Propositional Interval Neighborhood Logics: Expressiveness, Decidability, and Undecidable Extensions

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Abstract

In this paper, we investigate the expressiveness of the variety of propositional interval neighborhood logics (PNL), we establish their decidability on linearly ordered domains and some important sub-classes, and we prove undecidability of a number of extensions of PNL with additional modalities over interval relations. All together, we show that PNL form a quite expressive and nearly maximal decidable fragment of Halpern-Shoham's interval logic HS.

Key words: neighborhood interval logics, expressiveness, two-variable fragment, decidability, undecidability.

1 Introduction

The study of interval-based temporal logics on linearly ordered domains is an emerging research area of increasing importance in computer science and artificial intelligence. A recent survey of the main developments, results, and open

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problems in this area can be found in [11]. The main systems of propositional interval temporal logics studied so far include Moszkowski's Propositional Interval Logic (PITL) [22], Halpern and Shoham's modal logic of time intervals (HS) [15], and Venema's CDT logic [27] (extended to branching-time frames with linear intervals in [12]). Important fragments of HS studied in more detail include the logic of *begins/ends* (BE) [18], the logics of temporal neighborhood [5,7,10], and the logics of subinterval structures [2,3]. Unfortunately, even when restricted to the case of propositional languages and linear time, interval logics are usually undecidable. In particular, PITL was proved to be undecidable on the classes of discrete and finite frames in [22]; undecidability on dense linearly ordered sets was proved by Lodaya in [18]. Likewise, the logic HS (and therefore CDT) was shown to be (often highly) undecidable in most natural classes of frames in [15]. That result was sharpened by Lodaya in [18], where the undecidability of BE on dense orderings was proved (as noted in [11], this result carries over to the class of all linearly ordered sets).

Decidability results for interval logics are scarce. Moreover, most of them are obtained by imposing suitable restrictions on the semantics, such as the projection principles of locality and homogeneity [15,22], or on the family of intervals available in the model, as in the case of 'split structures' [20]. So far, very few unrestricted decidability results for fragments of HS are known, which are based on tableau methods, e.g., the NEXPTIME decision procedure for the future fragment of neighborhood logic interpreted on $\mathbb{N}$ [4,7], later extended to the class of all linear orderings [6] and to full neighborhood logic over $\mathbb{Z}$ [5], and the PSPACE decision procedures for the logics of strict and proper subinterval structures over dense linear orderings [2,3].

In this paper, we address expressiveness and decidability issues for propositional neighborhood logics (PNL). These are fragments of HS which feature the modalities corresponding to the relations of right-adjacent and left-adjacent intervals (in terms of Allen's relations, *meets/met by*), and (possibly) the modal constant π, which is true precisely on point-intervals (intervals with coinciding endpoints). We focus our attention on three variants of PNL, namely, PNL^-, based on strict semantics which excludes point-intervals, PNL^+, based on non-strict semantics which includes point-intervals, and $\mathrm{PNL}^{\pi+}$, which extends PNL^+ with π. Besides the above-mentioned decidability results for $\mathbb{N}$ and $\mathbb{Z}$, a number of representation theorems and sound and complete axiomatic systems for various classes of linear orderings, as well as a tableau-based semi-decision procedure, have been obtained for PNL [10].

The main results given in the present paper are:

(1) NEXPTIME-complete decidability of the satisfiability problem for $\mathrm{PNL}^{\pi+}$ on some important classes of linear orderings. This result hinges upon the decidability of the satisfiability problem for the two-variable fragment of

first-order logic $FO^2[<]$ for binary relational structures over ordered domains, due to Otto [23]. Thus, while the main technical work behind this result has already been done elsewhere, we emphasize here its conceptual importance, being the first decidable general case of natural and expressive interval languages interpreted in genuine, unrestricted interval-based semantics.

(2) Expressive completeness of $PNL^{\pi+}$ with respect to $FO^2[<]$, by means of a suitable faithful translation of the latter into the former. This result is in the spirit of the seminal Kamp's theorem [17]. Kamp proved the functional completeness of the *Since* (S) and *Until* (U) temporal logic with respect to first-order definable connectives on Dedekind-complete linear orderings. This result has been later re-proved and generalized in several ways (see [9,16]). In particular, Stavi extended Kamp's result to the class of all linear orderings by adding the binary operators S' and U' (see [9] for details), while Etessami et al. [8] proved the functional completeness of the linear-time temporal logic with future and past operators F, P with respect to the two-variable, unary-predicate fragment of first-order logic over $\mathbb{N}$. Finally, Venema proved the expressive completeness of CDT with respect to the three-variable fragment of first-order logic with at most two free variables $FO^3_{x,y}[<]$ on the class of all linear orderings [27]. These expressive completeness results are important from both perspectives: for propositional interval logics and for bounded-variable fragments of first-order logic for relational structures over ordered domains, as they open the perspective for cross-pollination of these fields, especially with regards to decision procedures.

(3) Undecidability of a number of extensions of $PNL^{\pi+}$ with various additional interval modalities from the HS repertoire. The technique used to obtain these results is a non-trivial reduction from the (undecidable) tiling problem for an octant of the integer plane. That technique is quite versatile and can be applied to a variety of extensions of $PNL^{\pi+}$ (in fact, there exist a few extensions, such as those involving the modalities for the Allen's relations *begins/begun-by*, for which the decidability of the satisfiability problem is still an open problem).

The rest of the paper is organized as follows. We start with some preliminaries in Section 2. In Section 3 we compare the expressive power of PNL^-, PNL^+, and $PNL^{\pi+}$. We show that $PNL^{\pi+}$ is strictly more expressive than PNL^+ and PNL^-, while the latter two are incomparable in terms of expressiveness. Then, in Section 4 we prove the decidability of the satisfiability problem for $PNL^{\pi+}$ on the classes of all linear orderings, all well-orders, all finite linear orderings, and $\mathbb{N}$ by a reduction to Otto's results. Next, in Section 5 we provide a translation of $FO^2[<]$ into $PNL^{\pi+}$, thus proving expressive completeness of the latter with respect to the former on the class of all linear orderings, and in Section 6 we show that $PNL^{\pi+}$ is a maximal fragment of HS that translates into $FO^2[<]$. In Section 7, we establish undecidability of various extensions of

$\mathrm{PNL}^{\pi+}$ with additional interval modalities. The paper ends with concluding remarks and open questions.

2 Preliminaries

2.1 Syntax and semantics of propositional neighborhood logics

We will distinguish three variants of propositional neighborhood logics. The language of *full Propositional Neighborhood Logic* ($\mathrm{PNL}^{\pi+}$) consists of a set $\mathcal{AP}$ of atomic propositions (or propositional variables), the propositional connectives $\neg, \vee$, the modal constant π, and the modal operators $\Diamond_r$ and $\Diamond_l$. The other propositional connectives, as well as the logical constants $\top$ (*true*) and $\bot$ (*false*), and the dual modal operators $\Box_r$ and $\Box_l$, are defined as usual. The *formulas* of $\mathrm{PNL}^{\pi+}$, typically denoted by $\varphi, \psi, \ldots$, are recursively defined as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \pi \mid \Diamond_r\varphi \mid \Diamond_l\varphi.$$

Removing the modal constant π from $\mathrm{PNL}^{\pi+}$ yields the language of *Non-strict Propositional Neighborhood Logic* (PNL^{+}), while the language of *Strict Propositional Neighborhood Logic* (PNL^{-}) is obtained from that of PNL^{+} by replacing the modalities $\Diamond_r$ and $\Diamond_l$ with the modalities $\langle A\rangle$ and $\langle\overline{A}\rangle$ (with dual modalities $[A]$ and $[\overline{A}]$), respectively[1]. We will use PNL to refer collectively to $\mathrm{PNL}^{\pi+}$, PNL^{+}, and PNL^{-}.

Propositional neighborhood logics are interpreted in interval structures on linear orderings, which are defined as follows. Let $\mathbb{D} = \langle D, <\rangle$ be a linearly ordered set. An *interval* in $\mathbb{D}$ is an ordered pair $[a, b]$, where $a, b \in D$ and $a \leq b$. An interval $[a, b]$ is a *strict interval* if $a < b$, while it is a *point interval* if $a = b$. We denote the set of all (resp., strict) intervals in $\mathbb{D}$ by $\mathbb{I}(\mathbb{D})^+$ (resp., $\mathbb{I}(\mathbb{D})^-$). The semantics of $\mathrm{PNL}^{\pi+}$/PNL^{+} is given in terms of *non-strict interval models* $\langle\mathbb{I}(\mathbb{D})^+, V\rangle$, while that of PNL^{-} is given in terms of *strict interval models* $\langle\mathbb{I}(\mathbb{D})^-, V\rangle$. The *valuation function* $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})^+}$ (resp., $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})^-}$) assigns to every propositional variable p the set of all (resp., strict) intervals $V(p)$ on which p holds. To explicitly distinguish valuations in non-strict and strict models, we will write V^+ and V^-, respectively; likewise, we will write $\mathbb{I}(\mathbb{D})$ for either of $\mathbb{I}(\mathbb{D})^+$ and $\mathbb{I}(\mathbb{D})^-$ and will denote non-strict and strict models respectively by $\mathbf{M}^+$ and $\mathbf{M}^-$, while using $\mathbf{M}$ to denote either type.

The *truth relation* of a formula of PNL at a given interval in a model $\mathbf{M}$ is

[1] We adopt different notation for the modalities of $\mathrm{PNL}^{\pi+}$/PNL^{+} and PNL^{-} to reflect their historical links and to make it easier to distinguish between the non-strict and strict semantics from the syntax.

defined by structural induction on formulas:

- $\mathbf{M}, [a,b] \Vdash p$ iff $[a,b] \in V(p)$, for all $p \in \mathcal{AP}$;
- $\mathbf{M}, [a,b] \Vdash \neg\psi$ iff it is not the case that $\mathbf{M}, [a,b] \Vdash \psi$;
- $\mathbf{M}, [a,b] \Vdash \varphi \vee \psi$ iff $\mathbf{M}, [a,b] \Vdash \varphi$ or $\mathbf{M}, [a,b] \Vdash \psi$;
- $\mathbf{M}, [a,b] \Vdash \Diamond_r \psi$ (resp., $\langle A \rangle \psi$) iff there exists c such that $c \geq b$ (resp., $c > b$) and $\mathbf{M}, [b,c] \Vdash \psi$;
- $\mathbf{M}, [a,b] \Vdash \Diamond_l \psi$ (resp., $\langle \overline{A} \rangle \psi$) iff there exists c such that $c \leq a$ (resp., $c < a$) and $\mathbf{M}, [c,a] \Vdash \psi$;
- $\mathbf{M}^+, [a,b] \Vdash \pi$ iff $a = b$.

A PNL-formula is *satisfiable* if it is true on some interval in some interval model for the respective language and it is *valid* if it is true on every interval in every interval model. It is worth noting that valuation sets represent *binary relations* and thus validity of a PNL-formula is *not* a *monadic second-order* property but a *dyadic* one.

As shown in [10], PNL can express meaningful temporal properties, e.g., constraints on the structure of the underlying linear ordering. In particular, in $\mathrm{PNL}^{\pi+}$ and PNL^{-} one can express the *difference* operator and thus simulate *nominals*.

2.2 The two-variable fragment of first-order logic

Let us denote by FO^2 (resp., $\mathrm{FO}^2[=]$) the fragment of a generic first-order language (resp., first-order language with equality) whose formulas contain only two fixed distinct variables. We denote formulas from these languages by $\alpha, \beta, \ldots$. For example, the formula $\forall x(P(x) \rightarrow \forall y \exists x Q(x,y))$ belongs to FO^2, while the formula $\forall x(P(x) \rightarrow \forall y \exists z(Q(z,y) \wedge Q(z,x)))$ does not. We focus our attention on the language $\mathrm{FO}^2[<]$ over a purely relational vocabulary $\{=, <, P, Q, \ldots\}$ including equality and a distinguished binary relation $<$ interpreted as a linear ordering (in fact, $=$ can be defined in terms of $<$). Since atoms in the two-variable fragment can involve at most two distinct variables, we may further assume without loss of generality that the arity of every relation is exactly 2.

Let x and y be the two variables of the language. Formulas of $\mathrm{FO}^2[<]$ can be defined recursively as follows:

$$\begin{aligned}
\alpha &::= A_0 \mid A_1 \mid \neg\alpha \mid \alpha \vee \beta \mid \exists x \alpha \mid \exists y \alpha \\
A_0 &::= x = x \mid x = y \mid y = x \mid y = y \mid x < y \mid y < x \\
A_1 &::= P(x,x) \mid P(x,y) \mid P(y,x) \mid P(y,y),
\end{aligned}$$

where A_1 deals with (uninterpreted) binary predicates. For technical convenience, we assume that both variables x and y occur as (possibly vacuous) free variables in every formula $\alpha \in \mathrm{FO}^2[<]$, that is, $\alpha = \alpha(x, y)$.

