Elementary canonical formulae: extending Sahlqvist's theorem

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Abstract

We generalize and extend the class of Sahlqvist formulae in arbitrary polyadic modal languages, to the class of so called *inductive formulae.* To introduce them we use a representation of modal polyadic languages in a combinatorial style and thus, in particular, develop what we believe to be a better syntactic approach to elementary canonical formulae at all. By generalizing the method of minimal valuations a la Sahlqvist - van Benthem and the topological approach of Sambin and Vaccaro we prove that all inductive formulae are elementary canonical and thus extend Sahlqvist's theorem over them. In particular, we give a simple example of an inductive formula which is not frame-equivalent to any Sahlqvist formula. Then, after a deeper analysis of the inductive formulae as set-theoretic operators in descriptive and Kripke frames, we establish a somewhat stronger model-theoretic characterization of these formulae in terms of a suitable equivalence to syntactically simpler formulae ('primitive regular formulae') in the extension of the language with reversive modalities. Lastly, we study and characterize the elementary canonical formulae in reversive languages with nominals, where the relevant notion of persistence is with respect to discrete frames.

Key words: modal logic, elementary canonical formulae, polyadic modal languages, Sahlqvist formulae, first-order definability, persistence, reversive languages, nominals, pure formulae.

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Introduction

Historical remarks

The quest for general frame-completeness results has driven research in modal logic ever since the emergence of the Kripke semantics, and particularly after the hopes for its universal adequacy were shattered by the discoveries of incomplete modal logics due to Thomason and Fine in the mid 1970's. One of the most general results of the sort was the celebrated Sahlqvist's theorem [40] where he proved two notable facts about a large, syntactically defined class of modal formulae, called now Sahlqvist formulae: the *first-order correspondence:* that they all define elementary conditions on Kripke frames and these conditions can be effectively "computed" from the modal formulae; and the *completeness via canonicity:* that all these formulae are valid in their respective canonical frames, and hence axiomatize completely the classes of frames satisfying their corresponding first-order conditions.

Sahlqvist's work was partly induced by Lemmon-Scott's conjecture, claiming first-order definability and canonicity of a subset of Sahlqvist formulae, confirmed by Goldblatt (see [23]). Sahlqvist's theorem (also proved independently, in a similar form, in [2]) substantially generalizes the set of formulae covered by that conjecture, while the class of Sahlqvist formulae, modulo inessential manipulations, has turned out to be remarkably robust, so much so that a widespread opinion has developed over the years that these cannot be extended further without exorbitant technical complications (see e.g. [4] and [33]).

The striving for better understanding of what makes Sahlqvist formulae tick and the pursuit of their further extension have been an active line of research in modal logic. Some landmarks in the study of Sahlqvist formulae include:

- the systematic development in [2,3] of an algorithmic approach for computing the first-order equivalents of Sahlqvist-type formulae, based on the method of substitutions with minimal valuations;
- the modern approach to Sahlqvist formulae developed in [42], based on the topological properties of descriptive frames, allowing for unified treatment of first-order definability and canonicity;
- Kracht's calculus developed in [32] where the class of first-order formulae corresponding to Sahlqvist formulae was studied and described;
- the extension of the class of Sahlqvist formulae to polyadic languages in [39];
- the algebraic proof of canonicity of Sahlqvist formulae, without using their first-order definability, in [30], building on ideas from the seminal paper [31]

where a restricted version of Sahlqvist's theorem was already established in algebraic terms;
- in some recent papers (which appeared while the current paper was under submission) canonicity has been generalized to a much wider setting than Boolean algebras with operators in [20], and Sahlqvist's theorem has been extended to distributive modal logics in [21].

Other important contributions related to the topic include [22] and [51] where alternative results on canonicity have been obtained for non first-order definable formulae, as well as the recent work by Goldblatt, Hodkinson, and Venema [26], refuting Fine's conjecture.

Good expositions of the ideas and technicalities around Sahlqvist's theorem, with different proofs, can be found in [42], [4], [6], [33], and [16].

