal{B}}[\gamma[X_1], \ldots, \gamma[X_n]] \subseteq Y$ *for some* $X_1 \in \phi_1, \ldots, X_n \in \phi_n\} =$

$\{Y \subseteq B \mid F^{\mathcal{B}}[Y_1, \ldots, Y_n] \subseteq Y$ *for some* $Y_1 \in \mathbf{f}(\gamma)(\phi_1), \ldots, Y_n \in \mathbf{f}(\gamma)(\phi_n)\} =$

$\mathbf{f}(F^{\mathcal{B}})(\mathbf{f}(\gamma)(\phi_1), \ldots, \mathbf{f}(\gamma)(\phi_n))$.

4.2) Let R be an $n+1$-ary relational symbol from σ and $\phi_1, \ldots, \phi_{n+1}$ be filters on A. Then we must show that $\mathbf{f}(R^{\mathcal{A}})(\phi_1, \ldots, \phi_{n+1})$ implies $\mathbf{f}(R^{\mathcal{B}})(\mathbf{f}(\gamma)(\phi_1), \ldots, \mathbf{f}(\gamma)(\phi_{n+1}))$.

Let $\mathbf{f}(R^{\mathcal{A}})(\phi_1, \ldots, \phi_{n+1})$ and $Y_i \in \mathbf{f}(\gamma)(\phi_i)$, $i \in [n]$. Then, for some $X_i \in \phi_i$ and every $i \in [n]$, $\gamma[X_i] \subseteq Y_i$, and $< R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_n) \in \phi_{n+1}$. For every $x \in \ < R^{\mathcal{A}} >_{n+1}$ $(X_1, \ldots, X_n)$ and $y_1 = \gamma(x_1) \in Y_1, \ldots, y_n = \gamma(x_n) \in Y_n, R^{\mathcal{B}}(y_1, \ldots, \ y_n \gamma(x))$ holds. Therefore $\gamma[< R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_n)] \subseteq < R^{\mathcal{B}} >_{n+1} (Y_1, \ldots, Y_n)$. Hence $< R^{\mathcal{B}} >_{n+1}$ $(Y_1, \ldots, Y_n) \in \mathbf{f}(\gamma)(\phi_{n+1})$. Thus, $\mathbf{f}(R^{\mathcal{B}})\mathbf{f}(\gamma)(\phi_1), \ldots, \mathbf{f}(\gamma)(\phi_{n+1})$.

Now, $\mathbf{f_p}(\gamma)$, being a restriction of the homomorphism $\mathbf{f_p}(\gamma)$ to the substructure $\mathbf{F_p}(\mathcal{A})$, is a homomorphism, too.

As for $\mathbf{u}(\gamma)$, we only have to take care of the functional symbols, since the interpretations of the relational ones in filter and ultrafilter extensions agree. Let F be an n-ary functional symbol from σ and $u_1, \ldots, u_n$ be ultrafilters on A. Then we must show that

$$\mathbf{f}(\gamma)(\mathbf{f}(F^{\mathcal{A}})(u_1, \ldots, u_n)) = \mathbf{f}(F^{\mathcal{B}})(\mathbf{f}(\gamma)(u_1), \ldots, \mathbf{f}(\gamma)(u_n)).$$

Since both sides are ultrafilters, it is enough to prove the inclusion $\subseteq$. For technical simplicity we shall consider the case $n = 2$. Let $Y \in \mathbf{f}(\gamma)(\mathbf{f}(F^{\mathcal{A}})(u_1, u_2))$, i.e. $\gamma[X] \subseteq Y$ for some $X \in \mathbf{f}(F^{\mathcal{A}})(u_1, u_2)$. Then $W = \{x_1 \in A \mid \{x_2 \in A \mid F^{\mathcal{A}}(x_1, x_2) \in X\} \in u_2\} \in u_1$. Let $x_1 \in W$ be fixed for a while. Then $W(x_1) = \{x_2 \in A \mid F^{\mathcal{A}}(x_1, x_2) \in X\} \in u_2$, hence $\gamma[W(x_1)] \in \mathbf{f}(\gamma)(u_2)$ and $\gamma[W(x_1)] = \{\gamma(x_2) \mid x_2 \in A$ and $\gamma(F^{\mathcal{A}}(x_1, x_2)) \in \gamma[X]\} = \{\gamma(x_2) : x_2 \in A$ and $F^{\mathcal{B}}(\gamma(x_1), \gamma(x_2)) \in \gamma[X]\} \subseteq \{y_2 \in B \mid F^{\mathcal{B}}(\gamma(x_1), y_2) \in Y\}$. Therefore the latter set belongs to $\mathbf{f}(\gamma)(u_2)$.

Further, $\gamma[W] \subseteq \{y_1 \in B \mid \{y_2 \in B \mid F^{\mathcal{B}}(y_1, y_2) \in Y\} \in \mathbf{f}(\gamma)(u_2)\}$ and $\gamma[W] \in \mathbf{f}(\gamma)(u_1)$, hence $Y \in \mathbf{f}(F^{\mathcal{B}})(\mathbf{f}(\gamma)(u_1), \mathbf{f}(\gamma)(u_2))$ by definition.

5. *If γ is an isomorphic embedding then $\mathbf{f}(\gamma)$ is injective and homomorphism due to 2. and 4. In addition, we have to show that $\mathbf{f}(\gamma)$ is a strong homomorphism. So, let R be an $n+1$-ary relational symbol and $\phi_1, \ldots, \phi_n$ be filters on A. Then we must show that*

$$\mathbf{f}(R^{\mathcal{B}})(\mathbf{f}(\gamma)(\phi_1), \ldots, \mathbf{f}(\gamma)(\phi_{n+1})) \text{ implies } \mathbf{f}(R^{\mathcal{A}})(\phi_1, \ldots, \phi_{n+1}).$$

Suppose $\mathbf{f}(R^{\mathcal{B}})(\mathbf{f}(\gamma)(\phi_1), \ldots, \mathbf{f}(\gamma)(\phi_{n+1}))$. We shall prove that $R_i^{\mathcal{A}}(\phi_1, \ldots, \phi_{n+1})$ holds for each $i \in [n+1]$. For technical convenience we shall consider the case $i = n+1$; the others are analogous. So, let $X_i \in \phi_i$, $i \in [n]$. Then $\gamma[X_i] \in \mathbf{f}(\gamma)(\phi_i)$, hence $< R^{\mathcal{B}} >_{n+1} (\gamma[X_1], \ldots, \gamma[X_n]) \in \mathbf{f}(\gamma)(\phi_{n+1})$. Therefore $\gamma[X] \subseteq < R^{\mathcal{B}} >_{n+1} (\gamma[X_1], \ldots, \gamma[X_n])$ for some $X \in \phi_{n+1}$. Now, the crucial fact to be used is that

$$\gamma[X] \subseteq < R^{\mathcal{B}} >_{n+1} (\gamma[X_1], \ldots, \gamma[X_n]) \text{ iff } X \subseteq < R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_n)$$

for γ being an injective strong homomorphism. It implies $< R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_n) \in \phi_{n+1}$, hence $R_{n+1}^{\mathcal{A}}(\phi_1, \ldots, \phi_{n+1})$ holds. Thus, $\mathbf{f}(R^{\mathcal{A}})(\phi_1, \ldots, \phi_{n+1})$.

It follows immediately that $\mathbf{f_p}(\gamma)$ and $\mathbf{u}(\gamma)$ are isomorphic embeddings, too.

Finally, if γ is an isomorphism, i.e. a surjective isomorphic embedding, then so are $\mathbf{f}(\gamma), \mathbf{f_p}(\gamma)$ and $\mathbf{u}(\gamma)$.