Formulas of $\mathrm{FO}^2[<]$ are interpreted in *relational models* of the form $\mathcal{A} = \langle \mathbb{D}, V_{\mathcal{A}} \rangle$, where $\mathbb{D} = \langle D, < \rangle$ is a linear ordering and $V_{\mathcal{A}}$ is a *valuation function* that assigns to every *binary relation* P a subset of $D \times D$. When we evaluate a formula $\alpha(x, y)$ on a pair of elements a, b, we write $\alpha(a, b)$ for $\alpha[x := a, y := b]$.

The satisfiability problem for FO^2 without equality was proved decidable by Scott [24] by using a satisfiability preserving reduction of any FO^2-formula to a formula of the form $\forall x \forall y \psi_0 \wedge \bigwedge_{i=1}^{m} \forall x \exists y \psi_i$, which belongs to Gödel's prefix-defined decidable class of first-order formulas [1]. Later, Mortimer extended this result by including equality in the language [21]. More recently, Grädel, Kolaitis, and Vardi improved Mortimer's result by lowering the complexity bound [14]. Finally, by building on techniques from [14] and performing an in-depth analysis of the basic 1-types and 2-types in $\mathrm{FO}^2[<]$-models, Otto proved the decidability of $\mathrm{FO}^2[<]$ over the class of all linear orderings, as well as over some natural subclasses of it [23].

Theorem 1 ([23]) *The satisfiability problem for* $\mathrm{FO}^2[<]$ *is decidable in NEXPTIME for each of the classes of structures where* $<$ *is interpreted as:*

(i) any linear ordering,
(ii) any well-ordering,
(iii) any finite linear ordering,
(iv) the linear ordering on natural numbers.

2.3 Comparing the expressive power of interval logics

There are various ways to compare the expressive power of different modal languages and logics. For instance, they can be compared with respect to frame validity, that is, with respect to the properties of frames that they can express. (Such a comparison for PNL can be found in [10].) Here we compare the considered logics with respect to expressing properties of a given interval in a model. We distinguish three different cases: the case in which we compare two interval logics on the same class of models, e.g., $\mathrm{PNL}^{\pi+}$ and PNL^{+}, the case in which we compare strict and non-strict interval logics, e.g., PNL^{-} and $\mathrm{PNL}^{\pi+}$, and the case in which we compare an interval logic with a first-order logic, e.g., $\mathrm{PNL}^{\pi+}$ and $\mathrm{FO}^2[<]$.

Given two interval logics L and L′ interpreted in the same class of models $\mathcal{C}$, we say that L′ is *at least as expressive as* L (with respect to $\mathcal{C}$), denoted by

$\mathrm{L} \preceq_{\mathcal{C}} \mathrm{L}'$ ($\mathcal{C}$ is omitted if clear from the context), if there exists an effective translation τ from L to L′ (inductively defined on the structure of formulas) such that for every model $\mathbf{M}$ in $\mathcal{C}$, any interval $[a,b]$ in $\mathbf{M}$, and any formula φ of L, we have $\mathbf{M},[a,b] \Vdash \varphi$ iff $\mathbf{M},[a,b] \Vdash \tau(\varphi)$. Furthermore, we say that L is *as expressive as* L′, denoted by $\mathrm{L} \equiv_{\mathcal{C}} \mathrm{L}'$, if both $\mathrm{L} \preceq_{\mathcal{C}} \mathrm{L}'$ and $\mathrm{L}' \preceq_{\mathcal{C}} \mathrm{L}$, while we say that L is *strictly more expressive than* L′, denoted by $\mathrm{L}' \prec_{\mathcal{C}} \mathrm{L}$, if $\mathrm{L}' \preceq_{\mathcal{C}} \mathrm{L}$ and $\mathrm{L} \not\preceq_{\mathcal{C}} \mathrm{L}'$.

When comparing an interval logic L^- interpreted in *strict* interval models with an interval logic L^+ interpreted in *non-strict* ones, we need to slightly revise the above definitions. Given a strict interval model $\mathbf{M}^- = \langle \mathbb{I}(\mathbb{D})^-, V^- \rangle$, we say that a non-strict interval model $\mathbf{M}^+ = \langle \mathbb{I}(\mathbb{D})^+, V^+ \rangle$ is *a non-strict extension* of $\mathbf{M}^-$ (and that $\mathbf{M}^-$ is *the strict restriction* of $\mathbf{M}^+$) if V^- and V^+ agree on the valuation of strict intervals, that is, if for every strict interval $[a,b] \in \mathbb{I}(\mathbb{D})^-$ and propositional variable $p \in \mathcal{AP}$, $[a,b] \in V^-(p)$ if and only if $[a,b] \in V^+(p)$. We say that L^+ is *at least as expressive as* L^-, and we denote it by $\mathrm{L}^- \preceq_I \mathrm{L}^+$, if there exists an effective translation τ from L^- to L^+ such that for any strict interval model $\mathbf{M}^-$, any interval $[a,b]$ in $\mathbf{M}^-$, and any formula φ of L^-, $\mathbf{M}^-,[a,b] \Vdash \varphi$ iff $\mathbf{M}^+,[a,b] \Vdash \tau(\varphi)$ for every non-strict extension $\mathbf{M}^+$ of $\mathbf{M}^-$. Conversely, we say that L^- is *at least as expressive as* L^+, and we denote it by $\mathrm{L}^+ \preceq_I \mathrm{L}^-$, if there exists an effective translation τ' from L^+ to L^- such that for any non-strict interval model $\mathbf{M}^+$, any strict interval $[a,b]$ in $\mathbf{M}^+$, and any formula φ of L^+, $\mathbf{M}^+,[a,b] \Vdash \varphi$ iff $\mathbf{M}^-,[a,b] \Vdash \tau'(\varphi)$, where $\mathbf{M}^-$ is the strict restriction of $\mathbf{M}^+$. $\mathrm{L}^- \equiv_I \mathrm{L}^+$, $\mathrm{L}^- \prec_I \mathrm{L}^+$, and $\mathrm{L}^+ \prec_I \mathrm{L}^-$ are defined in the usual way.

Finally, we compare interval logics with first-order logics interpreted in relational models. In this case, the above criteria are no longer adequate, since we need to compare logics which are interpreted in different types of models (interval models and relational models). We deal with this complication by following the approach outlined by Venema in [27]: we first define suitable model transformations (from interval models to relational models and vice versa) and then we compare the expressiveness of interval and first-order logics modulo these transformations.

To define the mapping from interval models to relational models, we associate a binary relation P with every propositional variable $p \in \mathcal{AP}$ of the considered interval logic [27].

Definition 2 *Given an interval model* $\mathbf{M} = \langle \mathbb{I}(\mathbb{D}), V_{\mathbf{M}} \rangle$*, the corresponding relational model* $\eta(\mathbf{M})$ *is a pair* $\langle \mathbb{D}, V_{\eta(\mathbf{M})} \rangle$*, where for all* $p \in \mathcal{AP}$*,* $V_{\eta(\mathbf{M})}(P) = \{(a,b) \in D \times D : [a,b] \in V_{\mathbf{M}}(p)\}$.

Note that the relational models above can be viewed as ‘point’ models for modal logics on $\mathbb{D}^2$ and the above transformation as a mapping of propositional variables of the interval logic, interpreted in $\mathbb{I}(\mathbb{D})$, into propositional variables of the target logic, interpreted in $\mathbb{D}^2$ [25,26].

To define the mapping from relational models to interval ones, we have to solve a technical problem: the truth of formulas in interval models is evaluated only on ordered pairs $[a, b]$, with $a \leq b$, while in relational models there is not such a constraint. To deal with this problem, we associate two propositional variables $p^{\leq}$ and $p^{\geq}$ of the interval logic with every binary relation P.

Definition 3 *Given a relational model $\mathcal{A} = \langle \mathbb{D}, V_{\mathcal{A}} \rangle$, the corresponding non-strict interval model $\zeta(\mathcal{A})$ is a pair $\langle \mathbb{I}(\mathbb{D})^+, V_{\zeta(\mathcal{A})} \rangle$ such that for any binary relation P and any interval $[a, b]$, $[a, b] \in V_{\zeta(\mathcal{A})}(p^{\leq})$ iff $(a, b) \in V_{\mathcal{A}}(P)$ and $[a, b] \in V_{\zeta(\mathcal{A})}(p^{\geq})$ iff $(b, a) \in V_{\mathcal{A}}(P)$.*

Given an interval logic L_I and a first-order logic L_{FO}, we say that L_{FO} is *at least as expressive as* L_I, denoted by $\mathrm{L}_I \preceq_R \mathrm{L}_{FO}$, if there exists an effective translation τ from L_I to L_{FO} such that for any interval model $\mathbf{M}$, any interval $[a, b]$, and any formula φ of L_I, $\mathbf{M}, [a, b] \Vdash \varphi$ iff $\eta(\mathbf{M}) \models \tau(\varphi)(a, b)$. Conversely, we say that L_I is *at least as expressive as* L_{FO}, denote by $\mathrm{L}_{FO} \preceq_R \mathrm{L}_I$, if there exists an effective translation τ' from L_{FO} to L_I such that for any relational model $\mathcal{A}$, any pair (a, b) of elements, and any formula φ of L_{FO}, $\mathcal{A} \models \varphi(a, b)$ iff $\zeta(\mathcal{A}), [a, b] \Vdash \tau'(\varphi)$ if $a \leq b$ or $\zeta(\mathcal{A}), [b, a] \Vdash \tau'(\varphi)$ otherwise. We say that L_I is *as expressive as* L_{FO}, denoted by $\mathrm{L}_I \equiv_R \mathrm{L}_{FO}$, if $\mathrm{L}_I \preceq_R \mathrm{L}_{FO}$ and $\mathrm{L}_{FO} \preceq_R \mathrm{L}_I$. $\mathrm{L}_I \prec_R \mathrm{L}_{FO}$ and $\mathrm{L}_{FO} \prec_R \mathrm{L}_I$ are defined in the usual way.

3 Comparing the expressiveness of $\mathbf{PNL}^{\pi+}$, $\mathbf{PNL}^{+}$, and $\mathbf{PNL}^{-}$

In this section we compare the relative expressive power of $\mathrm{PNL}^{\pi+}$, PNL^{+}, and PNL^{-}. We will prove that both PNL^{-} and PNL^{+} are strictly less expressive than $\mathrm{PNL}^{\pi+}$, while neither $\mathrm{PNL}^{+} \preceq_I \mathrm{PNL}^{-}$ nor $\mathrm{PNL}^{-} \preceq_I \mathrm{PNL}^{+}$.

In order to compare the expressive power of $\mathrm{PNL}^{\pi+}$ and PNL^{+}, we use bisimulation games [13]. More precisely, we apply a simple game-theoretic argument to exhibit two models that can be distinguished by a $\mathrm{PNL}^{\pi+}$ formula, but not by a PNL^{+} formula. To this end, we define the notion of a *k-round* PNL^{+}*-bisimulation game* to be played by two players, Player I and Player II, on a pair of PNL^{+} models $(\mathbf{M_0}^+, \mathbf{M_1}^+)$, with $\mathbf{M_0}^+ = \langle \mathbb{I}(\mathbb{D}_0)^+, V_0 \rangle$ and $\mathbf{M_1}^+ = \langle \mathbb{I}(\mathbb{D}_1)^+, V_1 \rangle$. The game starts from a given *initial configuration*, where a *configuration* is a pair of intervals $([a_0, b_0], [a_1, b_1])$, with $[a_0, b_0] \in \mathbb{I}(\mathbb{D}_0)^+$ and $[a_1, b_1] \in \mathbb{I}(\mathbb{D}_1)^+$. A configuration $([a_0, b_0], [a_1, b_1])$ is *matching* if $[a_0, b_0]$ and $[a_1, b_1]$ satisfy the same atomic propositions in their respective models.

At every round, given a current configuration $([a_0, b_0], [a_1, b_1])$, Player I can play one of the following two moves:

$\Diamond_r$**-move:** choose $\mathbf{M_i}^+$, where $i \in \{0,1\}$, and an interval $[b_i, c_i]$;
$\Diamond_l$**-move:** choose $\mathbf{M_i}^+$, where $i \in \{0,1\}$, and an interval $[c_i, a_i]$.

In the first case, Player II must reply by choosing an interval $[b_{1-i}, c_{1-i}]$ in $\mathbf{M_{1-i}}^+$, which leads to the new configuration $([b_0, c_0], [b_1, c_1])$; likewise, in the second case, Player II must choose an interval $[c_{1-i}, a_{1-i}]$ in $\mathbf{M_{1-i}}^+$, which leads to the new configuration $([c_0, a_0], [c_1, a_1])$.

If after any given round the current configuration is not matching, Player I wins the game; otherwise, after k rounds, Player II wins the game.

Intuitively, Player II has a *winning strategy* in the k-round PNL$^+$-bisimulation game on the models $\mathbf{M_0}^+$ and $\mathbf{M_1}^+$ with a given initial configuration if she can win regardless of the moves played by Player I; otherwise, Player I has a winning strategy. A formal definition of winning strategy can be found in [13]. The following key property of the k-round PNL$^+$-bisimulation game can be proved routinely, in analogy with similar results about bisimulation games in modal logic [13] [2].