Aims and content of the paper

This study was initiated as a systematic attempt to answer the question "*What are Sahlqvist formulae, after all?*". While defined in a purely syntactic manner which is vulnerable to otherwise innocuous transformations (including tautological equivalence), they bear a precise, but practically intractable semantic characterization. The two important features of Sahlqvist formulae, which together imply Sahlqvist's theorem, are *(locally) first-order definability* and *(local) d-persistence* in a sense which depends on the construction of the canonical models for the logical systems under consideration and implies canonicity, hence completeness. In the case of ordinary polyadic modal languages, with no special rules of inference added to the axiomatic systems, this is *d-persistence*. The formulae in a given polyadic modal language having these properties will be called *elementary*[1] *canonical formulae*. Thus, the concept of elementary canonical formulae is the ultimate semantic idea behind Sahlqvist formulae which in turn are only a syntactic approximation of it. From this more general perspective we use the term '*Sahlqvist theorem*' as a generic claim that all formulae from a given effectively defined class are elementary and canonical.

The current paper is intended as the first part of a comprehensive study of elementary canonical formulae from syntactic, computational, model-theoretic, topological and algebraic perspectives (for sequels see [9,12,13]). In this part of the study we take a new approach to the syntactic description of elementary canonical formulae, by means of syntactic re-shaping of the modal languages and introducing in Section 2 the so called *purely modal languages* whereby

[1] 'Elementary' here is used as a synonym of 'first-order definable', as customary in logic.

disjunctions, (respectively, conjunctions), are regarded as boxes (respectively, diamonds), and formulae are built from propositional variables, negations and polyadic boxes only, in a combinatorial style resembling the Propositional Dynamic Logic. This syntactic framework allows us to introduce in Section 3 a large and natural syntactic class of '*regular*' formulae (including not only Sahlqvist formulae, but also e.g. Gödel-Löb formula and Segerberg's induction axiom). We identify a subclass of *simple regular* formulae and show that conjunctions of such formulae subsume all so far defined polyadic Sahlqvist formulae (see [39]). Further, we extend these to the larger class **I** of *inductive* formulae [2], the syntactic description of which is based on a certain *dependency digraph* defined on the set of variables in the formula, and generating a partial 'dependency' ordering on these variables.

After a set-theoretic and topological analysis of the polyadic descriptive general frames in Section 5, we extend the method of *minimal valuations* [3] of Sahlqvist-van Benthem ([3]) and the topological approach of Sambin and Vaccaro (see [42], [4]) to establish respectively *local first-order definability* in Section 4 and *local d-persistence* of the formulae in **I** in Section 6, thereby proving the Sahlqvist theorem for them. In Section 4 we also show how the method of minimal valuations works for regular formulae, too, but in general produces effectively computable equivalents in first-order language extended with *least fixpoint operators*. (However, not all regular formulae are canonical, nor even complete.)

We show in Section 7 that the class **I** extends the class of Sahlqvist formulae in the basic modal language not only syntactically, but semantically, too (contrary to the common opinion mentioned above).

In the rest of the paper we further analyze, from topological perspective, the inductive formulae regarded as set-theoretic operators and eventually establish a somewhat stronger semantic characterization of them in suitably extended modal languages. This approach involves some ideas of Sambin and Vaccaro, Kracht, and especially Venema, as it uses a detour via the 'reversive' extension of the language (containing all inverses of the basic modal operators) where all inductive formulae can be reduced to '*primitive*' ones for which both parts of the Sahlqvist theorem are proved in a uniform way.

In particular, in Section 8 we introduce *reversive* extensions of polyadic modal

[2] In [27] we call these 'polyadic Sahlqvist formulae'. The change of terminology, here and elsewhere, reflects: first, our effort to avoid the arising ambiguity and confusion caused by the overuse of the term 'Sahlqvist formulae'; second, the shift of the focus in the study; and third, the (now established) fact that inductive formulae essentially extend the polyadic Sahlqvist formulae as previously defined, e.g. in [4].
[3] The minimal valuations of the variables in inductive formulae are defined inductively on the dependency ordering, whence the term.