Theorem 21 *If $\gamma : \mathcal{A} \rightarrow \mathcal{B}$ is a strong homomorphism then $\mathbf{f}(\gamma), \mathbf{f_p}(\gamma)$ and $\mathbf{u}(\gamma)$ are strong homomorphisms.*

Proof 11 *We begin with $\mathbf{f}(\gamma)$. It is enough to consider the case when γ is surjective. Then, in the general case we can decompose γ into a strong homomorphism $\gamma_1 : \mathcal{A} \rightarrow \gamma[\mathcal{A}]$, which is surjective, followed by the identity $\gamma_2 : \gamma[\mathcal{A}] \rightarrow \mathcal{B}$ which is a strong homomorphism, too. Then, by the particular case and th. 20.5, $\mathbf{f}(\gamma)$ will be represented as a composition of two strong homomorphisms which is a strong homomorphism itself.*

So, let γ be surjective. Then $\mathbf{f}(\gamma)$ is a homomorphism and we only have to show in addition that it is strong. Let R be an $n+1$-ary relational symbol and $\phi_1, \ldots, \phi_{n+1}$ be filters on A. Then we must show that if

$$\mathbf{f}(R^{\mathcal{B}})(\mathbf{f}(\gamma)(\phi_1), \ldots, \mathbf{f}(\gamma)(\phi_{n+1}))$$

then there exist filters $\omega_1, \ldots, \omega_{n+1}$ on A, such that

$$\mathbf{f}(\gamma)(\omega_i) = \mathbf{f}(\gamma)(\phi_i), \; i \in [n+1], \tag{5}$$

and

$$\mathbf{f}(R^{\mathcal{A}})(\omega_1, \ldots, \omega_{n+1}). \tag{6}$$

We start with considering the following sets:

$$\theta_i = \{X \subseteq A \;\mid \gamma^{-1}[\gamma[X']] \subseteq X \text{ for some } X' \in \phi_i\}, i \in [n+1].$$

For every $i \in [n+1]$, θ_i is a filter on A:

1. *If* $\gamma^{-1}[\gamma[X']] = \emptyset$ *then* $X' = \emptyset$, *hence* $\emptyset \notin \theta_i$.

2. $\gamma^{-1}[\gamma[A]] \subseteq A$, *hence* $A \in \theta_i$.

3. θ_i *is obviously closed under supersets.*

4. *Let* $X_1, X_2 \in \theta_i$, *hence* $\gamma^{-1}[\gamma[X_1']] \subseteq X_1$ *and* $\gamma^{-1}[\gamma[X_2']] \subseteq X_2$ *for some* $X_1', X_2' \in \phi_i$. *Then* $X_1' \cap X_2' = X' \in \phi_i$ *and* $\gamma^{-1}[\gamma[X']] \subseteq \gamma^{-1}[\gamma[X_1']] \cap \gamma^{-1}[\gamma[X_2']] \subseteq X_1 \cap X_2$, *hence* $X_1 \cap X_2 \in \theta_i$.

Now, some observations:

a) *For every* $X \subseteq A, X \subseteq \gamma^{-1}[\gamma[X]]$.

b) *Let* $X = \gamma^{-1}[\gamma[X']]$ *for some* $X' \subseteq A$. *Then* $\gamma^{-1}[\gamma[X]] = \gamma^{-1}[\gamma[\gamma^{-1}[\gamma[X']]]] = \gamma^{-1}[\gamma[X']]$ $= X$. *Thus* $X = \gamma^{-1}[\gamma[X]]$. *We shall call sets* X *for which* $X = \gamma^{-1}[\gamma[X]]$ regular.

c) *Thus, every* $Y \in \theta_i$ *contains a regular set from* $\theta_i \cap \phi_i$.

d) *For every regular set* $X \subseteq A$ *and* $x \in A, \gamma(x) \in \gamma[X]$ *iff* $x \in X$.

e) *If* X_1, X_2 *are regular then* $X_1 \cap X_2$ *is regular:*

$$X_1 \cap X_2 \subseteq \gamma^{-1}[\gamma[X_1 \cap X_2]] \subseteq \gamma^{-1}[\gamma[X_1]] \cap \gamma^{-1}[\gamma[X_2]] = X_1 \cap X_2.$$

f) *If all* $X_i \in \phi_i$, $i \in [n]$, *are regular then*

$$< R^{\mathcal{B}} >_{n+1} (\gamma[X_1], \ldots, \gamma[X_n]) = \gamma[< R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_n)].$$

Indeed, $< R^{\mathcal{B}} >_{n+1} (\gamma[X_1], \ldots, \gamma[X_n]) = \{\gamma(x) \mid x \in A$ *and* $R^{\mathcal{B}}(\gamma(x_1), \ldots, \gamma(x_n), \gamma(x)$ *for some* $x_1 \in X_1, \ldots, x_n \in X_n\}$ *(here we used that* γ *is surjective)* $= \{\gamma(x) \mid x \in A$ *and* $R^{\mathcal{A}} x_1, \ldots, x_n, x$ *for some* $x_1 \in X_1, \ldots, x_n \in X_n\}$ *(here we used the fact that* γ *is strong and the property* d) *of the regular sets)* $= \gamma[< R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_n)]$.

Now, we consider the sets

$$\psi_i = \{X \subseteq A \mid < R^{\mathcal{A}} >_i (X_1, .., X_{i-1}, X_{i+1}, .., X_n) \subseteq X \textit{ for some regular sets} X_j \in \theta_j, j \neq i\}$$

We shall show that for every $i \in [n+1]$, ψ_i *is a filter on* A. *For technical ease we shall consider the case* $i = n+1$; *the others are analogous.*

1. $\emptyset \notin \psi_{n+1}$. *Otherwise* $< R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_n) = \emptyset$ *for some regular sets* $X_1 \in \theta_1, \ldots X_n \in \theta_n$. *But then*

   $\emptyset = \gamma[< R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_n)] =< R^{\mathcal{B}} >_{n+1} (\gamma[X_1], \ldots, \gamma[X_n]) \in \mathbf{f}(\gamma)(\phi_{n+1})$ *since* $\gamma[X_i] \in \mathbf{f}(\gamma)(\phi_i)$ *for each* $i \in [n]$, *and* $\mathbf{f}(R^{\mathcal{B}})\mathbf{f}(\gamma)(\phi_1), \ldots, \mathbf{f}(\gamma)(\phi_{n+1})$. *Thus* $\emptyset \in \mathbf{f}(\gamma)(\phi_{n+1})$ *which is impossible.*

2. *Obviously,* $A \in \psi_{n+1}$.

3. *If* $X \in \psi_{n+1}$ *and* $X \subseteq Y$ *then clearly* $Y \in \psi_{n+1}$.

4. If $X', X'' \in \psi_{n+1}$, then $< R^{\mathcal{A}} >_{n+1} (X'_1, \ldots, X'_n) \subseteq X'$ and $< R^{\mathcal{A}} >_{n+1} (X''_1, \ldots, X''_1) \subseteq X''$ for some regular $X'_i, X''_i \in \theta_i$. Then for every $i \in [n], X_i = X'_i \cap X'_i$ is a regular set in θ_i and $< R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_n) \subseteq X' \cap X''$, hence $X' \cap X'' \in \psi_{n+1}$.

Now, we are ready to show that $\theta_i \cup \psi_i$ is a centered family of sets. Again the case $i = n+1$ is representative. For that purpose, having already proved that θ_{n+1} and ψ_{n+1} are both filters, and due to the observation c) above, it is enough to show that for every regular $X_i \in \theta_i, i \in [n+1]$,

$$X_{n+1} \cap < R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_n) \neq \emptyset.$$

Suppose otherwise. Then $\emptyset =$
$\gamma[X_{n+1} \cap < R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_n)]$
$\{\gamma(x) \ \mid x \in X_{n+1}$ *and* $R^{\mathcal{A}} x_1, \ldots, x_n, x$ *for some* $x_1 \in X_1, \ldots, x_n \in X_n\} =$
(again we shall use the fact that $X_1, \ldots, X_{n+1}$ are regular sets)
$\{\gamma(x) \ \mid x \in X_{n+1}$ *and* $R^{\mathcal{B}} \gamma(x_1), \ldots, \gamma(x_n), \gamma(x)$ *for some* $x_1 \in X_1, \ldots, x_n \in X_n\} =$
$\gamma[X_{n+1}] \cap < R^{\mathcal{B}} >_{n+1} (\gamma[X_1], \ldots, \gamma[X_n]) \in \mathbf{f}(\gamma)(\phi_{n+1})$,
which is impossible.