Proposition 4 *Let $\mathcal{P}$ be a finite set of propositional variables. For all $k \geq 0$, Player II has a winning strategy in the k-round* PNL$^+$*-bisimulation game on* $\mathbf{M_0}^+$ *and* $\mathbf{M_1}^+$ *with initial configuration* $([a_0, b_0], [a_1, b_1])$ *iff* $[a_0, b_0]$ *and* $[a_1, b_1]$ *satisfy the same* PNL$^+$*-formulas over $\mathcal{P}$ with modal depth at most k.*

To begin with, we will use Proposition 4 to prove that the interval constant π cannot be expressed in PNL$^+$. For that, it suffices to construct two models $\mathbf{M_0}^+$ and $\mathbf{M_1}^+$ that can be distinguished with a PNL$^{\pi+}$ formula (which makes an essential use of π), but not by a PNL$^+$ formula. The latter claim is proved by showing that, for all k, Player II has a winning strategy in the k-round PNL$^+$-bisimulation game on $\mathbf{M_0}^+$ and $\mathbf{M_1}^+$.

Theorem 5 *The interval constant π cannot be defined in* PNL$^+$.

Proof.

Let $\mathbf{M}^+ = \langle \mathbb{I}(\mathbb{Z})^+, V \rangle$, where V is such that p holds everywhere, be a non-strict model. Consider the k-round PNL$^+$-bisimulation game on $(\mathbf{M}^+, \mathbf{M}^+)$ with initial configuration $([0,1],[1,1])$. The intervals $[0,1]$ and $[1,1]$ are easily distinguished in PNL$^{\pi+}$, since π holds in $[1,1]$ but not in $[0,1]$. We show that

[2] In Proposition 4, we refer to the notion of modal depth of a PNL$^+$-formula φ, which is defined in the usual way. Let us denote by $mdepth(\varphi)$ the modal depth of φ. It can be inductively defined as follows: (i) mdepth(p) = 0, for each $p \in \mathcal{AP}$; (ii) $mdepth(\neg\varphi) = mdepth(\varphi), mdepth(\varphi \vee \psi) = max\{mdepth(\varphi), mdepth(\psi)\}, mdepth(\Diamond_r \varphi) = mdepth(\Diamond_l \varphi) = mdepth(\varphi) + 1$.

this pair of intervals cannot be distinguished in PNL^+ by providing a simple winning strategy for Player II in the k-round PNL^+-bisimulation game on $(\mathbf{M}^+, \mathbf{M}^+)$ with initial configuration $([0,1],[1,1])$. If Player I plays a $\Diamond_r$-move on a given structure, then Player II chooses arbitrarily a right neighbor of the current interval on the other structure. Likewise, if Player I plays a $\Diamond_l$-move on a given structure, then Player II chooses arbitrarily a left-neighbor of the current interval on the other structure. Since the valuation V is such that p holds everywhere, in any case the new configuration is matching. $\square$

Theorem 6 $\text{PNL}^- \prec_I \text{PNL}^{\pi+}$.

Proof.

We prove the claim by showing that $\text{PNL}^- \preceq_I \text{PNL}^{\pi+}$ and $\text{PNL}^{\pi+} \not\preceq_I \text{PNL}^-$. To prove the former, we provide a translation τ from PNL^- to $\text{PNL}^{\pi+}$. Consider the mapping τ_0 defined as follows:

$$\tau_0(p) = p \qquad \tau_0(\langle A\rangle\varphi) = \Diamond_r(\neg\pi \wedge \tau_0(\varphi))$$
$$\tau_0(\neg\varphi) = \neg\tau_0(\varphi) \qquad \tau_0(\langle\overline{A}\rangle\varphi) = \Diamond_l(\neg\pi \wedge \tau_0(\varphi))$$
$$\tau_0(\varphi_1 \vee \varphi_2) = \tau_0(\varphi_1) \vee \tau_0(\varphi_2)$$

For every PNL^--formula φ, let $\tau(\varphi) = \neg\pi \wedge \tau_0(\varphi)$. Given a strict model $\mathbf{M}^- = \langle \mathbb{I}(\mathbb{D})^-, V^-\rangle$, let $\mathbf{M}^+ = \langle \mathbb{I}(\mathbb{D})^+, V^+\rangle$ be a non-strict extension of $\mathbf{M}^-$. It is immediately apparent that for any interval $[a,b]$ in $\mathbf{M}^-$ and any PNL^--formula φ, $\mathbf{M}^-, [a,b] \Vdash \varphi$ if and only if $\mathbf{M}^+, [a,b] \Vdash \tau(\varphi)$. The proof is an easy induction on the structure of φ. This proves that $\text{PNL}^- \preceq_I \text{PNL}^{\pi+}$.

To prove that $\text{PNL}^{\pi+} \not\preceq_I \text{PNL}^-$, suppose by contradiction that there exists a translation τ' from $\text{PNL}^{\pi+}$ to PNL^- such that, for any non-strict model $\mathbf{M}^+$, any strict interval $[a,b]$, and any formula φ of $\text{PNL}^{\pi+}$, $\mathbf{M}^+, [a,b] \Vdash \varphi$ iff $\mathbf{M}^-, [a,b] \Vdash \tau'(\varphi)$, where $\mathbf{M}^-$ is the strict restriction of $\mathbf{M}^+$. Consider the non-strict models $\mathbf{M}^+_0 = \langle \mathbb{I}(\mathbb{Z})^+, V_0\rangle$ and $\mathbf{M}^+_1 = \langle \mathbb{I}(\mathbb{Z})^+, V_1\rangle$, where $V_0(p) = \{[a,b] \in \mathbb{I}(\mathbb{Z})^+ : a \leq b\}$ and $V_1(p) = \{[a,b] \in \mathbb{I}(\mathbb{Z})^+ : a < b\}$. It is immediately apparent that $\mathbf{M}^+_0, [0,1] \Vdash \Box_r p$, while $\mathbf{M}^+_1, [0,1] \not\Vdash \Box_r p$. Let $\mathbf{M}^- = \langle \mathbb{I}(\mathbb{Z})^-, V^-\rangle$ be a strict interval model such that p holds everywhere in $\mathbb{I}(\mathbb{Z})^-$. We have that $\mathbf{M}^-$ is the strict restriction of both $\mathbf{M}^+_0$ and $\mathbf{M}^+_1$. Hence, we conclude that $\mathbf{M}^-, [0,1] \Vdash \tau'(\Box_r p)$ and $\mathbf{M}^-, [0,1] \not\Vdash \tau'(\Box_r p)$, which is a contradiction. $\square$

Theorem 7 *The expressive powers of* PNL^+ *and* PNL^- *are incomparable, namely,* $\text{PNL}^- \not\preceq_I \text{PNL}^+$ *and* $\text{PNL}^+ \not\preceq_I \text{PNL}^-$.

Proof.

We first prove that $\mathrm{PNL}^- \not\preceq_I \mathrm{PNL}^+$. Let $\mathbf{M_0}^+ = \langle \mathbb{I}(\mathbb{Z})^+, V_0\rangle$ and $\mathbf{M_1}^+ = \langle \mathbb{I}(\mathbb{Z} \setminus \{2\})^+, V_1\rangle$, where V_0 is such that $V_0(p) = \{[1,1],[1,2],[2,2]\}$ and V_1 is such that $V_1(p) = \{[1,1]\}$, be two PNL^+-models. For any $k \geq 0$, consider the k-round PNL^+-bisimulation game between $\mathbf{M_0}^+$ and $\mathbf{M_1}^+$, with initial configuration $([0,1],[0,1])$. Player II has the following winning strategy: at any round, if Player I chooses an interval $[a,b] \in \mathbb{I}(\mathbb{Z} \setminus \{2\})^+$ in one of the models, then Player II chooses the same interval on the other model, while if Player I chooses an interval $[a,2]$, with $a < 2$ (resp., $[2,2],[2,b]$, with $b > 2$) in $\mathbf{M_0}^+$, then Player II chooses the interval $[a,1]$ (resp., $[1,1],[1,b]$) in $\mathbf{M_1}^+$. On the other hand, the strict restrictions $\mathbf{M_0}^-$ of $\mathbf{M_0}^+$ and $\mathbf{M_1}^-$ of $\mathbf{M_1}^+$ can be easily distinguished by PNL^-: we have that $\mathbf{M_0}^-, [0,1] \Vdash \langle A\rangle p$, while $\mathbf{M_1}^-, [0,1] \not\Vdash \langle A\rangle p$. Since $\mathbf{M_0}^+$ and $\mathbf{M_1}^+$ satisfy the same formulas on the interval $[0,1]$, there cannot exist a translation τ' from PNL^- to PNL^+ such that $\mathbf{M_0}^+, [0,1] \Vdash \tau'(\langle A\rangle p)$ and $\mathbf{M_1}^+, [0,1] \not\Vdash \tau'(\langle A\rangle p)$.

As for $\mathrm{PNL}^+ \not\preceq_I \mathrm{PNL}^-$, we can use the very same proof we gave to show that $\mathrm{PNL}^{\pi+} \not\preceq_I \mathrm{PNL}^-$ (it suffices to note that $\Box_r p$ is a PNL^+ formula). $\Box$

4 Decidability of PNL

In this section, we prove the decidability of $\mathrm{PNL}^{\pi+}$, and consequently that of its fragments PNL^+ and PNL^-, by embedding it into the two-variable fragment of first-order logic $\mathrm{FO}^2[<]$ as follows. Let $\mathcal{AP}$ be the set of propositional variables in $\mathrm{PNL}^{\pi+}$. The signature for $\mathrm{FO}^2[<]$ includes a binary relational symbol P for every $p \in \mathcal{AP}$. The translation function $ST_{x,y}$ is defined as follows:

$$ST_{x,y}(\varphi) = x \leq y \wedge ST'_{x,y}(\varphi),$$

where x, y are two first-order variables and

$$\begin{aligned} ST'_{x,y}(p) &= P(x,y) & ST'_{x,y}(\varphi \vee \psi) &= ST'_{x,y}(\varphi) \vee ST'_{x,y}(\psi) \\ ST'_{x,y}(\pi) &= (x = y) & ST'_{x,y}(\Diamond_r \varphi) &= \exists x (y \leq x \wedge ST'_{y,x}(\varphi)) \\ ST'_{x,y}(\neg\varphi) &= \neg ST'_{x,y}(\varphi) & ST'_{x,y}(\Diamond_l \varphi) &= \exists y (y \leq x \wedge ST'_{y,x}(\varphi)) \end{aligned}$$

Two variables are thus sufficient to translate $\mathrm{PNL}^{\pi+}$ into $\mathrm{FO}^2[<]$. As we will show later, this is not the case with any proper extension of $\mathrm{PNL}^{\pi+}$ in HS or CDT. The next theorem proves that $\mathrm{FO}^2[<]$ is at least as expressive as $\mathrm{PNL}^{\pi+}$. (Recall that η is the model transformation defined in Section 2.)

Theorem 8 *For any* $\mathrm{PNL}^{\pi+}$*-formula* φ*, any non-strict interval model* $\mathbf{M}^+ = \langle \mathbb{I}(\mathbb{D})^+, V \rangle$*, and any interval* $[a, b]$ *in* $\mathbf{M}^+$*:*

$$\mathbf{M}^+, [a, b] \Vdash \varphi \textit{ iff } \eta(\mathbf{M}^+) \models ST_{x,y}(\varphi)[x := a, y := b].$$

Proof.

The proof is by structural induction on φ. The base case and the cases of the Boolean connectives are straightforward and thus omitted. Let $\varphi = \Diamond_r \psi$. From $\mathbf{M}^+, [a, b] \Vdash \varphi$, it follows that there exists an element $c \geq b$ such that $\mathbf{M}^+, [b, c] \Vdash \psi$. By inductive hypothesis, we have that $\eta(\mathbf{M}^+) \models ST_{y,x}(\psi)[y := b, x := c]$. By definition of $ST_{y,x}(\psi)$, this is equivalent to $\eta(\mathbf{M}^+) \models y \leq x \wedge ST'_{y,x}(\psi)[y := b, x := c]$. This implies that $\eta(\mathbf{M}^+) \models \exists x(y \leq x \wedge ST'_{y,x}(\psi))[y := b]$. Since $[a, b]$ in $\mathbf{M}^+$, we have $a \leq b$, hence $\eta(\mathbf{M}^+) \models ST_{x,y}(\Diamond_r \psi)[x := a, y := b]$. The converse direction can be proved in a similar way. The case $\varphi = \Diamond_l \psi$ is completely analogous and thus omitted. $\square$

Corollary 9 *A* $\mathrm{PNL}^{\pi+}$*-formula* φ *is satisfiable in a class of non-strict interval structures built over a class of linear orderings* $\mathcal{C}$ *iff* $ST_{x,y}(\varphi)$ *is satisfiable in the class of all* $\mathrm{FO}^2[<]$*-models expanding linear orderings from* $\mathcal{C}$*.*

Since the above translation is polynomial in the size of the input formula, the complexity of the satisfiability of $\mathrm{PNL}^{\pi+}$ follows from Theorem 1.