languages and show that the inverse operators in such extensions are closed in descriptive frames of the basic languages. In Section 9 we show that inductive formulae are persistent with respect to passing from descriptive frames to their 'closure extensions' and then, following ideas from [28], we prove that every inductive formula in a given polyadic modal language can be effectively transformed into an equivalent in a suitable semantic sense, primitive regular formula in the reversive extension of the language, thus eventually re-proving Sahlqvist theorem for inductive formulae in arbitrary polyadic languages. Then, in Section 11 we consider polyadic modal languages with *nominals* and introduce *discrete-canonical formulae* which is the right notion of canonicity in such languages, where the relevant persistence is '*di-persistence*', with respect to *discrete* general frames (with all singletons admissible). We show that in reversive languages with nominals every primitive elementary canonical formula is equivalent to a *pure formula* (containing only nominals), which can be computed *within* the minimal logic for that language, and which in a rather direct way encodes the corresponding first-order equivalent. This yields an analogue of Sahlqvist theorem for inductive formulae in reversive polyadic languages with nominals. Eventually we obtain a simple and natural characterization of the discretely-canonical formulae in such languages: they are precisely those, locally equivalent over discrete frames to inductive formulae, and hence to pure formulae.

1 Preliminaries

We assume basic familiarity with the syntax and semantics of the standard polyadic modal languages, a state-of-the-art reference for which is e.g. [4], from where we quote some of the definitions below and give a few additional definitions, not explicitly mentioned in that book.

1.1 Some syntactic and semantic notions

Hereafter we consider an arbitrarily fixed polyadic modal language $\mathcal{L}$.

Definition 1 *Formulas A and B from $\mathcal{L}$ are:*

- ***tautologically equivalent**, if $A \leftrightarrow B$ is a Boolean tautology;*
- ***semantically equivalent**, hereafter denoted $A \equiv B$, if $A \leftrightarrow B$ is a valid formula.*
- ***locally equivalent**, if they are valid at the same states in the same general frames for $\mathcal{L}_\tau$.*

- ***locally frame-equivalent****, if they are valid at the same states in the same Kripke frames for $\mathcal{L}_\tau$.*
- ***frame-equivalent****, if they are valid in the same Kripke frames for $\mathcal{L}_\tau$.*
- ***axiomatically equivalent****, if the logics $\mathbf{K}_\tau + A$ and $\mathbf{K}_\tau + B$ have the same theorems, equivalently, if $\mathbf{K}_\tau + A \vdash B$ and $\mathbf{K}_\tau + B \vdash A$, where $\mathbf{K}_\tau + A$ means the $\mathcal{L}_\tau$-logic obtained by adding the axiom A to $\mathbf{K}_\tau$.*

Hereafter, the term 'equivalent formulae' will mean 'semantically equivalent formulae', unless otherwise specified.

Positive and negative formulae are defined as usual: a formula is positive (resp. negative) if every occurrence of a variable is in the scope of an even number of even (resp. odd) number of negations.

1.2 Sahlqvist formulae in classical polyadic languages

The following definitions are combined from [4] and [39].

Definition 2 ***Boxed atom*** *is a formula $\Box_1 \cdots \Box_n p$ where $\Box_1, \ldots, \Box_n$ is a (possibly empty) string of unary boxes and p is a propositional variable.*

Sahlqvist antecedent: *a formula constructed from propositional constants, boxed atoms and negative formulae by applying $\vee$, $\wedge$, and diamonds of arbitrary arities.*

Definite Sahlqvist antecedent *is a Sahlqvist antecedent obtained without applying $\vee$ (i.e. constructed from propositional constants, boxed atoms and negative formulae by applying only $\wedge$ and diamonds of arbitrary arities).*

(Definite) Sahlqvist implication: *$A \to B$ where A is a (definite) Sahlqvist antecedent and B is a positive formula. The Sahlqvist implication is* ***monadic*** *if no polyadic modalities occur in it.*

Definition 3 ***(Definite) Sahlqvist formula ((D)SF):*** *a formula constructed from (definite) Sahlqvist implications by freely applying unary boxes and conjunctions, and applying polyadic boxes and disjunctions to formulae sharing no common variables. The Sahlqvist formula is* ***monadic*** *if no polyadic modalities occur in it.*

Basic Sahlqvist formula *is a definite Sahlqvist formula obtained without applying conjunctions to Sahlqvist implications.*

This class of polyadic Sahlqvist formulae, so defined by de Rijke and Venema, will be denoted by **dRV**.

We note that every Sahlqvist implication is tautologically equivalent to a formula of the type $\neg A$ where A is a Sahlqvist antecedent, and therefore every Sahlqvist formula is equivalent to a negated Sahlqvist antecedent, too.