Thus, $\theta_{n+1} \cup \psi_{n+1}$ is centered. The same holds for $\theta_i \cup \psi_i$, for each $i \in [n]$. Therefore the families

$$\omega_i = \{X \subseteq A \ \mid Y \cap Z \subseteq X \text{ for some } Y \in \theta_i \text{ and } Z \in \psi_i\}$$

are filters on A.

We shall prove that they satisfy (5) and (6) .

Let us prove (5) for $i = n+1$. If $W \in \mathbf{f}(\gamma)(\phi_{n+1})$ then $\gamma[X] \subseteq W$ for some $X \in \phi_{n+1}$. Therefore $\gamma^{-1}[\gamma[X]] \in \theta_{n+1} \subseteq \omega_{n+1}$, hence $\gamma[X] = \gamma[\gamma^{-1}[\gamma[X]]] \in \mathbf{f}(\gamma)(\omega_{n+1})$, so $W \in \mathbf{f}(\gamma)(\omega_{n+1})$. For the opposite inclusion it is enough to show that for every $Y \in \theta_{n+1}$ and $Z \in \psi_{n+1}$, $\gamma[Y \cap Z] \in \mathbf{f}(\gamma)(\phi_{n+1})$. It is enough to consider a regular Y and $Z =< R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_n)$ where $X_i \in \theta_i$ are regular. As we saw before, in such a case
$\gamma[Y \cap < R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_n)] = \gamma[Y] \cap < R^{\mathcal{B}} >_{n+1} (\gamma[X_1], \ldots, \gamma[X_n]) \in \mathbf{f}(\gamma)(\phi_{n+1})$.

Thus $\mathbf{f}(\gamma)(\omega_{n+1}) = \mathbf{f}(\gamma)(\phi_{n+1})$ and likewise $\mathbf{f}(\gamma)(\omega_i) = \mathbf{f}(\gamma)(\phi_i)$ for $i \in [n]$.

In order to prove (6) we have to show that $R_i^{\mathcal{A}}(\omega_1, \ldots, \omega_{n+1})$ for $i \in [n+1]$). Again we shall consider the case $i = n+1$. Let $Z_1 \in \omega_1, \ldots, Z_n \in \omega_n$. We have to show that $< R^{\mathcal{A}} >_{n+1} (Z_1, \ldots, Z_n) \in \omega_{n+1}$.

It is enough to consider all Z_i of the kind $Z_i = X_i \cap < R^{\mathcal{A}} >_i (Y^i_1, .., Y^i_{i-1}, Y^i_{i+1}, .., Y^i_{n+1})$, where $X_i \in \theta_i$ are regular and $Y^i_j \in \theta_j$ are regular, too. Let

$$Y_j = Y_j^1 \cap \ldots Y_j^{j-1} \cap Y_j^{j+1} \cap \ldots Y_j^n \cap X_j,$$

for each $j \in [n]$, and

$$Y_{n+1} = Y^1_{n+1} \cap \ldots \cap Y^n_{n+1}.$$

Then Y_j is a regular set from θ_j, due to the observation (e) and $Y_j \subseteq X_j$ for all $j \in [n]$. For each $i \in [n]$ we denote

$$W_i = X_i \cap < R^{\mathcal{A}} >_i (Y_1, .., Y_{i-1}, Y_{i+1}, .., Y_{n+1}).$$

$W_i \in \omega_i$ and $W_i \subseteq Z_i$. Therefore, it is enough to show that

$$< R^{\mathcal{A}} >_{n+1} (W_1, \ldots, W_n) \in \omega_{n+1}.$$

By definition, $Y_{n+1} \cap < R^{\mathcal{A}} >_{n+1} (Y_1, \ldots, Y_n) \in \omega_{n+1}$. *So, if we show that*

$$Y_{n+1} \cap < R^{\mathcal{A}} >_{n+1} (Y_1, \ldots, Y_n) \subseteq < R^{\mathcal{A}} >_{n+1} (W_1, \ldots, W_n),$$

we are done. Let us do it. If $y \in Y_{n+1} \cap < R^{\mathcal{A}} >_{n+1} (Y_1, \ldots, Y_n)$ *then for some* $y_1 \in Y_1, \ldots, y_n \in Y_n$, $R^{\mathcal{A}}(y_1, \ldots, y_n, y)$. *Then* $y_i \in X_i$ *since* $Y_i \subseteq X_i$, *and* $y_i \in < R^{\mathcal{A}} >_i (Y_1, .., Y_{i-1}, Y_{i+1}, .., Y_{n+1})$ *since* $y \in Y_{n+1}$.

Thus, $y_i \in W_i$, *hence* $y \in < R^{\mathcal{A}} >_{n+1} (W_1, \ldots, W_n)$. *So, the above inclusion is shown, hence* $R^{\mathcal{A}}_{n+1}(\omega_1, \ldots, \omega_{n+1})$ *is proved and likewise* $R^{\mathcal{A}}_i(\omega_1, \ldots, \omega_{n+1})$ *holds for each* $i \in [n]$.

Thus, (5) and (6) hold for $\omega_1, \ldots, \omega_{n+1}$ *which completes the case.*

Now, it easily follows that $\mathbf{f_p}(\gamma)$ *is a strong homomorphism, too. For, if* $\phi_1, \ldots, \phi_{n+1}$ *are principal filters,* $\phi_i = \mathbf{f}[X_i]$, $i \in [n+1]$, *then the filters* θ_i, ψ_i *and* ω_i *constructed as above are principal filters, too. Viz.*

$$\theta_i = \mathbf{f}[\gamma^{-1}[\gamma[X_i]]],$$

$$\psi_i = \mathbf{f}[< R^{\mathcal{A}} >_i (\gamma^{-1}[\gamma[X_1]], .., \gamma^{-1}[\gamma[X_{i-1}]], \gamma^{-1}[\gamma[X_{i+1}]], .., \gamma^{-1}[\gamma[X_n]])],$$

and ω_i *is principal, being generated by the intersection of the generating sets of these two principal filters.*

As for $\mathbf{u}(\gamma)$, *the beginning of the proof repeats, mutatis mutandis, that for* $\mathbf{f}(\gamma)$. *Thus, we have constructed filters* $\omega_1, \ldots, \omega_{n+1}$ *satisfying the analog of (5):*

$$\mathbf{u}(\gamma)(\omega_i) = \mathbf{u}(\gamma)(\phi_i), i \in [n+1], \tag{7}$$

and (6), hence its weakened version:

$$R^{\mathcal{A}}_{n+1}\omega_1, \ldots, \omega_{n+1}. \tag{8}$$

We shall show that, provided $\phi_1, \ldots, \phi_{n+1}$ *are ultrafilters,* $\omega_1, \ldots, \omega_{n+1}$ *can be extended to ultrafilters* $U_1, \ldots, U_{n+1}$ *with the same properties. (Note that (8), applied to ultrafilters, is equivalent, due to prop. 3, to the analog of (6) :* $\mathbf{u}(R^{\mathcal{A}})(\omega_1, \ldots, \omega_{n+1})$.*)*

For that purpose we consider the family $\mathcal{F}$ *of all* $(n+1)$*-tuples* $< F_1, \ldots, F_{n+1} >$ *of filters on* A, *such that* $\omega_i \subseteq F_i$ *for each* $i \in [n+1]$, *and* $F_1, \ldots, F_{n+1}$, *substituted for* $\omega_1, \ldots, \omega_{n+1}$, *satisfy (7) and (8).*

The following hold for $\mathcal{F}$:

1. $\mathcal{F}$ *is non-empty;*

2. $\mathcal{F}$ *is partially ordered by a coordinate-wise inclusion:*
   $< F'_1, \ldots, F'_{n+1} > \subseteq < F''_1, \ldots, F''_{n+1} >$ *iff* $F'_i \subseteq F''_i$ *for all* $i \in [n+1]$.