Corollary 10 *The satisfiability problem for* $\mathrm{PNL}^{\pi+}$ *is decidable in NEXPTIME for each of the classes of non-strict interval structures built over:*

(i) any linear ordering,
(ii) any well-ordering,
(iii) any finite linear ordering,
(iv) the linear ordering on natural numbers.

Since $\mathrm{PNL}^+ \prec \mathrm{PNL}^{\pi+}$ and $\mathrm{PNL}^- \prec_I \mathrm{PNL}^{\pi+}$, both PNL^+ and PNL^- are decidable in NEXPTIME (at least) on the same classes of orderings as $\mathrm{PNL}^{\pi+}$. Moreover, a translation from PNL^+ to $\mathrm{FO}^2[<]$ can be obtained from that for $\mathrm{PNL}^{\pi+}$ by simply removing the rule for π, while a translation from PNL^- to $\mathrm{FO}^2[<]$ can be obtained from that for $\mathrm{PNL}^{\pi+}$ by removing the rule for π, substituting $<$ for $\leq$, and replacing $\Diamond_r$ by $\langle A \rangle$ and $\Diamond_l$ by $\langle \overline{A} \rangle$.

The NEXPTIME-hardness of the satisfiability problem for $\mathrm{PNL}^{\pi+}$, PNL^+, and PNL^- can be proved by exploiting the reduction from the exponential tiling problem given by Bresolin et al. for the future fragment of PNL [7] (the proof refers to the linear ordering on natural numbers, but it basically works for all the considered orderings). Together with Corollary 10, such a reduction allows us to prove the following theorem.

Basic formulas	Non-basic formulas
$\tau[x,y](x=x) = \tau[x,y](y=y) = \top$	$\tau[x,y](\neg\alpha) = \neg\tau[x,y](\alpha)$
$\tau[x,y](x=y) = \tau[x,y](y=x) = \pi$	$\tau[x,y](\alpha \vee \beta) = \tau[x,y](\alpha) \vee \tau[x,y](\beta)$
$\tau[x,y](y<x) = \bot$	$\tau[x,y](\exists x\beta) =$
$\tau[x,y](x<y) = \neg\pi$	$\Diamond_r(\tau[y,x](\beta)) \vee \Box_r\Diamond_l(\tau[x,y](\beta))$
$\tau[x,y](P(x,x)) = \Diamond_l(\pi \wedge p^{\leq} \wedge p^{\geq})$	$\tau[x,y](\exists y\beta) =$
$\tau[x,y](P(y,y)) = \Diamond_r(\pi \wedge p^{\leq} \wedge p^{\geq})$	$\Diamond_l(\tau[y,x](\beta)) \vee \Box_l\Diamond_r(\tau[x,y](\beta))$
$\tau[x,y](P(x,y)) = p^{\leq}$	
$\tau[x,y](P(y,x)) = p^{\geq}$	

Fig. 1. Translation τ from $\mathrm{FO}^2[<]$ to $\mathrm{PNL}^{\pi+}$.

Theorem 11 *The satisfiability problem for* PNL^-, PNL^+, *and* $\mathrm{PNL}^{\pi+}$ *interpreted in the class of all linear orderings (resp., all well-orderings, all finite linear orderings, and the linear ordering on natural numbers) is NEXPTIME-complete.*

This result can be extended to the satisfiability problem for $\mathrm{PNL}^{\pi+}$ in any class of linear orderings definable in $\mathrm{FO}^2[<]$ within any of the above, e.g., the class of all bounded or unbounded (above, below) linear orderings. Moreover, the case of the linear ordering on integer numbers has been positively solved by Bresolin et al. in [5]. The decidability of the satisfiability problem for $\mathrm{PNL}^{\pi+}$ in the class of all discrete (resp., dense, Dedekind complete) linear orderings is still open.

5 Expressive completeness of $\mathrm{PNL}^{\pi+}$ for $\mathrm{FO}^2[<]$

In this section we define a truth preserving translation of $\mathrm{FO}^2[<]$ into $\mathrm{PNL}^{\pi+}$, thus showing that $\mathrm{PNL}^{\pi+}$ is at least as expressive as $\mathrm{FO}^2[<]$. Combining this result with the standard translation of $\mathrm{PNL}^{\pi+}$ into $\mathrm{FO}^2[<]$ presented in the previous section, we conclude that $\mathrm{PNL}^{\pi+}$ is as expressive as $\mathrm{FO}^2[<]$. A similar result was obtained by Venema in [27], viz., the expressive completeness of CDT with respect to the fragment $\mathrm{FO}^3_{x,y}[<]$ of first-order logic interpreted in linear orderings whose language contains only three, possibly reused variables and at most two of them, x and y, can be free in a formula. Both results can be viewed as interval-based counterparts of Kamp's expressive completeness theorem for the propositional point-based linear time temporal logic LTL with respect to the monadic first-order logic over Dedekind complete linear orderings [17]. The translation τ from $\mathrm{FO}^2[<]$ to $\mathrm{PNL}^{\pi+}$ is given in Figure 1.

As formally stated by Theorem 12 below, every $\mathrm{FO}^2[<]$-formula $\alpha(x,y)$ is mapped into two distinct $\mathrm{PNL}^{\pi+}$-formulas $\tau[x,y](\alpha)$ and $\tau[y,x](\alpha)$. The first captures precisely those models (if any) of $\alpha(x,y)$ where $x \le y$, while the second captures precisely those models (if any) of $\alpha(x,y)$ where $y \le x$.

Example 1 *Consider the formula* $\alpha = \exists x \neg \exists y (x < y)$, *which constrains the model to be bounded above. Let* $\beta = \exists y(x < y)$. *We have that*

$$\begin{aligned}\tau[x,y](\beta) &= \Diamond_l(\tau[y,x](x<y)) \vee \Box_l\Diamond_r(\tau[x,y](x<y)) \;= \\ &= \Diamond_l\bot \vee \Box_l\Diamond_r\neg\pi \;(\equiv\; \Box_l\Diamond_r\neg\pi)\end{aligned}$$

and that

$$\begin{aligned}\tau[y,x](\beta) &= \Diamond_r(\tau[x,y](x<y)) \vee \Box_r\Diamond_l(\tau[y,x](x<y)) \;= \\ &= \Diamond_r\neg\pi \vee \Box_r\Diamond_l\bot \;(\equiv\; \Diamond_r\neg\pi).\end{aligned}$$

The resulting translation of α *is:*

$$\begin{aligned}\tau[x,y](\alpha) &= \Diamond_r(\tau[y,x](\neg\beta)) \vee \Box_r\Diamond_l(\tau[x,y](\neg\beta)) \;= \\ &= \Diamond_r(\neg\tau[y,x](\beta)) \vee \Box_r\Diamond_l(\neg\tau[x,y](\beta)) \;= \\ &= \Diamond_r\neg\Diamond_r\neg\pi \vee \Box_r\Diamond_l\neg\Box_l\Diamond_r\neg\pi \;= \\ &= \Diamond_r\Box_r\pi \vee \Box_r\Diamond_l\Diamond_l\Box_r\pi \\ &\quad (\equiv \Diamond_r\Box_r\pi \vee \Box_r\pi),\end{aligned}$$

which is a $\mathrm{PNL}^{\pi+}$*-formula that likewise constrains the model to be bounded above.*

Given an $\mathrm{FO}^2[<]$-model $\mathcal{A} = \langle \mathbb{D}, V_{\mathcal{A}}\rangle$, let $\zeta(\mathcal{A}) = \langle \mathbb{I}(\mathbb{D})^+, V_{\zeta(\mathcal{A})}\rangle$ be the corresponding $\mathrm{PNL}^{\pi+}$-model (cf. Section 2).

Theorem 12 *For every* $\mathrm{FO}^2[<]$*-formula* $\alpha(x,y)$, *every* $\mathrm{FO}^2[<]$*-model* $\mathcal{A} = \langle \mathbb{D}, V_{\mathcal{A}}\rangle$, *and every pair* $a, b \in D$, *with* $a \le b$:

(i) $\mathcal{A} \models \alpha(a,b)$ *if and only if* $\zeta(\mathcal{A}), [a,b] \Vdash \tau[x,y](\alpha)$, *and*
(ii) $\mathcal{A} \models \alpha(b,a)$ *if and only if* $\zeta(\mathcal{A}), [a,b] \Vdash \tau[y,x](\alpha)$.

Proof.

The proof is by simultaneous induction on the complexity of α.

- $\alpha = (x = x)$ or $\alpha = (y = y)$. Both α and $\tau[x,y](\alpha) = \top$ are true.
- $\alpha = (x < y)$.
  Claim (i): $\mathcal{A} \models \alpha(a,b)$ iff $a < b$ iff $\zeta(\mathcal{A}), [a,b] \Vdash \neg\pi$.

Claim (ii): $\mathcal{A} \not\models \alpha(b,a)$, since $a \leq b$, and $\zeta(\mathcal{A}), [a,b] \not\Vdash \tau[y,x](x < y)(= \bot)$. Likewise, for $\alpha = (y < x)$.

- $\alpha = P(x,y)$ or $\alpha = P(y,x)$. Both claims follow from the valuation of $p^{\leq}$ and $p^{\geq}$ (given in Section 2).
- $\alpha = P(x,x)$.
  Claim (i): $\mathcal{A} \models \alpha(a,b)$ iff $\mathcal{A} \models P(a,a)$ iff $\zeta(\mathcal{A}), [a,a] \Vdash \pi \wedge p^{\leq} \wedge p^{\geq}$ iff $\zeta(\mathcal{A}), [a,b] \Vdash \Diamond_l(\pi \wedge p^{\leq} \wedge p^{\geq})$.
  A similar argument can be used to prove claim (ii). Likewise for $\alpha = P(y,y)$.
- The Boolean cases are straightforward.
- $\alpha = \exists x \beta$.
  Claim (i): suppose that $\mathcal{A} \models \alpha(a,b)$. Then there is $c \in \mathcal{A}$ such that $\mathcal{A} \models \beta(c,b)$. There are two (non-exclusive) cases: $b \leq c$ and $c \leq b$. If $b \leq c$, by the inductive hypothesis, we have that $\zeta(\mathcal{A}), [b,c] \Vdash \tau[y,x](\beta)$ and thus $\zeta(\mathcal{A}), [a,b] \Vdash \Diamond_r(\tau[y,x](\beta))$. Likewise, if $c \leq b$, by the inductive hypothesis, we have that $\zeta(\mathcal{A}), [c,b] \Vdash \tau[x,y](\beta)$ and thus for every d such that $b \leq d$, $\zeta(\mathcal{A}), [b,d] \Vdash \Diamond_l(\tau[x,y](\beta))$, that is, $\zeta(\mathcal{A}), [a,b] \Vdash \Box_r \Diamond_l(\tau[x,y](\beta))$. Hence $\zeta(\mathcal{A}), [a,b] \Vdash \Diamond_r(\tau[y,x](\beta)) \vee \Box_r \Diamond_l(\tau[x,y](\beta))$, that is, $\zeta(\mathcal{A}), [a,b] \Vdash \tau[x,y](\alpha)$. For the converse direction, it suffices to note that the interval $[a,b]$ has at least one right neighbor, viz. $[b,b]$, and thus the above argument can be reversed.
  Claim (ii) can be proved similarly.
- $\alpha = \exists y \beta$. Analogous to the previous case. □

Corollary 13 *For every formula $\alpha(x,y)$ and every $\mathrm{FO}^2[<]$-model $\mathcal{A} = \langle \mathbb{D}, V_{\mathcal{A}} \rangle$, $\mathcal{A} \models \forall x \forall y \alpha(x,y)$ if and only if $\zeta(\mathcal{A}) \Vdash \tau[x,y](\alpha) \wedge \tau[y,x](\alpha)$.*

Definition 14 *We say that a $\mathrm{PNL}^{\pi+}$-model $\mathbf{M}$ of the considered language is* synchronized on a pair of variables $(p^{\leq}, p^{\geq})$ *if these variables are equally true at any point interval $[a,a]$ in $\mathbf{M}$; $\mathbf{M}$ is* synchronized for a $\mathrm{FO}^2[<]$-formula α *if it is synchronized on every pair of variables $(p^{\leq}, p^{\geq})$ corresponding to a predicate p occurring in α; $\mathbf{M}$ is* synchronized *if it is synchronized on every pair $(p^{\leq}, p^{\geq})$.*

It is immediate to see that every model $\zeta(\mathcal{A})$, where $\mathcal{A}$ is a $\mathrm{FO}^2[<]$-model, is synchronized. Conversely, every synchronized $\mathrm{PNL}^{\pi+}$-model $\mathbf{M}$ can be represented as $\zeta(\mathcal{A})$ for some model $\mathcal{A}$ for $\mathrm{FO}^2[<]$: the linear ordering of $\mathcal{A}$ is inherited from $\mathbf{M}$ and the interpretation of every binary predicate P is defined in accordance with Theorem 12, that is, for any $a, b \in \mathcal{A}$ we set $P(a,b)$ to be true precisely when $a \leq b$ and $\mathbf{M}, [a,b] \Vdash p^{\leq}$ or $b \leq a$ and $\mathbf{M}, [b,a] \Vdash p^{\geq}$. Due to synchronization, these two conditions agree when $a = b$. Furthermore, the condition that a $\mathrm{PNL}^{\pi+}$-model $\mathbf{M}$ is synchronized on a pair of variables $p^{\leq}$ and $p^{\geq}$ can be expressed by the validity in $\mathbf{M}$ of the formula $[U](\pi \rightarrow (p^{\leq} \leftrightarrow p^{\geq}))$, where $[U]$ is the *universal modality*, which is definable in $\mathrm{PNL}^{\pi+}$ as follows

$$\begin{array}{ccc} \mathrm{CDT} & \overset{\equiv_R}{\text{———}} & \mathrm{FO}^3_{x,y}[<] \\ \diagdown \prec & & \diagdown \prec \\ \mathrm{PNL}^{\pi+} & \overset{\equiv_R}{\text{———}} & \mathrm{FO}^2[<] \end{array}$$

Fig. 2. Expressive completeness results for interval logics.