Some examples: $\Diamond\Box p \to \Box p$, $\Box((\Box(\Diamond\neg p \vee \Diamond\Box\neg q) \wedge \Diamond\Box p) \to \Box\Diamond\Box(p \vee \Diamond\Box q))$, and $\langle \mathbf{2} \rangle(p,q) \to [\mathbf{2}](p,q)$, where $[\mathbf{2}]$ is a binary box and $\langle \mathbf{2} \rangle$ is its dual diamond, are Sahlqvist formulae, while $\Box\Diamond p \to p$, $\Box(p \vee q) \to (p \vee q)$, and $[\mathbf{2}](p,p) \to \langle \mathbf{2} \rangle(p,p))$ are not. Even the K-axiom $\Box(p \to q) \to (\Box p \to \Box q)$, or its equivalent $\Box p \wedge \Box(\neg p \vee q) \to \Box q$ are (syntactically) not Sahlqvist formulae.

The following easy observations will be used in the next section.

Proposition 4

(1) Every Sahlqvist implication is equivalent to a conjunction of definite Sahlqvist implications.
(2) Every Sahlqvist formula from dRV is equivalent to a conjunction of basic Sahlqvist formulae.

Remark 5 *[34] defines a similar class of polyadic Sahlqvist formulae, using also definable operators like* $\clubsuit(A,B) = \Box(A,B) \wedge \Box(\neg A, B) \wedge \Box(A, \neg B)$ *which is actually equivalent to* $\Box(A,\bot) \wedge \Box(\bot, B)$*, so that class does not extend* ***dRV****.*

2 Purely modal polyadic languages and logics

2.1 Purely modal polyadic languages: syntax

Definition 6 *A purely modal polyadic language* $\mathcal{L}_\tau$ *contains a countably infinite set of propositional variables VAR, negation* $\neg$*, and a* ***modal similarity type*** τ *consisting of a set of* ***basic modal terms*** (modalities) *with pre-assigned finite arities, including a 0-ary modality* ι_0 *a unary one* ι_1 *and a binary one* ι_2*.*

The intuition behind the 3 distinguished modalities above is simple: ι_0 will be interpreted as the constant $\top$ and its dual as $\bot$; ι_1 will be the self-dual identity; ι_2 will be $\vee$ and its dual — $\wedge$. Treating these connectives as modalities, besides allowing for elegance and uniformity, will provide suitable technical framework for working with elementary canonical formulae.

Definition 7 *By simultaneous mutual induction we define the set of* ***modal terms*** $MT(\tau)$ *and their* ***arity function*** ρ*, and the set of* ***(purely) modal formulae*** $MF(\tau)$ *as follows:*

(MT i) Every basic modal term is a modal term of the predefined arity.
(MT ii) Every formula containing no variables (hereafter called a ***constant formula****) is a 0-ary modal term.*
(MT iii) If $n > 0$, $\alpha, \beta_1, \ldots, \beta_n \in MT(\tau)$ *and* $\rho(\alpha) = n$, *then* $\alpha(\beta_1, \ldots, \beta_n) \in MT(\tau)$ *and* $\rho(\alpha(\beta_1, \ldots, \beta_n)) = \rho(\beta_1) + \cdots + \rho(\beta_n)$.

Modal terms of arity 0 will be called ***modal constants****.*

(MF i) Every propositional variable is a modal formula.
(MF ii) Every modal constant is a modal formula.
(MF iii) If A *is a formula then* $\neg A$ *is a formula;*
(MF iv) If $A_1, \ldots, A_n$ *are formulae,* α *is a modal term and* $\rho(\alpha) = n > 0$, *then* $[\alpha](A_1, \ldots, A_n)$ *is a modal formula.*

Definition 8 *The modal term* α *in the modal formula* $A = [\alpha](A_1, \ldots, A_n)$ *is called* ***the leading term of*** A.

Note that constant formulae and 0-ary terms are regarded as both modal terms and formulae. This ambiguity of the syntax should not cause confusion if properly handled, and we have put up with it for the sake of technical simplicity and convenience.