3. $\mathcal{F}$ *satisfies the condition of Zorn's lemma: every ascending chain in* $\mathcal{F}$ *has an upper bound in* $\mathcal{F}$. *Indeed, if* $\left\{\mathbf{F}_j = < F^j_1, \ldots, F^j_{n+1} >\right\}_{j \in J}$ *is a chain in* $\mathcal{F}$, *then*

   $\mathbf{F} = \left\langle \bigcup_{j \in \mathcal{N}} F^j_1, \ldots, \bigcup_{j \in \mathcal{N}} F^j_{n+1} \right\rangle$ *belongs to* $\mathcal{F}$: *it is a routine task to check that* $\mathcal{F}$ *satisfies (7) and (8).*

Therefore, by Zorn's lemma, $\mathcal{F}$ has a maximal element $< U_1, \ldots, U_{n+1} >$. We shall prove that $U_1, \ldots, U_{n+1}$ are ultrafilters on A. Let us begin with U_{n+1}. Suppose it is not an ultrafilter. Then for some $Z \subseteq A, Z \notin U_{n+1}$ and $A \setminus Z \notin U_{n+1}$. Therefore $U'_{n+1} = \{Y \subseteq A \mid X \cap Z \subseteq Y$ for some$X \in U_{n+1}\}$ is a filter containing U_{n+1}. Obviously U'_{n+1} satisfies (8) for $\omega_1 = U_1, \ldots,\ \omega_n = U_n, \omega_{n+1} = U'_{n+1}$. It also satisfies (7). Clearly, $\mathbf{u}(\gamma)(u_{n+1}) \subseteq \mathbf{u}(\gamma)(U'_{n+1})$, hence these are equal since $\mathbf{u}(\gamma)(u_{n+1})$ is an ultrafilter and $\mathbf{u}(\gamma)(U'_{n+1})$ is a filter on B. Thus, $< U_1, \ldots, U_n, U'_{n+1} > \in \mathcal{F}$ and is greater than the maximal $< U_1, \ldots, U_{n+1} >$ which is impossible. Therefore U_{n+1} is an ultrafilter.

Now, consider U_i, for any $i \in [n]$. Suppose U_i is not an ultrafilter. Then for some $Z \subseteq A, Z \notin U_i$ and $A \setminus Z \notin U_i$. We shall show that either $Y = Z$ or $Y = A \setminus Z$ satisfies the condition:

$$< R^{\mathcal{A}} >_{n+1} (X_1, ..X_{i-1}, X_i \cap Y, .., X_n) \in U_{n+1}$$

for every $X_i \in U_i,\ i \in [n]$.

Assume the contrary. Then for some $X'_j \in U_j,\ j \in [n]$:

$$< R^{\mathcal{A}} >_{n+1} (X'_1, \ldots, X'_{i-1}, X'_i \cap Z, \ldots, X'_n) \notin U_{n+1},$$

and for some $X''_j \in U_j,\ j \in [n]$:

$$< R^{\mathcal{A}} >_{n+1} (X''_1, \ldots, X''_{i-1}, X''_i \cap (A \setminus Z), \ldots, X''_n) \notin U_{n+1}.$$

Let $X_j = X'_j \cap X''_j,\ j \in [n]$. Then $X_j \in U_j$,

$$< R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_{i-1}, X_i \cap Z, \ldots, X_n) \notin U_{n+1}$$

and

$$< R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_{i-1}, X_i \cap (A \setminus Z), \ldots, X_n) \notin U_{n+1},$$

by monotonicity of $< R^{\mathcal{A}} >_{n+1}$.

Therefore $< R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_{i-1}, X_i \cap Z, \ldots, X_n) \cup < R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_{i-1}, X_i \cap (A \setminus Z), \ldots, X_n) =$

$< R^{\mathcal{A}} >_{n+1} (X_1, \ldots, X_{i-1}, X_i, \ldots, X_n)$ does not belong to U_{n+1}, which contradicts to (8).

Thus, 4. holds either for Z or for $A \setminus Z$; assume it holds for Z. Then we consider the filter $U'_i = \{Y \subseteq A \mid X \cap Z \subseteq Y$ for some $X \in U_i\}$. As before we show that $\mathbf{u}(\gamma)(U'_i) = \mathbf{u}(\gamma)(u_i)$. Thus, we have constructed an $(n+1)$-tuple $< U_1, \ldots, U'_i, \ldots, U_{n+1} > \in \mathcal{F}$, greater than $< U_1, \ldots, U_{n+1} >$ which contradicts to the maximality of the latter. Therefore U_i is an ultrafilter.

In the long run we have constructed ultrafilters $U_1, \ldots, U_{n+1}$ satisfying (7) and (8) and thus have proved that $\mathbf{u}(\gamma)$ is a strong homomorphism.

This completes the proof of the theorem.

Corollary 22 *If $\gamma : \mathcal{A} \to \mathcal{B}$ is a bounded homomorphism then $\mathbf{f}(\gamma)$, $\mathbf{f_p}(\gamma)$, and $\mathbf{u}(\gamma)$ are bounded homomorphisms.*

Proof 12 *A slight modification of the previous proof. Let us begin with* $\mathbf{f}(\gamma)$*. Let* R *be an* $n+1$*-ary relational symbol and* $\phi_1, \dots, \phi_{n+1}$ *filters on* A*. Choose any* $j \in [n+1]$*, say* $j = 1$*. We must show that if* $\mathbf{f}(R^{\mathcal{B}})(\mathbf{f}(\gamma)(\phi_1), \dots, \mathbf{f}(\gamma)(\phi_{n+1}))$ *then there exist filters* $\omega_1, \dots, \omega_{n+1}$ *on* A*, such that* $\omega_1 = \phi_1$ *and the conditions (5) and (6) hold. Again we construct the filters* $\theta_2, \dots, \theta_{n+1}$ *as in the previous proof; then we define* $\psi_2, \dots, \psi_{n+1}$ *as before, but let* X_1 *be any set from* ϕ_1*. The crucial point here is that the property* (f) *from the previous proof remains true for arbitrary* $X_1 \in \phi_1$*, due to the fact that* γ *is bounded. Now, choose* $\omega_1 = \phi_1$ *and* $\omega_2, \dots, \omega_{n+1}$ *as before. Further everything goes the same way until we reach the point where (6) is being proved. The proof of* $R_1^{\mathcal{A}}(\omega_1, \dots, \omega_{n+1})$ *remains unchanged, since for each* $i = 2, \dots, n+1$*, every regular set from* θ_i *belongs to* ϕ_i*, too, and hence* $Y_1 \cap < R^{\mathcal{A}} >_1 (Y_2, \dots, Y_{n+1}) \in \omega_1$*. As for* $R_i^{\mathcal{A}}(\omega_1, \dots, \omega_{n+1})$*,* $i = 2, \dots, n+1$*, the only difference in this proof is that* $Z_1 \in \phi_1$ *is arbitrary,* Y_1 *is taken to be* $Y_1^2 \cap \dots \cap Y_1^{n+1} \cap Z_1$*, and* W_1 *is* Y_1*.*

The proof is easily amended for $\mathbf{f_p}(\gamma)$*, and for* $\mathbf{u}(\gamma)$ *practically remains the same.*