[10]:

$$[U]\varphi ::= \Box_r\Box_r\Box_l\varphi \wedge \Box_r\Box_l\Box_l\varphi \wedge \Box_l\Box_l\Box_r\varphi \wedge \Box_l\Box_r\Box_r\varphi.$$

Building on this observation, we associate with every $\mathrm{FO}^2[<]$-formula α the formulas

$$\sigma_v(\alpha) = \left(\bigwedge_{p^{\leq}, p^{\geq}} [U](\pi \rightarrow (p^{\leq} \leftrightarrow p^{\geq})) \right) \rightarrow (\tau[x,y](\alpha) \wedge \tau[y,x](\alpha))$$

and

$$\sigma_s(\alpha) = \left(\bigwedge_{p^{\leq}, p^{\geq}} [U](\pi \rightarrow (p^{\leq} \leftrightarrow p^{\geq})) \right) \wedge (\tau[x,y](\alpha) \vee \tau[y,x](\alpha)),$$

where the conjunctions range over all pairs $p^{\leq}, p^{\geq}$ corresponding to predicates occurring in α.

Corollary 15 *For any* $\mathrm{FO}^2[<]$*-formula* α*:*

(i) α *is valid in all* $\mathrm{FO}^2[<]$*-models iff* $\sigma_v(\alpha)$ *is a valid* $\mathrm{PNL}^{\pi+}$*-formula, and*
(ii) α *is satisfiable in some* $\mathrm{FO}^2[<]$*-model iff* $\sigma_s(\alpha)$ *is a satisfiable* $\mathrm{PNL}^{\pi+}$*-formula.*

Note that the proposed translation from $\mathrm{FO}^2[<]$ to $\mathrm{PNL}^{\pi+}$ is exponential in the size of the input formula, due to the clause for the existential quantifier (at the moment, we do not know whether there exists a polynomial translation).

In Figure 2, we combine the expressive completeness results for CDT and $\mathrm{PNL}^{\pi+}$ using the notation introduced in Section 2. Since $\mathrm{FO}^2[<]$ is a proper fragment of $\mathrm{FO}^3_{x,y}[<]$, from the equivalences between CDT and $\mathrm{FO}^3_{x,y}[<]$ and between $\mathrm{PNL}^{\pi+}$ and $\mathrm{FO}^2[<]$, it immediately follows that CDT is strictly more expressive than $\mathrm{PNL}^{\pi+}$.

6 Comparing $\mathrm{PNL}^{\pi+}$ with other fragments of HS

In this section, we compare $\mathrm{PNL}^{\pi+}$ with other fragments of HS and show that $\mathrm{PNL}^{\pi+}$ is essentially the maximal fragment of HS which translates to $\mathrm{FO}^2[<]$. More precisely, we consider the interval modalities $\langle B\rangle$, $\langle E\rangle$, $\langle O\rangle$, $\langle D\rangle$, $\langle L\rangle$, and their inverses, which correspond to Allen's relations *begins, ends, overlaps, during, after*, and their inverse relations. The standard translations of these modalities into first-order logic are as follows:

$$\begin{aligned}
ST_{x,y}(\langle B\rangle\varphi) &= x \leq y \wedge \exists z(z < y \wedge ST_{x,z}(\varphi))\\
ST_{x,y}(\langle E\rangle\varphi) &= x \leq y \wedge \exists z(x < z \wedge ST_{z,y}(\varphi))\\
ST_{x,y}(\langle O\rangle\varphi) &= x \leq y \wedge \exists z(x < z < y \wedge \exists y(y < x \wedge ST_{y,z}(\varphi)))\\
ST_{x,y}(\langle D\rangle\varphi) &= x \leq y \wedge \exists z(x < z < y \wedge \exists y(x < y \wedge ST_{y,z}(\varphi)))\\
ST_{x,y}(\langle L\rangle\varphi) &= x \leq y \wedge \exists x(y < x \wedge \exists y ST_{x,y}(\varphi)).
\end{aligned}$$

Note that the standard translation of $\langle L\rangle$ is a two-variable formula, while the standard translations of the other modalities are three-variable formulas. However, $\langle L\rangle$ can be defined in $\mathrm{PNL}^{\pi+}$ as follows: $\langle L\rangle\varphi = \Diamond_r(\neg\pi \wedge \Diamond_r\varphi)$. Likewise, the inverted modality $\langle \overline{L}\rangle$ is definable in $\mathrm{PNL}^{\pi+}$.

We will show that none of the other interval modalities listed above can be defined in $\mathrm{PNL}^{\pi+}$ by using game-theoretic arguments similar to those in the proof of Theorem 5. To this end, we define the *k-round* $\mathrm{PNL}^{\pi+}$*-bisimulation game* played on a pair of $\mathrm{PNL}^{\pi+}$ models $(\mathbf{M_0}^+, \mathbf{M_1}^+)$ starting from a given initial configuration as follows. The rules of the game are the same as those of the k-round PNL^+-bisimulation game described in Section 3; the only difference is that a configuration $([a_0, b_0], [a_1, b_1])$ is matching if and only if:

(i) $[a_0, b_0]$ and $[a_1, b_1]$ share the same valuation of propositional variables, and
(ii) $a_0 = b_0$ iff $a_1 = b_1$, that is, $\mathbf{M_0}^+, [a_0, b_0] \Vdash \pi$ iff $\mathbf{M_1}^+, [a_1, b_1] \Vdash \pi$.

The following result is analogous to Proposition 4.

Proposition 16 *Let $\mathcal{P}$ be a finite set of propositional variables. For all $k \geq 0$, Player II has a winning strategy in the k-round* $\mathrm{PNL}^{\pi+}$*-bisimulation game on* $\mathbf{M_0}^+$ *and* $\mathbf{M_1}^+$ *with initial configuration* $([a_0, b_0], [a_1, b_1])$ *iff* $[a_0, b_0]$ *and* $[a_1, b_1]$ *satisfy the same formulas of* $\mathrm{PNL}^{\pi+}$ *over $\mathcal{P}$ with operator depth at most k.*

We exploit Proposition 16 to prove that none of the interval modalities $\langle B\rangle$, $\langle E\rangle$, $\langle O\rangle$, and $\langle D\rangle$ is expressible in $\mathrm{PNL}^{\pi+}$. The proof structure is always the same: for every operator $\langle X\rangle$, we choose two models $\mathbf{M_0}^+$ and $\mathbf{M_1}^+$ that can be distinguished with a formula containing $\langle X\rangle$ and we prove that Player II has a winning strategy in the k-rounds $\mathrm{PNL}^{\pi+}$-bisimulation game.

Theorem 17 *Neither of $\langle B\rangle$, $\langle E\rangle$, $\langle O\rangle$, and $\langle D\rangle$, or their inverses, can be defined in* $\mathrm{PNL}^{\pi+}$.

Proof.

We prove the claim for $\langle B\rangle$ and $\langle D\rangle$; the other cases are analogous. Consider the PNL$^{\pi+}$-models $\mathbf{M_0}^+ = \langle \mathbb{I}(\mathbb{Z} \setminus \{1,2\})^+, V_0\rangle$ and $\mathbf{M_1}^+ = \langle \mathbb{I}(\mathbb{Z})^+, V_1\rangle$, where V_1 is such that p holds for all intervals $[a,b]$ such that $a < b$ and V_0 is the restriction of V_1 to $\mathbb{I}(\mathbb{Z} \setminus \{1,2\})^+$. Note that $\mathbf{M_1}^+, [0,3] \Vdash \langle B\rangle p$, while $\mathbf{M_0}^+, [0,3] \not\Vdash \langle B\rangle p$; likewise for $\langle D\rangle p$. Thus, to prove the claims, it suffices to show that Player II has a winning strategy for the k-round PNL$^{\pi+}$-bisimulation game between $\mathbf{M_0}^+$ and $\mathbf{M_1}^+$ with initial configuration $([0,3],[0,3])$. In fact, Player II has a *uniform* strategy to play that game forever: at any position, assuming that Player I has not yet won, if he chooses a $\Diamond_r$-move, then Player II arbitrarily chooses a right neighbor of the current interval on the other structure, with the only constraint being to take a point-interval if and only if Player I has taken a point-interval as well. If Player I chooses a $\Diamond_l$-move, Player II acts likewise. □

7 Undecidable extensions of PNL

A natural question now arises: is it possible to extend PNL$^{\pi+}$ with other modal operators (such as those listed in the previous section) without losing decidability? In this section, we address and partly answer this question negatively by considering the extensions of PNL$^{\pi+}$ within HS. First of all, we show that adding the modal operator $\langle D\rangle$, or its inverse $\langle \overline{D}\rangle$, to PNL suffices to cross the undecidability border.

The technique used here is based on a non-trivial reduction from the *unbounded tiling problem* for the second octant $\mathcal{O}$ of the integer plane [1]. This is the problem of establishing whether a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ can tile $\mathcal{O} = \{(i,j) : i,j \in \mathbb{N} \wedge 0 \leq i \leq j\}$. For every tile type $t_i \in \mathcal{T}$, let $right(t_i)$, $left(t_i)$, $up(t_i)$, and $down(t_i)$ be the colors of the corresponding sides of t_i. To solve the problem, one must find a function $f : \mathcal{O} \to \mathcal{T}$ such that $right(f(n,m)) = left(f(n+1,m))$ with $n < m$, and $up(f(n,m)) = down(f(n,m+1))$.

Such a reduction works for the class of all linear orderings and for a number of interesting subclasses of it. Moreover, it turns out to be quite versatile, being applicable to a variety of extensions of PNL$^{\pi+}$. In summary, we will show that the satisfiability problem for any extension of PNL$^{\pi+}$ containing at least one of the following is undecidable: $\langle D\rangle$, $\langle \overline{D}\rangle$, $\langle D\rangle_{\sqsubset}$, $\langle \overline{D}\rangle_{\sqsubset}$, $\langle B\rangle$ and $\langle \overline{E}\rangle$, $\langle \overline{B}\rangle$ and $\langle E\rangle$, where $\langle D\rangle_{\sqsubset}$ is the modal operator of the *proper subinterval relation* (and $\langle \overline{D}\rangle_{\sqsubset}$ is its inverse), studied in more detail in [2,3], which is defined as follows [3]:

$$ST_{x,y}(\langle D\rangle_{\sqsubset}\varphi) = x < y \wedge \exists z \exists w(x \leq z \wedge w \leq y \wedge (x < z \vee w < y) \wedge ST_{z,w}(\varphi)).$$

These cases cover a huge majority of all fragments of HS containing $\mathrm{PNL}^{\pi+}$. In the following, we will provide a detailed analysis of the representative case of $\mathrm{PNL}^{\pi+}+\langle D\rangle$; at the end of the section, we will show how to adapt the formulas used in the proof to the remaining cases. Beside the original undecidability result for HS, the present one can be paired with two other undecidability results, namely, that for the BE-fragment [11,18] and that for Compass Logic [19], which can be seen as a generalized propositional logic of intervals.

7.1 Undecidability of $\mathrm{PNL}^{\pi+}+\langle D\rangle$

Language and point-intervals. Let us fix an arbitrary finite set of tiles $\mathcal{T} = \{t_1, \ldots, t_k\}$ and assume that the set of atomic propositions $\mathcal{AP}$ is finite (but arbitrary) and contains, inter alia, the following propositional variables: u, Id, tile, $\mathtt{t}_1, \ldots, \mathtt{t}_k$, bb, be, eb, and corr. For the sake of convenience, we define the PNL^- operator $\langle A\rangle$ in terms of $\Diamond_r$ and π:

$$\langle A\rangle p = \Diamond_r(\neg\pi \wedge p). \tag{1}$$

The inverse operator $\langle \overline{A}\rangle$ can be defined likewise.