Now, let $\mathcal{A}$ be a substructure of $\mathcal{B}$. Then the identity $\iota_{\mathcal{A}}$ of $\mathcal{A}$ is an isomorphic embedding of $\mathcal{A}$ into $\mathcal{B}$. Therefore, due to th. 20, it is uplifted to isomorphic embeddings

$$\mathbf{f}(\iota_{\mathcal{A}}) : \mathbf{F}(\mathcal{A}) \to \mathbf{F}(\mathcal{B}), \quad \mathbf{f_p}(\iota_{\mathcal{A}}) : \mathbf{F_p}(\mathcal{A}) \to \mathbf{F_p}(\mathcal{B}), \quad \text{and} \quad \mathbf{u}(\iota_{\mathcal{A}}) : \mathbf{U}(\mathcal{A}) \to \mathbf{U}(\mathcal{B})$$

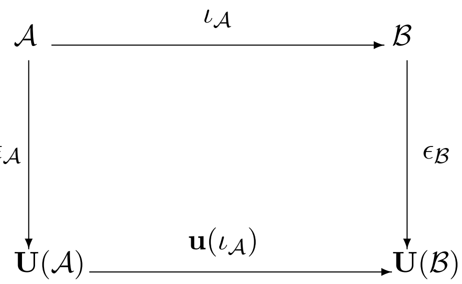
defined (according to prop. 18) respectively by:

$$\mathbf{f}(\iota_{\mathcal{A}})(\phi) = \{Y \subseteq |\mathcal{B}| \mid X \subseteq Y \text{ for some } X \in \phi\},$$

$$\mathbf{f_p}(\iota_{\mathcal{A}})(f_{\mathcal{A}}[Z]) = \{Y \subseteq |\mathcal{B}| \mid X \subseteq Y \text{ for some } X \in \mathbf{f}_{\mathcal{A}}[Z]\} = \mathbf{f}_{\mathcal{B}}[Z],$$

$$\mathbf{u}(\iota_{\mathcal{A}})(u) = \{Y \subseteq |\mathcal{B}| \mid X \subseteq Y \text{ for some } X \in u\}.$$

At that, the following diagram is commutative:



6 Filter and Ultrafilter Extensions of Congruences

Let $\mathcal{A}$ be a structure, $|\mathcal{A}| = A$, and θ be a binary relation in $\mathcal{A}$. Then we define a binary relation $\mathbf{f}[\theta]$ in $\mathbf{F}(\mathcal{A})$ as follows:

$$\phi_1 \mathbf{f}[\theta] \phi_2$$

if

$$\text{for every } X \subseteq A, \theta[X] \in \phi_1 \text{ iff } \theta[X] \in \phi_2,$$

where $\theta[X] = \{y \in A \mid x\theta y \text{ for some } x \in X\}$.

The restriction of $\mathbf{f}[\theta]$ on $\mathbf{F_p}(\mathcal{A})$ will be denoted by $\mathbf{f_p}[\theta]$ and the one on $\mathbf{U}(\mathcal{A})$ by $\mathbf{u}[\theta]$.

Note that $\mathbf{f}[X_1]\mathbf{f_p}[\theta]\mathbf{f}[X_2]$ if and only if $\theta[X_1] = \theta[X_2]$.

Proposition 23 *Let θ be a congruence in $\mathcal{A}$ and η_θ be the canonical homomorphism of $\mathcal{A}$ onto $\mathcal{A}/\theta$. Then*

1. *$\mathbf{f}[\theta]$ is the kernel congruence of $\mathbf{f}(\eta_\theta) : \mathbf{F}(\mathcal{A}) \to \mathbf{F}(\mathcal{A}/\theta)$.*
2. *$\mathbf{f_p}[\theta]$ is the kernel congruence of $\mathbf{f_p}(\eta_\theta) : \mathbf{F_p}(\mathcal{A}) \to \mathbf{F_p}(\mathcal{A}/\theta)$.*
3. *$\mathbf{u}[\theta]$ is the kernel congruence of $\mathbf{u}(\eta_\theta) : \mathbf{U}(\mathcal{A}) \to \mathbf{U}(\mathcal{A}/\theta)$.*

Proof 13 *By definition, for every $\phi \in \mathbf{F}(\mathcal{A})$,*

$$\mathbf{f}(\eta_\theta)(\phi) = \{Y/\theta \mid X/\theta \subseteq Y/\theta \text{ for some } X \in \phi\}.$$

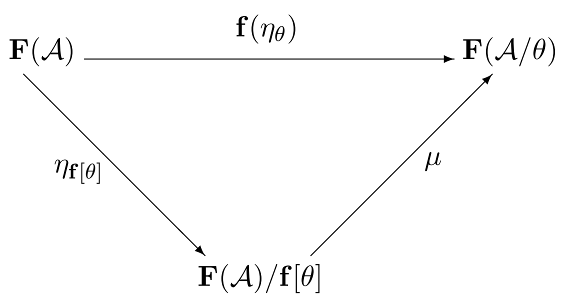
Let us observe that $X/\theta \subseteq Y/\theta$ iff $X \subseteq \theta[Y]$. Therefore $\mathbf{f}(\eta_\theta)(\phi) = \{Y/\theta \mid \theta[Y] \in \phi\}$. Then, obviously $\phi_1\mathbf{f}[\theta]\phi_2$ iff $\mathbf{f}(\eta_\theta)(\phi_1) = \mathbf{f}(\eta_\theta)(\phi_2)$.

2. and 3. are analogous.

Corollary 24

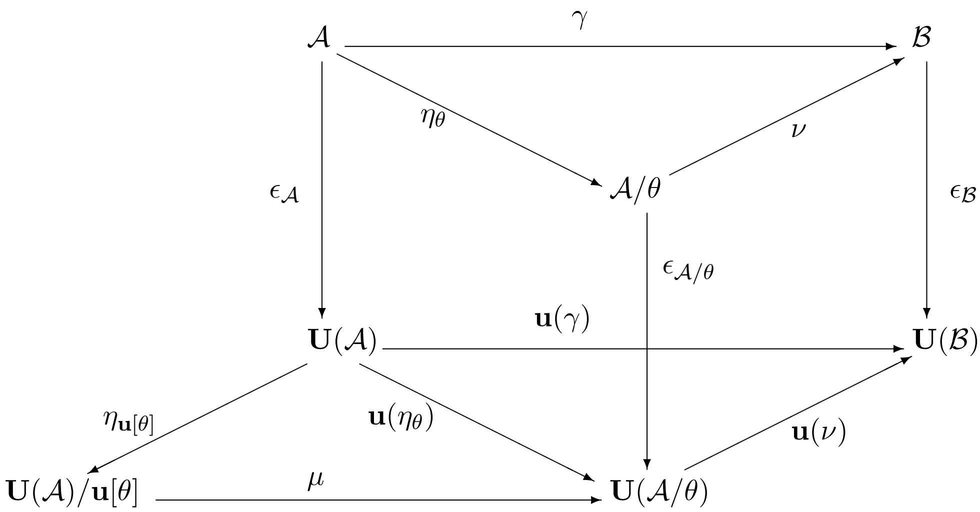
1. *$\mathbf{F}(\mathcal{A})/\mathbf{f}[\theta] \cong \mathbf{F}(\mathcal{A}/\theta)$.*
2. *$\mathbf{F_p}(\mathcal{A})/\mathbf{f_p}[\theta] \cong \mathbf{F_p}(\mathcal{A}/\theta)$.*
3. *$\mathbf{U}(\mathcal{A})/\mathbf{u}[\theta] \cong \mathbf{U}(\mathcal{A}/\theta)$.*

Proof 14 *η_θ is a strong surjective homomorphism, hence $\mathbf{f}(\eta_\theta), f_\mathbf{p}(\eta_\theta)$ and $\mathbf{u}(\eta_\theta)$ are strong surjective homomorphisms. Therefore, by the Homomorphism theorem there is an isomorphism μ such that the diagram*



is commutative. Likewise for the principal_filter and ultrafilter extensions.

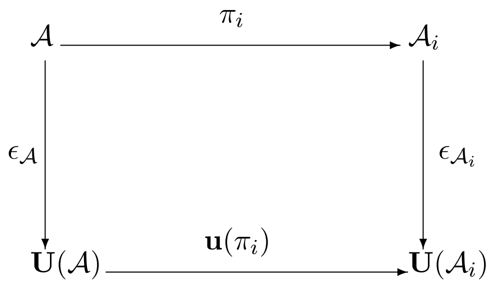
Combining the results from this and the previous section, we obtain the following situation. Let $\gamma : \mathcal{A} \to \mathcal{B}$ be a homomorphism with a kernel congruence θ. Then the following diagram is commutative:



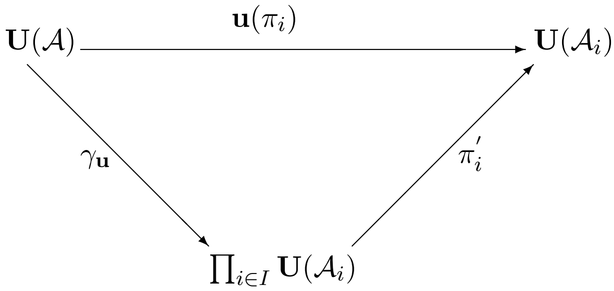
Likewise for the filter and principal_filter extensions.

7 Filter and Ultrafilter Extensions of Direct Products.

Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of structures of the same signature σ. Let $\mathcal{A} = \prod_{i\in I}\{\mathcal{A}_i\}$ and for every $i \in I$, π_i be the projecting homomorphism of $\mathcal{A}$ onto $\mathcal{A}_i$. Then for every $i \in I$, the following diagram is commutative:



where $\mathbf{u}(\pi_i)$ are surjective homomorphisms. Therefore, by the universal property of the direct products, there exists a homomorphism $\gamma_\mathbf{u} : \mathbf{U}(\mathcal{A}) \to \prod_{i\in I}\{\mathbf{U}(\mathcal{A}_i)\}$, such that the diagram



is commutative ($\pi_i^{'}$ is the corresponding projection). Likewise, there exist homomorphisms

$$\gamma_{\mathbf{f}} : \mathbf{F}(\mathcal{A}) \to \prod_{i\in I}\{\mathbf{F}(\mathcal{A}_i)\} \quad \text{and} \quad \gamma_{\mathbf{pf}} : \mathbf{F_p}(\mathcal{A}) \to \prod_{i\in I}\{\mathbf{F_p}(\mathcal{A}_i)\}$$

which make the corresponding diagrams commutative. In fact, $\gamma_{\mathbf{pf}}$ is a restriction of $\gamma_{\mathbf{f}}$ to $\mathbf{F_p}(\mathcal{A})$.

Theorem 25 *The homomorphisms $\gamma_{\mathbf{f}}$ and $\gamma_{\mathbf{pf}}$ are retractions, i.e. there exist homomorphisms $\delta_{\mathbf{f}} : \prod_{i\in I}\{\mathbf{F}(\mathcal{A}_i)\} \to \mathbf{F}(\mathcal{A})$ and $\delta_{\mathbf{pf}} : \prod_{i\in I}\{\mathbf{F_p}(\mathcal{A}_i)\} \to \mathbf{F_p}(\mathcal{A})$, such that $\gamma_{\mathbf{f}}\delta_{\mathbf{f}}$ is the identity of $\prod_{i\in I}\{\mathbf{F}(\mathcal{A}_i)\}$, and likewise for $\gamma_{\mathbf{pf}}\delta_{\mathbf{pf}}$.*

Proof 15 *Let $|\mathcal{A}| = A$, $|\mathcal{A}_i| = A_i$, $\mathcal{P} = \prod_{i\in I}\{\mathbf{F}(\mathcal{A}_i)\}$, and $|\mathcal{P}| = P$.*

Take any $g \in P$, i.e. $g : I \to \bigcup_{i\in I}\{\mathbf{F}(\mathcal{A}_i)\}$ is such that $g(i) \in \mathbf{F}(\mathcal{A}_i)$ for each $i \in I$. Put

$$\delta_{\mathbf{f}}(g) = \left\{ Y \subseteq A \;\middle|\; \prod_{i\in I}\{Z(i)\} \subseteq Y \text{ for some } Z \in \prod_{i\in I}\{g(i)\} \right\}.$$

We shall first check that $\delta_{\mathbf{f}}(g)$ is a filter on A.

1. *$\emptyset \notin \delta_{\mathbf{f}}(g)$ since $\prod_{i\in I}\{Z(i)\} = \emptyset$ iff $Z(i) = \emptyset$ for some $i \in I$ (by AC), but $Z(i)$ is non-empty since it belongs to the filter $g(i)$.*
2. *$A_i \in g(i)$ for every $i \in I$, hence $A = \prod_{i\in I}\{A_i\} \in \delta_{\mathbf{f}}(g)$.*
3. *Clearly, $\delta_{\mathbf{f}}(g)$ is closed under supersets.*
4. *Let $Y_1, Y_2 \in \delta_{\mathbf{f}}(g)$. Then for some $Z_1, Z_2 \in \prod_{i\in I}\{g(i)\}$, $\prod_{i\in I}\{Z_1(i)\} \subseteq Y_1$ and $\prod_{i\in I}\{Z_2(i)\} \subseteq Y_2$. Then $\prod_{i\in I}\{Z_1(i) \cap Z_2(i)\} \subseteq Y_1 \cap Y_2$ and $Z_1(i) \cap Z_2(i) \in g(i)$ for every $i \in I$. Hence $Y_1 \cap Y_2 \in \delta_{\mathbf{f}}(g)$.*

Now shall show that $\gamma_{\mathbf{f}}(\delta_{\mathbf{f}}(g)) = g$.
Let $j \in I$ and $X \in g(j)$. Further, let $Z \in \prod_{i\in I}\{Z(i)\}$ be defined by

$$Z(i) = \begin{cases} X & \text{if } i = j, \\ A_i & \text{if } i \neq j. \end{cases}$$

Then $\prod_{i\in I}\{Z(i)\} \in \delta_{\mathbf{f}}(g)$.

Note that, due to the latter diagram above, $\gamma_{\mathbf{f}}(\delta_{\mathbf{f}}(g))(i) = \mathbf{f}(\pi_i)(\delta_{\mathbf{f}}(g))$. Therefore $X = Z(j) = \pi_j[\prod_{i\in I}\{Z(i)\}] \in \gamma_{\mathbf{f}}(\delta_{\mathbf{f}}(g))(i)$. Thus $g(i) \subseteq \gamma_{\mathbf{f}}(\delta_{\mathbf{f}}(g))(i)$.

For the opposite inclusion, let $X \in \gamma_{\mathbf{f}}(\delta_{\mathbf{f}}(g))(i)$. Then $\pi_i[Y] \subseteq X$ for some $Y \in \delta_{\mathbf{f}}(g)$. Hence, for some $Z \in \prod_{i\in I}\{g(i)\}$, Y contains $\pi_i(\prod_{i\in I}\{Z(i)\}) = Z(i) \in g(i)$. Thus, $Y \in g(i)$.

Therefore $\gamma_{\mathbf{f}}(\delta_{\mathbf{f}}(g))(i) = g(i)$ for every $i \in I$, hence $\gamma_{\mathbf{f}}(\delta_{\mathbf{f}}(g)) = g$.