Unit-intervals. We set our framework by forcing the existence of a unique infinite chain of so-called *unit-intervals* (for short, u-*intervals*) on the linear ordering, which covers an initial segment of the model. These u-intervals will be used as cells to arrange the tiling. They will be labeled by the propositional variable u. Formally, we define the formula

$$\mathsf{UnitChain} ::= \mathsf{u} \wedge [\overline{A}][\overline{A}][A]\neg\mathsf{u} \wedge [U](\mathsf{u} \rightarrow (\neg\pi \wedge \langle A\rangle\mathsf{u} \wedge \neg\langle D\rangle\langle A\rangle\mathsf{u})). \tag{2}$$

Lemma 18 *Suppose that* $\mathbf{M}, [a,b] \Vdash \mathsf{UnitChain}$. *Then, there exists an infinite sequence of points* $b_0 < b_1 < \ldots$ *in* $\mathbf{M}$ *such that* $a = b_0, b = b_1$, *for each* i, $\mathbf{M}, [b_i, b_{i+1}] \Vdash \mathsf{u}$, *and no other interval* $[c,d]$ *in* $\mathbf{M}$ *satisfies* u, *unless* $c > b_i$ *for every* $i \in \mathbb{N}$.

[3] In fact, three variables suffice to define $\langle D\rangle_{\sqsubset}$; four variables makes it possible to define it in a more compact way.

Proof.

Clearly, $\mathbf{M}, [a,b] \Vdash \mathsf{u} \wedge \neg\pi$, so $a \neq b$. The existence of the chain of endpoints of u-intervals $b_1 < b_2 < \ldots$ starting from a, b is easy, because every u-interval has a right neighbor u-interval. We still have to show that no other point either ends or begins a u-interval. Indeed, suppose that for some c, d with $c \neq b_i$ for every $i = 0, 1, \ldots$, $\mathbf{M}, [c,d] \Vdash \mathsf{u}$ holds. Because $\mathbf{M}, [a,b] \Vdash [\overline{A}][\overline{A}][A]\neg\mathsf{u}$, we have that $b_0 < c$, hence either $b_i < c < b_{i+1}$ for some i or $c > b_i$ for every $i = 0, 1, \ldots$. In the former case, $\mathbf{M}, [c,c] \Vdash \langle A\rangle\mathsf{u}$, hence $\mathbf{M}, [b_i, b_{i+1}] \Vdash \mathsf{u} \wedge \langle D\rangle\langle A\rangle\mathsf{u}$ which contradicts $\mathbf{M}, [a,b] \Vdash [U](\mathsf{u} \rightarrow \neg\langle D\rangle\langle A\rangle\mathsf{u})$. Finally, note that the case in which $c = b_i$ and $d = b_{i+q}$, where $q > 1$, contradicts $\mathbf{M}, [a,b] \Vdash [U](\mathsf{u} \rightarrow \neg\langle D\rangle\langle A\rangle\mathsf{u})$, since $\mathbf{M}, [b_{i+1}, b_{i+1}] \Vdash \langle A\rangle\mathsf{u}$. □

Then, to restrict our domain of 'legitimate intervals' to those composed of u-intervals, we impose that every interval of importance begins and ends with a u-interval:

$$[U] \bigwedge_{p \in \mathcal{AP}} (p \rightarrow [\overline{A}]\langle A\rangle\mathsf{u} \wedge [A]\langle\overline{A}\rangle\mathsf{u}). \tag{3}$$

Encoding a tile. Every u-interval will represent either a tile or a special marker, denoted by $*$, indicating the border between two ld-intervals, that will be defined later. Thus, we put:

$$[U](\mathsf{u} \leftrightarrow (* \vee \mathsf{tile}) \wedge (* \leftrightarrow \neg\mathsf{tile})), \tag{4}$$

$$[U](\mathsf{tile} \leftrightarrow (\bigvee_{i=1}^{k} \mathtt{t_i} \wedge \bigwedge_{i,j=1, j\neq i}^{k} \neg(\mathtt{t_i} \wedge \mathtt{t_j}))). \tag{5}$$

If a tile is placed on a u-interval $[a,b]$, we call a and b respectively the *beginning point* and the *ending point* of that tile.

Encoding rows of the tiling. Each ld-interval (or just ld) represents a row (level) of the tiling of $\mathcal{O}$. An ld-interval is an interval consisting of a finite sequence of at least two u-subintervals. The first u-subinterval in an ld is a $*$-interval and every following u-subinterval is the encoding of a tile. Moreover, the ld-intervals representing the bottom-up consecutive levels of the tiling of $\mathcal{O}$ are arranged one after another in a chain. So:

$$[U](\mathsf{ld} \rightarrow \neg\mathsf{u} \wedge \langle\overline{A}\rangle\langle A\rangle * \wedge \langle A\rangle\mathsf{ld}). \tag{6}$$

To prevent the existence of interleaving sequences of ld-intervals, we do not allow occurrences of $*$-subintervals inside an ld by means of the following formula:

$$[U](\mathsf{ld} \rightarrow \neg\langle D\rangle\langle A\rangle*). \tag{7}$$

The next formula states that the first ld is composed by a single tile:

$$\mathsf{First} = [\overline{A}][\overline{A}][A]\neg \mathsf{u} \wedge \mathsf{ld} \wedge \langle \overline{A} \rangle \langle A \rangle (* \wedge \langle A \rangle (\mathsf{tile} \wedge \langle A \rangle *)). \quad (8)$$

Finally, we put:

$$\mathsf{IdDef} = \mathsf{First} \wedge (2) \wedge (3) \wedge (4) \wedge (5) \wedge (6) \wedge (7). \quad (9)$$

Lemma 19 *Suppose that $\mathbf{M}, [a, b] \Vdash \mathsf{IdDef}$. Then there is a sequence of points $a = b_1^0 < b_1^1 < \ldots b_1^{k_1} = b_2^0 < b_2^1 < \ldots < b_2^{k_2} = b_3^0 < \ldots$, such that $k_1 = 2$ and for every j:*

(1) $\mathbf{M}, [b_j^0, b_j^{k_j}] \Vdash \mathsf{Id}$ and no other interval $[c, d]$ in $\mathbf{M}$ is an Id-interval, unless possibly for $c > b_j^{k_j}$ for every $j \in \mathbb{N}$;

(2) $\mathbf{M}, [b_j^0, b_j^1] \Vdash *$ and no other interval $[c, d]$ in $\mathbf{M}$ is a $*$-interval, unless possibly for $c > b_j^{k_j}$ for every $j \in \mathbb{N}$;

(3) for every i such that $0 < i < k_j$, $\mathbf{M}, [b_j^i, b_j^{i+1}] \Vdash \mathsf{tile}$, and no other interval $[c, d]$ in $\mathbf{M}$ is a tile-interval, unless possibly for $c > b_j^{k_j}$ for every $j \in \mathbb{N}$.

Proof.

The existence of the infinite sequence of points follows from First and formula (6) which together imply existence of an infinite sequence of consecutive Id-intervals $[b_1^0, b_1^{k_1}], [b_2^0, b_2^{k_2}] \ldots$. Now, let the endpoints of the u-subintervals of $[b_j^0, b_j^{k_j}]$ be $b_j^0 < b_j^1 < \ldots < b_j^{k_j}$. Thus, the first part of claim (1) holds by construction.

Now suppose that another interval $[c, d]$ satisfies Id. The left endpoint c cannot be less than a, because an Id-interval begins with a u-interval and, by First, no u-interval begins to the left of a. Assuming that $b_j^0 < c < b_j^{k_j}$ for some j leads to a contradiction with (7), because the beginning of $[c, d]$ is a $*$-interval properly contained in the Id-interval $[b_j^0, b_j^{k_j}]$. Finally, assuming that $c = b_j^0$ for some j leads to a contradiction as well. Since $[c, d]$ ($\neq [b_j^0, b_j^{k_j}]$) must be followed immediately by another Id-interval $[d, e]$, the beginning $*$-subinterval of $[d, e]$ must be strictly inside $[b_j^0, b_j^{k_j}]$ or the $*$-interval $[b_{j+1}^0, b_{j+1}^1]$ must be strictly inside $[c, d]$, either of which is impossible due to condition (7). Thus, claim (1) is established.

Claims (2) and (3) can be proved in a similar way, using the respective conjuncts in IdDef. □

Definition 20 *Let $\mathbf{M}, [a, b] \Vdash \mathsf{IdDef}$ and $b_1^0 < b_1^1 < \ldots b_1^{k_1} = b_2^0 < b_2^1 < \ldots < b_2^{k_2} = b_3^0 \ldots$ be the sequence of points whose existence is guaranteed by*

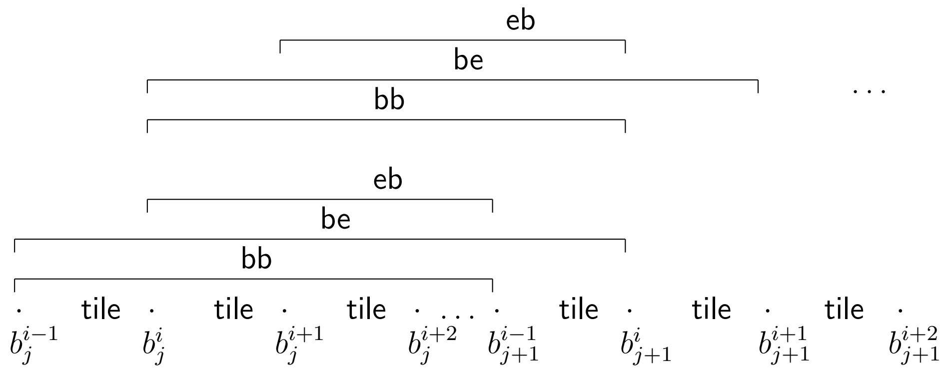

Fig. 3. A representation of bb, be, and eb-intervals.

Lemma 19. For any j, the interval $[b_j^0, b_j^{k_j}]$ is the j-th Id-interval *of the sequence and for any $i \geq 1$, the interval $[b_j^i, b_j^{i+1}]$ is the i*-th tile *of the* Id*-interval $[b_j^0, b_j^{k_j}]$.*

Corresponding tiles. So far we have that, given a starting interval, the formula IdDef forces the underlying linearly ordered set to be, in the future of the current interval, a sequence of Id's, the first one of which containing exactly one tile. Now, we want to make sure that each tile at a certain *level* in $\mathcal{O}$ (i.e., Id) always has its corresponding tile at the immediate upper level. We will use auxiliary propositional variables in order to guarantee this property, namely: bb, which is to connect the beginning point of a tile to the beginning point of the corresponding tile above; be, which is to connect the beginning point of a tile to the ending point of the corresponding tile above; and eb, which is to connect the ending point of a tile to the beginning point of the corresponding tile above. If an interval is labeled with either of these three propositional variables, we call it a *corresponding interval*, abbreviated corr-*interval*. A pictorial representation is given in Figure 3. In the following, we force corr-intervals to respect suitable properties so that all models satisfying them encode correct tiling.

First, we put the propositional variable corr wherever one among bb, be and eb holds:

$$[U]((\mathsf{bb} \vee \mathsf{be} \vee \mathsf{eb}) \leftrightarrow \mathsf{corr}). \tag{10}$$

Then we prevent any corr-interval to coincide with an Id-interval:

$$[U]\neg(\mathsf{corr} \wedge \mathsf{Id}). \tag{11}$$

In addition, we impose that neither is a corr-interval properly contained in an Id-interval, nor the other way around. This means that a corr-interval must contain a unique separating marker $*$ and that it cannot be followed immediately by $*$:

$$[U](\mathsf{corr} \rightarrow (\neg \mathsf{u} \wedge \langle D\rangle(\langle A\rangle * \vee \langle \overline{A}\rangle *) \wedge \neg\langle D\rangle(\langle A\rangle * \wedge \langle \overline{A}\rangle *) \wedge \neg\langle A\rangle *). \tag{12}$$

We put

$$\mathsf{CorrDef} = (10) \wedge (11) \wedge (12). \tag{13}$$

Lemma 21 *Let* $\mathbf{M}, [a, b] \Vdash \mathsf{IdDef} \wedge \mathsf{CorrDef}$. *Then no* Id*-interval in* $\mathbf{M}$ *coincides with a* corr*-interval, nor is it properly contained in a* corr*-interval, nor is a* corr*-interval properly contained in an* Id*-interval unless it begins with a* $*$.

Proof.

A corr-interval cannot coincide with an Id-interval because of (11); it cannot properly contain an Id-interval because of (12), and it cannot be properly contained in an Id-interval unless it begins with a $*$, again by (12). □

To guarantee that every tile in every Id corresponds, via bb, be, and eb, to some tile of the next Id and that every tile but the last one of every Id corresponds, via bb, be, and eb, to some tile of the previous Id, we take advantage of the following formulas:

$$[U](\mathsf{u} \rightarrow (\neg * \leftrightarrow \langle \overline{A} \rangle \langle A \rangle \mathsf{bb})), \tag{14}$$

$$[U](\mathsf{u} \rightarrow ((\neg \langle A \rangle * \wedge \neg \langle A \rangle (\mathsf{u} \wedge \langle A \rangle *)) \leftrightarrow \langle A \rangle \langle \overline{A} \rangle \mathsf{bb})), \tag{15}$$

$$[U](\mathsf{u} \rightarrow (\neg * \leftrightarrow \langle \overline{A} \rangle \langle A \rangle \mathsf{be})), \tag{16}$$

$$[U](\mathsf{u} \rightarrow ((\neg * \wedge \neg \langle A \rangle *) \leftrightarrow \langle A \rangle \langle \overline{A} \rangle \mathsf{be})), \tag{17}$$

$$[U](\mathsf{u} \rightarrow (\neg \langle \overline{A} \rangle * \leftrightarrow \langle \overline{A} \rangle \langle A \rangle \mathsf{eb})), \tag{18}$$

$$[U](\mathsf{u} \rightarrow ((\neg \langle A \rangle * \wedge \neg \langle A \rangle (\mathsf{u} \wedge \langle A \rangle *)) \leftrightarrow \langle A \rangle \langle \overline{A} \rangle \mathsf{eb})). \tag{19}$$

Now, we put

$$\mathsf{CorrBound} = (14) \wedge (15) \wedge (16) \wedge (17) \wedge (18) \wedge (19). \tag{20}$$

Lemma 22 *Let* $\mathbf{M}, [a, b] \Vdash \mathsf{IdDef} \wedge \mathsf{CorrBound}$, *and let* $b_1^0 < b_1^1 < \ldots b_1^{k_1} = b_2^0 < b_2^1 < \ldots < b_2^{k_2} = b_3^0 < \ldots$ *be a sequence of points whose existence is guaranteed by Lemma 19. Then for every* $i \geq 0, j \geq 1$:

(1) b_j^i *is the beginning point of a* bb*-interval and a* be*-interval if and only if* $1 \leq i \leq k_j - 1$;
(2) b_j^i *is the beginning point of a* eb*-interval if and only if* $2 \leq i \leq k_j$;
(3) b_j^i *is the ending point of a* bb*-interval and a* eb*-interval if and only if* $1 \leq i \leq k_j - 2$;
(4) b_j^i *is the ending point of a* be*-interval if and only if* $2 \leq i \leq k_j - 1$.

Proof.

Claim (1). By Lemma 19, we know that $\mathbf{M}, [b_j^i, b_j^{i+1}] \Vdash *$ iff $i = 0$. So, if $1 \leq i \leq k_j - 1$, any interval ending in b_j^i is such that the formula $\neg \langle A \rangle *$ is satisfied on it. Therefore, by (14) and of (16), the formulas $\langle \overline{A} \rangle \langle A \rangle \mathsf{bb}$ and $\langle \overline{A} \rangle \langle A \rangle \mathsf{be}$

must be satisfied as well. This means that the point b_j^i is the beginning point of some bb-interval and of some be-interval.

The other claims can be proved by similar arguments. □

Definition 23 *Given two* tile*-intervals* $[c, d]$ *and* $[e, f]$ *in a model* $\mathbf{M}$*,* $[c, d]$ corresponds to $[e, f]$ *if* $\mathbf{M}, [c, e] \Vdash \mathsf{bb}$ *and* $\mathbf{M}, [c, f] \Vdash \mathsf{be}$ *and* $\mathbf{M}, [d, e] \Vdash \mathsf{eb}$*.*

The following formulas specify the basic relationships between the three types of correspondence:

$$[U] \bigwedge_{c,c' \in \{\mathsf{bb},\mathsf{eb},\mathsf{be}\}, c \neq c'} \neg(c \wedge c'), \tag{21}$$

$$[U](\mathsf{bb} \rightarrow \neg\langle D\rangle\mathsf{bb} \wedge \neg\langle D\rangle\mathsf{eb} \wedge \neg\langle D\rangle\mathsf{be}), \tag{22}$$

$$[U](\mathsf{eb} \rightarrow \neg\langle D\rangle\mathsf{bb} \wedge \neg\langle D\rangle\mathsf{eb} \wedge \neg\langle D\rangle\mathsf{be}), \tag{23}$$

$$[U](\mathsf{be} \rightarrow \langle D\rangle\mathsf{eb} \wedge \neg\langle D\rangle\mathsf{bb} \wedge \neg\langle D\rangle\mathsf{be}). \tag{24}$$

Let us put

$$\mathsf{CorrProp} = (21) \wedge (22) \wedge (23) \wedge (24). \tag{25}$$

Lemma 24 *Let* $\mathbf{M}, [a, b] \Vdash \mathsf{IdDef} \wedge \mathsf{CorrDef} \wedge \mathsf{CorrBound} \wedge \mathsf{CorrProp}$*. Then, for any* $j \geq 1$ *and* $i \geq 1$*:*

(1) the i*-th tile of the* j*-th* Id*-interval corresponds to the* i*-th tile of the* $j+1$*-th* Id*-interval;*
(2) there are exactly $j + 1$ *tiles in the* $j + 1$*-th* Id*-interval;*
(3) no tile of the j*-th* Id*-interval corresponds to the last tile of the* $j + 1$*-th* Id*-interval.*

Proof.

To prove the first claim, we proceed by nested induction, first on j, then on i.

Let $j = 1$ (base case). The base case of the i-induction directly follows from Lemmas 19, 21, and 22; the inductive step is trivial.

Let $j > 1$ (inductive step). The proof of the base case of the i-induction uses the same argument of the inductive step but is simpler than it. Thus, we concentrate our attention on the latter. Let $h > 1$ and suppose (inductive hypothesis) that for all $i < h$, the i-th tile of the j-th Id-interval corresponds to the i-th tile of the $j + 1$-th Id-interval.

(bb) Let us show that $[b_j^h, b_{j+1}^h]$ is a bb-interval.
Consider the point b_j^h. By Lemma 22, it must begin some bb-interval that must end at some point c such that $b_{j+1}^1 \leq c \leq b_{j+1}^{k_{j+1}-1}$. Now suppose, for contradiction, that $c \neq b_{j+1}^h$. We must distinguish two cases.

i) Suppose $c < b^h_{j+1}$. Since, by inductive hypothesis, the interval $[b^{h-1}_j, b^h_{j+1}]$ is a be-interval, the bb-interval $[b^h_j, c]$ turns out to be a strict subinterval of such a be-interval, which contradicts $\mathsf{CorrProp}$.

ii) Suppose $c > b^h_{j+1}$. By Lemma 22, the point b^h_{j+1} must end some eb-interval that must begin at some point $d \geq b^1_j$. If $d > b^h_j$, then the eb-interval $[d, b^h_{j+1}]$ is a strict subinterval of the bb-interval $[b^h_j, c]$, which contradicts $\mathsf{CorrProp}$. If $d < b^h_j$, then (by inductive hypothesis) the eb-interval $[b^h_j, b^{h-1}_{j+1}]$ turns out to be a strict subinterval of the eb-interval $[d, b^h_{j+1}]$, which contradicts $\mathsf{CorrProp}$. The last possibility is $d = b^h_j$. By Lemma 22, the point b^h_{j+1} must end some bb-interval $[e, b^h_{j+1}]$ with $e \geq b^1_j$. If $e > b^h_j$, the bb-interval $[e, b^h_{j+1}]$ is a strict subinterval of the bb-interval $[b^h_j, c]$ which contradicts $\mathsf{CorrProp}$. If $e = b^h_j$, both bb and be (by the above argument) hold over the interval $[b^h_j, b^h_{j+1}]$ which contradicts $\mathsf{CorrProp}$. Finally, if $e < b^h_j$, the eb-interval $[b^h_j, b^{h-1}_{j+1}]$ is a strict subinterval of the bb-interval $[e, b^h_{j+1}]$ which contradicts $\mathsf{CorrProp}$.

This allows us to conclude that $c = b^h_{j+1}$.

(be) Let us show that $[b^h_j, b^{h+1}_{j+1}]$ is a be-interval.

As in the previous case, by Lemma 22, the point b^h_j must be the beginning point of some be-interval that must end at some point c, with $b^1_{j+1} \leq c \leq b^{k_{j+1}-1}_{j+1}$. If $c = b^{h+1}_{j+1}$ we are done. Suppose, for contradiction, that $c \neq b^{h+1}_{j+1}$. We must distinguish two cases.

i) Suppose $c < b^{h+1}_{j+1}$. If $c = b^h_{j+1}$, then both bb and be hold over $[b^h_j, b^h_{j+1}]$ which contradicts $\mathsf{CorrProp}$; if $c < b^h_{j+1}$, the be-interval $[b^h_j, c]$ turns out to be a strict subinterval of the be-interval $[b^{h-1}_j, b^h_{j+1}]$ which contradicts $\mathsf{CorrProp}$ as well.

ii) Suppose that $c > b^{h+1}_{j+1}$. By Lemmas 22 and 21, the point b^{h+1}_{j+1} must end some be interval $[d, b^{h+1}_{j+1}]$, with $d \geq b^1_j$ (and $d \neq b^h_j$). If $d > b^h_j$, then the be-interval $[d, b^{h+1}_{j+1}]$ is a strict subinterval of the be-interval $[b^h_j, c]$ which contradicts $\mathsf{CorrProp}$. If $d < b^h_j$, the bb-interval $[b^h_j, b^h_{j+1}]$ is a strict subinterval of the be-interval $[d, b^{h+1}_{j+1}]$ which contradicts $\mathsf{CorrProp}$ as well.

(eb) Let us show that $[b^{h+1}_j, b^h_{j+1}]$ is an eb-interval.

Consider the point b^h_{j+1}. By Lemma 22, it must be the ending point of some eb-interval $[c, b^h_{j+1}]$, with $b^2_j \leq c \leq b^{k_j}_j$. If $c = b^{h+1}_j$ we are done. Suppose, for contradiction, that $c \neq b^{h+1}_j$. We must distinguish two cases.

i) Suppose $c > b^{h+1}_j$. By Lemma 22, the point b^{h+1}_j must begin some bb-interval $[b^{h+1}_j, d]$ with $b^1_{j+1} \leq d \leq b^{k_j-2}_{j+1}$. If $d \leq b^h_{j+1}$, then the bb-interval $[b^{h+1}_j, d]$ is a strict subinterval of the be-interval $[b^h_j, b^{h+1}_{j+1}]$ which contradicts $\mathsf{CorrProp}$. If $d > b^h_{j+1}$, then the eb-interval $[c, b^h_{j+1}]$ is a strict subinterval of the bb-interval $[b^{h+1}_j, d]$ which contradicts $\mathsf{CorrProp}$ as well.

ii) Suppose $c < b^{h+1}_j$. If $c = b^h_j$, then both eb and bb (by the previous point) hold over the interval $[b^h_j, b^h_{j+1}]$ which contradicts $\mathsf{CorrProp}$. If $c < b^h_j$, then the eb-interval $[b^h_j, b^{h-1}_{j+1}]$ is a strict subinterval of the eb-interval

$[c, b_{j+1}^h]$ which contradicts $\mathsf{CorrProp}$.

As for the second claim, we proceed by induction on j. The base case is straightforward, since, by Lemma 19, there is only one tile for $j = 1$. Suppose now that $j = n$ and for all $l < n$, the l-th Id-interval has exactly l tiles. Assume, for contradiction, that there are $m > n$ tiles in the n-th Id-interval (the case $m < n$ is excluded by the first claim). If $m > n$, the n-th tile is not the last one of the n-th Id-interval and thus, by Lemmas 21 and 22, the point b_n^n must be the ending point of some bb-interval beginning at some point c, with $b_{n-1}^1 \leq c \leq b_{n-1}^{k_{n-1}-1}$. Since $c < b_{n-1}^{k_{n-1}}$, the eb-interval $[b_{n-1}^{k_n}, b_n^{n-1}]$ is a strict subinterval of the bb-interval $[c, b_n^n]$ which contradicts $\mathsf{CorrProp}$. Hence $m = n$.