Finally, it remains to be proved that $\delta_{\mathbf{f}}$ is a homomorphism.

a) Let F be an n-ary functional symbol and $g_1, \ldots, g_n \in \mathcal{P}$. Then we must check that

$$\delta_{\mathbf{f}}(F^{\mathcal{P}}(g_1, \ldots, g_n)) = \mathbf{f}(F^{\mathcal{A}})(\delta_{\mathbf{f}}(g_1), \ldots, \delta_{\mathbf{f}}(g_n)).$$

The key observation to be used is that for every $i \in I, j \in [n]$, and $X^i_j \in \mathcal{A}_i$:

$$\prod_{i\in I}\{F^{\mathcal{A}i}[X^i_1, \ldots, X^i_n]\} = F^{\mathcal{A}}\left[\prod_{i\in I}\{X^i_1\}_{i\in I}, \ldots, \prod\{X^i_n\}\right].$$

The verification of this is routine.

Now, $F^{\mathcal{P}}(g_1, \ldots, g_n)(i) = \mathbf{f}(F^{\mathcal{A}i})(g_1(i), \ldots, g_n(i)) = \{Y_i \subseteq A_i \mid F^{\mathcal{A}i}[X^i_1, \ldots, X^i_n] \subseteq Y_i$ for some $X^i_j \in g_j(i), j \in [n]\}$.

Then $\delta_{\mathbf{f}}(F^{\mathcal{P}}(g_1, \ldots, g_n)) =$
$\{Y \subseteq A \mid \prod_{i\in I}\{F^{\mathcal{A}i}[X_1(i), \ldots, X_n(i)]\} \subseteq Y$ for some $X_j \in \prod_{i\in I}\{g_j(i)\}), j \in [n]\} =$
$\{Y \subseteq A \mid F^{\mathcal{A}}[\prod_{i\in I}\{X_1(i)\}, \ldots, \prod_{i\in I}\{X_n(i)\}] \subseteq Y$ for some $X_j \in \prod_{i\in I}\{g_j(i)\}), j \in [n]\} =$
$\{Y \subseteq A \mid F^{\mathcal{A}}[X_1, \ldots, X_n] \subseteq Y$ for some $X_j \in \delta_{\mathbf{f}}(g_j), j \in [n]\} =$
$\mathbf{f}(F^{\mathcal{A}})(\delta_{\mathbf{f}}(g_1), \ldots, \delta_{\mathbf{f}}(g_n))$.

b) Let R be an $n+1$-ary relational symbol and $g_1, \ldots, g_{n+1} \in \mathcal{P}$. We must verify that

$$R^{\mathcal{P}}(g_1, \ldots, g_{n+1}) \text{ implies } \mathbf{f}(R^{\mathcal{A}})(\delta_{\mathbf{f}}(g_1), \ldots, \delta_{\mathbf{f}}(g_{n+1})).$$

Suppose $R^{\mathcal{P}}(g_1, \ldots, g_{n+1})$. We shall show that $R^{\mathcal{A}}_j(\delta_{\mathbf{f}}(g_1), \ldots, \delta_{\mathbf{f}}(g_{n+1}))$ for every $j \in [n+1]$. As usual, we consider the typical case of $j = n+1$. Let $Z_1 \in \delta_{\mathbf{f}}(g_1), \ldots, Z_n \in \delta_{\mathbf{f}}(g_n)$. We must prove that

$$< R^{\mathcal{A}} >_{n+1} (Z_1, \ldots, Z_n) \in \delta_{\mathbf{f}}(g_{n+1}).$$

By definition of $\delta_{\mathbf{f}}$, for each $j \in [n]$ there exists $X_j \in \prod_{i\in I}\{g_j(i)\})$ such that $\prod_{i\in I}\{X_j(i)\} \subseteq Z_j$. For any $i \in I$, $R^{\mathcal{P}}(g_1, \ldots, g_{n+1})$ implies $\mathbf{f}(R^{\mathcal{A}i})(g_1(i), \ldots, g_{n+1}(i))$, hence $R^{\mathcal{A}i}_{n+1}(g_1(i), \ldots, g_{n+1}(i))$. Therefore $< R^{\mathcal{A}} >_{n+1} (X_1(i), \ldots, X_n(i)) \in g_{n+1}(i)$. We put for every $i \in I$:

$$Y_i =< R^{\mathcal{A}} >_{n+1} (X_1(i), \ldots, X_n(i)).$$

Now, it is sufficient to show that

$$\prod_{i\in I}\{Y_i\} \subseteq < R^{\mathcal{A}} >_{n+1} (Z_1, \ldots, Z_n),$$

Let $x \in \prod_{i\in I}\{Y_i\}$. Then $x(i) \in Y_i$, hence $R^{\mathcal{A}_i}(x^i_1, \ldots, x^i_n, x(i))$ for some $x^i_1 \in X_1(i), \ldots, x^i_n \in X_n(i)$, by the definition of Y_i. We define $x_1, \ldots, x_n \in A$ as follows: $x_j(i) = x^i_j$. Then $x_j \in \prod_{i\in I}\{X_j(i)\} \subseteq Z_j$ and $R^{\mathcal{A}}(x_1, \ldots, x_n, x)$. Thus, $x \in< R^{\mathcal{A}} >_{n+1} (Z_1, \ldots, Z_n)$, hence the above inclusion holds. The proof for $\gamma_{\mathbf{f}}$ is completed.

As for $\gamma_{\mathbf{pf}}$, it is enough to notice that if $g \in \prod_{i \in I}\{\mathbf{F_p}(\mathcal{A}_i)\}$, i.e. $g(i) = \mathbf{f}[X_i]$ for some $X_i \subseteq A_i, i \in I$, then $\delta_{\mathbf{f}}(g) = \mathbf{f}[\prod_{i \in I}\{X_i\}]$, hence $\delta_{\mathbf{f}}(g) \in \mathbf{F_p}(\mathcal{A}_i)$. Thus, we can choose $\delta_{\mathbf{pf}}$ to be the restriction of $\delta_{\mathbf{f}}$ to $\prod_{i \in I}\{\mathbf{F_p}(\mathcal{A}_i)\}$.

Corollary 26

1. *$\gamma_{\mathbf{f}}$ and $\gamma_{\mathbf{pf}}$ are strong surjective homomorphisms.*
2. *$\delta_{\mathbf{f}}$ and $\delta_{\mathbf{pf}}$ are isomorphic embeddings.*

Corollary 27 *The homomorphism $\gamma_{\mathbf{u}}$ is surjective.*

Proof 16 *For every $g \in \prod_{i \in I}\{\mathbf{U}(\mathcal{A}_i)\}$, the filter $\delta_{\mathbf{f}}(g)$ is contained in some ultrafilter u on A. As in the proof of th. 25 we show that $g(i) \subseteq \gamma_{\mathbf{u}}(u)(i)$, whence $g(i) = \gamma_{\mathbf{u}}(u)(i)$ since both sides are ultrafilters on $\mathcal{A}_i$. This holds for every $i \in I$, hence $g = \gamma_{\mathbf{u}}(u)$.*

8 Identities in Filter Extensions

Recall that σ is an arbitrarily fixed signature containing the equality symbol $=$.

Definition 28 Atomic formula *(in the signature σ) is a formula of the type*

$$R(t_1(v_1, \dots, v_n), \dots, t_m(v_1, \dots, v_n)),$$

where R is a relational symbol of σ and $t_1, \dots, t_m$ are terms in σ. An atomic formula of σ is an identity *in a structure $\mathcal{A}$ of σ if it is universally valid in $\mathcal{A}$, i.e. every valuation of the variables of the formula in $\mathcal{A}$ yields a true assertion in $\mathcal{A}$.*

Lemma 29 *If $\mathcal{A}$ is a structure of σ and $T(v_1, \dots, v_n)$ is a term in σ, representing the function $T^{\mathcal{A}}$ in $\mathcal{A}$, then:*

1. *for every $\phi_1, \dots, \phi_n \in \mathbf{F}(\mathcal{A})$,*

$$\mathbf{f}(T^{\mathcal{A}})(\phi_1, \dots, \phi_n) = \{Y \subseteq |\mathcal{A}| \ \mid T^{\mathcal{A}}[X_1, \dots, X_n] \subseteq Y \text{ for some } X_j \in \phi_j,\ j \in [n]\}.$$

2. *for every non-empty $X_1, \dots, X_n \subseteq |\mathcal{A}|$,*

$$\mathbf{f}(T^{\mathcal{A}})(f[X_1], \dots, \mathbf{f}[X_n]) = \mathbf{f}[T^{\mathcal{A}}[X_1, \dots, X_n]].$$

Proof 17 1. *By induction on T, using prop. 8.*

2. *Follows from 1.*

Theorem 30 *Let $\mathcal{A}$ be a structure of the signature σ, $T_1(v_1,\ldots,v_n),\ldots,T_{m+1}(v_1,\ldots,v_n)$ be terms in σ and R be an $m+1$-ary relational symbol of σ. Then the identity*

$$R(T_1(v_1,\ldots,v_n),\ldots,T_{m+1}(v_1,\ldots,v_n)) \quad (9)$$

holds in $\mathbf{F}(\mathcal{A})$ if and only if it holds in $\mathbf{F_p}(\mathcal{A})$.

Proof 18 *Every identity in $\mathbf{F}(\mathcal{A})$ holds in its substructure $\mathbf{F_p}(\mathcal{A})$.*

Conversely, let (9) hold in $\mathbf{F_p}(\mathcal{A})$ and $\phi_1,\ldots,\phi_n \in \mathbf{F}(\mathcal{A})$. Let us denote for each $i \in [m+1]$:

$$\psi_i = T_i^{\mathbf{F}(\mathcal{A})}(\phi_1,\ldots,\phi_n) = \mathbf{f}(T_i^{\mathcal{A}})(\phi_1,\ldots,\phi_n).$$

In order to show $R^{\mathbf{F}(\mathcal{A})}(\psi_1,\ldots,\psi_{m+1})$, i.e. $\mathbf{f}(R^{\mathcal{A}})(\psi_1,\ldots,\psi_{m+1})$, we shall prove that $R^{\mathcal{A}}_{m+1}(\psi_1,\ldots,\psi_{m+1})$ holds; the other cases are analogous.

Let $X_1 \in \psi_1,\ldots,X_m \in \psi_m$. Then $T_1^{\mathcal{A}}[Y_1^1,\ldots,Y_n^1] \subseteq X_1$ for some $Y_j^1 \in \phi_j, j \in [n],\ldots,$ $T_m^{\mathcal{A}}[Y_1^m,\ldots,Y_n^m] \subseteq X_m$ for some $Y_j^m \in \phi_j, j \in [n]$. Let

$$Y_1 = Y_1^1 \cap \ldots \cap Y_1^m,\ldots,\ Y_n = Y_n^1 \cap \ldots \cap Y_n^m.$$

Then $Y_j \in \phi_j$, for each $j \in [n]$, and $T_1^{\mathcal{A}}[Y_1,\ldots,Y_n] \subseteq X_1,\ldots,\ T_m^{\mathcal{A}}[Y_1,\ldots,Y_n] \subseteq X_m$. Moreover, $T_{m+1}^{\mathcal{A}}[Y_1,\ldots,Y_n] \in \psi_{m+1}$ and

$\mathbf{f_p}(R^{\mathcal{A}})(\mathbf{f}(T_1^{\mathcal{A}})(f[Y_1],\ldots,\mathbf{f}[Y_n]),\ldots,\mathbf{f}(T_{m+1}^{\mathcal{A}})(f[Y_1],\ldots,\mathbf{f}[Y_n])),$

i.e.

$\mathbf{f_p}(R^{\mathcal{A}})(\mathbf{f}[T_1^{\mathcal{A}}[Y_1,\ldots,Y_n]],\ldots,\mathbf{f}[T_{m+1}^{\mathcal{A}}[Y_1,\ldots,Y_n]])$

holds, due to lemma 29.

Therefore,

$T_{m+1}^{\mathcal{A}}[Y_1,\ldots,Y_n] \subseteq$
$< R^{\mathcal{A}} >_{m+1} (T_1^{\mathcal{A}}[Y_1,\ldots,Y_n],\ldots,T_m^{\mathcal{A}}[Y_1,\ldots,Y_n]) \subseteq$
$< R^{\mathcal{A}} >_{m+1} (X_1,\ldots,X_m),$
hence $< R^{\mathcal{A}} >_{m+1} (X_1,\ldots,X_m) \in \psi_{m+1}$.

Corollary 31 *For every structure $\mathcal{A}$, the structures $\mathbf{F}(\mathcal{A}), \mathbf{F_p}(\mathcal{A})$ and $\mathbf{P}^+(\mathcal{A})$ satisfy the same identities.*

Proof 19 *Follows immediately from th. 30 and prop. 15.*

Using this corollary and applying results about power structures from [Grätzer & Whitney, 1984] and [Grätzer & Lakser, 1988], we can now describe the identities of $\mathbf{F}(\mathcal{A})$, having those of $\mathcal{A}$.

Definition 32 *[Grätzer & Whitney, 1984] A term is* linear *if every variable occurs no more than once in it. An atomic formula $R(T_1^*,\ldots,T_m^*)$ is a* linearization *of the atomic formula $R(T_1,\ldots,T_m)$ if $T_1^*,\ldots,T_m^*$ are linear and $R(T_1,\ldots,T_m)$ is obtained from $R(T_1^*,\ldots,T_m^*)$ by identifying variables simultaneously in all terms.*

For instance $F_1(v_1,v_3) = F_2(v_2,v_3,v_4)$ is a linearization of $F_1(v_1,v_1) = F_2(v_2,v_1,v_2)$.

Given a class of structures $\mathcal{K}$, we denote $\mathbf{F}(\mathcal{K}) = \{\mathbf{F}(\mathcal{A}) \mid \mathcal{A} \in \mathcal{K}\}$ and $\mathbf{F_p}(\mathcal{K}) = \{\mathbf{F_p}(\mathcal{A}) \mid \mathcal{A} \in \mathcal{K}\}$. The variety generated by $\mathcal{K}$ is denoted by $\mathbf{var}(\mathcal{K})$.

Corollary 33 (From th. 1 in [Grätzer & Lakser, 1988]) *Let $\mathcal{V}$ be a variety of structures. Then the identities of* $\mathbf{var}(\mathbf{F_p}(\mathcal{V}))$ *and* $\mathbf{var}(\mathbf{F}(\mathcal{V}))$ *are precisely those resulting through identification of variables in linear identities of* $\mathcal{V}$.

Corollary 34 *(From th. 2 in [Grätzer & Whitney, 1984]) For every variety $\mathcal{V}$ of finitary structures the following conditions are equivalent:*

1. *$\mathcal{V}$ is closed under filter extensions.*
2. *$\mathcal{V}$ is closed under principal_filter extensions.*
3. *$\mathcal{V}$ is definable by a set of linear identities.*

Corollary 35 *Let $\mathcal{A}$ be a structure. An atomic formula φ is an identity in* $\mathbf{F}(\mathcal{A})$ *iff a linearization of φ is an identity in $\mathcal{A}$.*

9 A Concluding Remark

If non-proper filters (containing the empty set) are admitted then all introduced notions are smoothly extended and all results, except for prop 9, th. 25 and cor. 26.2), remain, mutatis mutandis, true. In this case the principal_filter extension of a structure $\mathcal{A}$ is isomorphic to the full power structure $\mathbf{P}(\mathcal{A})$ based on the set of *all* subsets of the underlying structure and th. 30 again yields a description of the identities in the filter extensions, employing the corresponding results about $\mathbf{P}(\mathcal{A})$ from [Grätzer & Whitney, 1984] and [Grätzer & Lakser, 1988] which involve the additional condition of "regularity" on the terms occurring in the identities.

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