As for the third claim, suppose that some tile of the j-th Id-interval corresponds to the $j+1$-th tile of the $j+1$-th Id-interval. Then, by definition, $b_{j+1}^{k_{j+1}-1}$ is the ending point of some bb-interval. Since, by Lemma 19, the u-interval $[b_{j+1}^{k_{j+1}}, b_{j+2}^1]$ is a $*$-interval, this contradicts $\mathsf{CorrBound}$ (more precisely, formula (15)). □

Encoding the tiling problem. We are now ready to show how to encode the octant tiling problem. First of all, we force the horizontal and the vertical matching of colors by means of the following two formulas:

$$[U]((\mathsf{tile} \wedge \langle A \rangle \mathsf{tile}) \rightarrow \bigvee_{right(t_i)=left(t_j)} (\mathtt{t_i} \wedge \langle A \rangle \mathtt{t_j})), \tag{26}$$

$$[U](\mathsf{bb} \rightarrow \bigvee_{up(t_i)=down(t_j)} (\langle \overline{A} \rangle \langle A \rangle \mathtt{t_i} \wedge \langle A \rangle \mathtt{t_j})). \tag{27}$$

Given the set of tiles $\mathcal{T} = \{t_1, \ldots, t_k\}$, we define

$$\Phi_{\mathcal{T}} = \mathsf{IdDef} \wedge \mathsf{CorrDef} \wedge \mathsf{CorrBound} \wedge \mathsf{CorrProp} \wedge (26) \wedge (27). \tag{28}$$

Theorem 25 *Given any finite set of tiles $\mathcal{T} = \{t_1, \ldots, t_k\}$, the formula $\Phi_{\mathcal{T}}$ is satisfiable if and only if $\mathcal{T}$ can tile the second octant $\mathcal{O}$.*

Proof.

(Only if:): Suppose that $\mathbf{M}, [a, b] \Vdash \Phi_{\mathcal{T}}$. Then there is a sequence of points $a = b_1^0 < b_1^1 < b_1^2 = b_2^0 < \ldots b_j^0 < b_j^1 < \ldots < b_j^{j+1} < \ldots$ that satisfy the claims of Lemmas 19, 21, 22, 24. In particular, for every i, j, with $i \leq j$, we have $\mathbf{M}, [b_j^i, b_j^{i+1}] \Vdash \mathsf{tile}$ and hence $\mathbf{M}, [b_j^i, b_j^{i+1}] \Vdash \mathtt{t}_k$ for a unique k. We put $f(i, j) = t_k$. From Lemma 24 (and formulas 26 and 27), it follows that the function $f : \mathcal{O} \mapsto \mathcal{T}$ defines a correct tiling of $\mathcal{O}$.

(If:) Let $f : \mathcal{O} \mapsto \mathcal{T}$ be a tiling function. We show that there exist a model $\mathbf{M}$

and an interval $[a,b]$ such that $\mathbf{M},[a,b] \Vdash \Phi_{\mathcal{T}}$. Let $\mathbf{M} = \langle \mathbb{I}(\mathbb{N}), V \rangle$ be a model whose valuation function V is defined as follows. First of all, for each $i, j \in \mathbb{N}$, we put:

$$\mathsf{u} \in V([i,j]) \Leftrightarrow 0 \leq i = j-1,$$

which guarantees that (2) is satisfied. Now, let $g : \mathbb{N} \mapsto \mathbb{N}$ be such that $g(n) = (n+1)(n+2)/2 - 1$. For each $(i,j) \in \mathcal{O}$,

$$* \in V([g(j), g(j)+1]),$$

and

$$f(i,j), \mathsf{tile} \in V([g(j)+i+1, g(j)+i+2]).$$

Tiles and $*$s are assigned to unit intervals only and thus (4) is satisfied too. Since $[g(j)+i+1, g(j)+i+2] = [g(j')+i'+1, g(j')+i'+2]$ only if $(i,j) = (i',j')$, no interval is assigned to two different tiles, and thus (5) is satisfied as well.

Now, for each $j \geq 0$, we put

$$\mathsf{Id} \in V([g(j), g(j+1)]).$$

By definition, the symbol $*$ is associated with the first unit interval of every Id-interval, no Id interval properly begins or ends another Id-interval, and every Id-interval is immediately followed by another Id-interval. Hence, the formula IdDef is satisfied over the interval $[0,2]$.

Finally, for every $i \leq j$, we put

$$\begin{aligned}
\mathsf{bb} &\in V([g(j)+i+1, g(j+1)+i+1]),\\
\mathsf{be} &\in V([g(j)+i+1, g(j+1)+i+2]),\\
\mathsf{eb} &\in V([g(j)+i+2, g(j+1)+i+1]).
\end{aligned}$$

It is straightforward to check that formulas CorrBound and CorrProp are satisfied. Moreover, since f is a tiling function, formulas (26) and (27) are satisfied as well, whence the thesis. □

As a matter of fact, the model construction in the above proof can be carried out on any linear ordering containing an infinite ascending chain of points. Thus, we obtain the following.

Corollary 26 *The satisfiability problem for any extension of* $\mathrm{PNL}^{\pi+}$ *which is expressive enough to define the operator* $\langle D \rangle$*, interpreted in any class of linear orderings containing a linear ordering with an infinite ascending chain, is undecidable.*

In the rest of the section, we briefly illustrate how the various formulas can be adapted to the other cases.

7.2 Other undecidable extensions of $\mathrm{PNL}^{\pi+}$

Undecidability of $\mathbf{PNL}^{\pi+}$+$\langle\overline{D}\rangle$. Consider now any extension of $\mathrm{PNL}^{\pi+}$ featuring the modality $\langle\overline{D}\rangle$ capturing the relation of strict superinterval, which is the inverse of $\langle D\rangle$. To describe u-intervals, we can rewrite (2) as follows:

$$\mathsf{u} \wedge [\overline{A}][\overline{A}][A]\neg\mathsf{u} \wedge [U]((\mathsf{u} \rightarrow (\neg\pi \wedge \langle A\rangle\mathsf{u})) \wedge ((\mathsf{u} \vee \langle A\rangle\mathsf{u}) \rightarrow \neg\langle\overline{D}\rangle\mathsf{u})). \quad (29)$$

Similarly, to describe Id-intervals, we can rewrite (7) as follows:

$$[U](\langle A\rangle * \rightarrow \neg\langle\overline{D}\rangle\mathsf{Id}). \quad (30)$$

The relation between corr-intervals and Id-intervals can be expressed by rewriting (12) as follows:

$$[U](\mathsf{corr} \rightarrow (\neg\mathsf{u} \wedge \neg\pi \wedge \neg\langle\overline{D}\rangle\mathsf{Id} \wedge \neg\langle A\rangle\mathsf{Id} \wedge \neg\langle\overline{A}\rangle\langle A\rangle\mathsf{First})), \quad (31)$$

$$[U](\mathsf{Id} \rightarrow \neg\langle\overline{D}\rangle\mathsf{corr}). \quad (32)$$

Finally, the relations between be, eb, and bb can be expressed by replacing formulas (22), (23), and (24) with the following ones, where the operator $\langle D\rangle$ has been replaced with $\langle\overline{D}\rangle$:

$$[U](\mathsf{be} \rightarrow \neg\langle\overline{D}\rangle\mathsf{bb}), \quad (33)$$

$$[U](\mathsf{eb} \rightarrow \neg\langle\overline{D}\rangle\mathsf{eb} \wedge \langle\overline{D}\rangle\mathsf{be}), \quad (34)$$

$$[U]((\mathsf{be} \vee \mathsf{bb}) \rightarrow \neg\langle\overline{D}\rangle\mathsf{be}). \quad (35)$$

Undecidability of $\mathbf{PNL}^{\pi+}$+$\langle D\rangle_\sqsubset$ and $\mathbf{PNL}^{\pi+}$+$\langle\overline{D}\rangle_\sqsubset$. If we replace the modality for the strict subinterval relation $\langle D\rangle$ (resp., superinterval relation $\langle\overline{D}\rangle$) by that for the proper subinterval relation $\langle D\rangle_\sqsubset$ (resp., superinterval relation $\langle\overline{D}\rangle_\sqsubset$), the encoding becomes much simpler. In particular, by using any of these operators, it is easy to express the relations between u-intervals, Id-intervals, and corr-intervals. For example, (12) can be expressed as follows:

$$[U]((\mathsf{corr} \rightarrow \neg\langle D\rangle_\sqsubset\mathsf{Id}) \wedge (\mathsf{Id} \rightarrow \neg\langle D\rangle_\sqsubset\mathsf{corr})), \quad (36)$$

or

$$[U]((\mathsf{corr} \rightarrow \neg\langle\overline{D}\rangle_\sqsubset\mathsf{Id}) \wedge (\mathsf{Id} \rightarrow \neg\langle\overline{D}\rangle_\sqsubset\mathsf{corr})). \quad (37)$$

The remaining formulas can be modified in a similar way. For the encoding of the tiling problem, we do not need three types of correspondence intervals anymore. It suffices to use only one propositional variable bb and to express the relation between different bb-intervals as follows:

$$[U]((\mathsf{bb} \rightarrow \neg\langle D\rangle_{\sqsubset}\mathsf{bb}), \tag{38}$$

or

$$[U]((\mathsf{bb} \rightarrow \neg\langle \overline{D}\rangle_{\sqsubset}\mathsf{bb}). \tag{39}$$

Undecidability of PNL$^{\pi+}$+$\langle B\rangle\langle \overline{E}\rangle$ and PNL$^{\pi+}$+$\langle E\rangle\langle \overline{B}\rangle$. The remaining two cases, namely, the extensions of PNL$^{\pi+}$ containing at least one of the pairs $\langle B\rangle, \langle \overline{E}\rangle$ and $\langle \overline{B}\rangle, \langle E\rangle$, are symmetric. Let us consider the former. The formula (2) becomes:

$$\mathsf{u} \wedge [\overline{A}][\overline{A}][A]\neg\mathsf{u} \wedge [U](\mathsf{u} \rightarrow \neg\langle B\rangle(\neg\pi \wedge \langle A\rangle\mathsf{u})). \tag{40}$$

Similarly, formula (7) can be rewritten as follows:

$$[U](\mathsf{ld} \rightarrow \neg\langle B\rangle(\neg\pi \wedge \langle A\rangle\mathsf{ld})). \tag{41}$$

To encode the tiling problem, two types of correspondence suffice. Indeed, using bb-intervals we can force the existence of eb-intervals by means of the following formula:

$$[U](\mathsf{bb} \wedge \neg\langle \overline{A}\rangle * \rightarrow \langle B\rangle(\mathsf{eb} \wedge \langle \overline{E}\rangle\mathsf{bb} \wedge [\overline{E}](\langle \overline{A}\rangle\mathsf{u} \rightarrow \neg\langle \overline{E}\rangle\mathsf{bb})) \wedge [B](\langle A\rangle\mathsf{u} \rightarrow \neg\langle B\rangle\mathsf{eb})). \tag{42}$$

The tiling problem can be encoded by specifying suitable conditions on bb-intervals and be-intervals.

By using $\langle B\rangle$ only, one can express the relation between corr-intervals and ld-intervals rewriting (12) as follows:

$$[U](corr \rightarrow (\langle B\rangle(\neg\pi \wedge \langle A\rangle *) \wedge [B](\langle A\rangle * \rightarrow \neg\langle B\rangle\langle A\rangle *) \wedge \neg\langle A\rangle *)). \tag{43}$$

Combining the above results, we have the following theorem.

Theorem 27 *The satisfiability problem for any extension of* PNL$^{\pi+}$ *expressive enough to define one of the following combinations of modal operators:* $\langle D\rangle$, $\langle \overline{D}\rangle$, $\langle D\rangle_{\sqsubset}$, $\langle \overline{D}\rangle_{\sqsubset}$, $\langle B\rangle$ *and* $\langle \overline{E}\rangle$, *and* $\langle \overline{B}\rangle$ *and* $\langle E\rangle$, *interpreted in any class of linear orderings containing a linear ordering with an infinite ascending chain, is undecidable.*

Note that in most of the considered extensions the inclusion of π is not necessary, since it is definable in the language, e.g. the case when $\langle B\rangle$ belongs to the language. It immediately follows that in such cases the corresponding extensions of the language of PNL^+ are undecidable as well. The remaining cases, as well as PNL^- extensions, are still open.

8 Concluding remarks

We have explored expressiveness and decidability issues for a variety of propositional interval neighborhood logics. First, we have compared $\mathrm{PNL}^{\pi+}$ with PNL^+ and PNL^- and have shown that the former is strictly more expressive than the other two which are, in a sense, incomparable. Then we have proved that $\mathrm{PNL}^{\pi+}$ is decidable by embedding it into $\mathrm{FO}^2[<]$ and it is essentially the maximal fragment of HS with that property. Furthermore, we have proved that $\mathrm{PNL}^{\pi+}$ is as expressive as $\mathrm{FO}^2[<]$. Finally, we have proved that most extensions of $\mathrm{PNL}^{\pi+}$ with other interval modalities are undecidable.

A number of questions still remain open. The most important ones are:

(1) Is the satisfiability problem for $\mathrm{PNL}^{\pi+}$ in the class of discrete (resp., dense, Dedekind complete) linear orderings decidable?
(2) Is there any decidable extension of PNL with modalities from the set $\{\langle B\rangle, \langle E\rangle, \langle O\rangle, \langle \overline{B}\rangle, \langle \overline{E}\rangle, \langle \overline{O}\rangle\}$?

Various natural further developments can stem from the present work. In particular, the tableau systems that have been developed in [4,5,7] for PNL on specific structures such as $\mathbb{N}$ and $\mathbb{Z}$ can be considered for adaptation to deal with $\mathrm{FO}^2[<]$ on these and related classes of linear orderings.

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