For technical purposes we extend the series of ι's with n-ary modalities ι_n: inductively as follows: $\iota_{n+1} = \iota_2(\iota_1, \iota_n)$ for $n > 1$. Furthermore, again for technical convenience, we can assume that the language contains *transposers*: operators θ_{ij} which swap the i-th and j-th argument of a modal term, i.e. $[\theta_{ij}(\alpha)](A_1, \ldots, A_i, \ldots, A_j, \ldots, A_n) = [\alpha](A_1, \ldots, A_j, \ldots, A_i, \ldots, A_n)$. We will not treat these transposers formally, but assuming them in the language will allow us not to be concerned with the specific ordering of the arguments in a modal formula.

Some notation on formulae:

$\langle\alpha\rangle(A_1, \ldots, A_n) := \neg[\alpha](\neg A_1, \ldots, \neg A_n)$;

$\top := \iota_0, \bot := \neg\iota_0$;

$A \vee B := [\iota_2](A, B)$, $A \wedge B := \langle\iota_2\rangle(A, B)$, and respectively

$A_1 \vee \ldots \vee A_n := [\iota_n](A_1, \ldots, A_n)$, $A_1 \wedge \ldots \wedge A_n := \langle\iota_n\rangle(A_1, \ldots, A_n)$;

$A \to B := \neg A \vee B$; $A \leftrightarrow B := (A \to B) \wedge (B \to A)$.

Positive and negative occurrences of variables and **positive and negative formulae** are defined as usual.

One effect of the mutual definition of modal terms and formulae is that it allows construction of *parameterized* modal terms, to be formally introduced later. For instance, if α is a unary term and β is a binary one, then $\gamma = \beta(\neg[\beta]([\alpha](\bot), \top), \iota_1)$ is a unary modal term, and the formula $[\gamma]p$ can be essentially identified with $[\beta](\neg[\beta]([\alpha](\bot), \top), p)$. The same transformation will be allowed further to non-constant arguments, too, where under certain conditions, some variables can be treated as parameters and imported into the modal terms.

2.2 Purely modal polyadic languages: semantics

The **semantics** of purely modal languages is a straightforward combination of the standard Kripke semantics for polyadic modal languages and the semantics of PDL-type polymodal languages, after taking into account the fact that conjunctions and disjunctions are now treated as modalities.

Let us fix an arbitrary purely modal language $\mathcal{L}_\tau$.

Definition 9 *A* ***(Kripke)*** τ***-frame*** *is a structure* $F = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)} \right\rangle$ *where the relations* R_α *are defined recursively by:*

- $R_{\iota_0} = W$, $R_{\iota_1} = \{(w, w) | w \in W\}$, $R_{\iota_2} = \{(w, w, w) | w \in W\}$.
- *for every basic modal term* α, $R_\alpha \subseteq W^{\rho(\alpha)+1}$.
- $R_{\alpha(\beta_1,\ldots,\beta_n)} = \{(w, w_{11}, \ldots, w_{1b_1}, \ldots, w_{n1}, \ldots, w_{nb_n}) \in W^{b_1+\cdots+b_n+1} |$ $\exists u_1 \ldots \exists u_n (R_\alpha w u_1 \ldots u_n \wedge \bigwedge_{i=1}^n R_{\beta_i} u_i w_{i1} \ldots w_{ib_i})\}$, *where* $\rho(\beta_i) = b_i, i = 1, \ldots, n$.

Note that $R_{\iota_n} = \{(w, \ldots, w) \in W^{n+1} | w \in W\}$ and that $R_\alpha \subseteq W^{\rho(\alpha)+1}$ for every modal term α.

Definition 10 *Given a* τ*-frame* $F = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)} \right\rangle$, *a* ***(Kripke)*** τ***-model over*** F *is a pair* $M = \langle F, V \rangle$ *where* $V : VAR \to \mathcal{P}(W)$ *is a* ***valuation*** *of the propositional variables in* F.

Definition 11 *The* ***truth definition*** *of a formula at a state* w *of a Kripke model* M *is defined through the following clauses:*

- $M, w \models p$ *iff* $w \in V(p)$,
- $M, w \models \neg A$ *iff not* $M, w \models A$,
- $M, w \models [\alpha](A_1, \ldots, A_n)$ *iff for all* $u_1, \ldots, u_n \in W$ *such that* $R_\alpha w u_1 \ldots u_n$, $M, u_i \models A_i$ *holds for* some $i \in \{1, \ldots, n\}$.

In particular, $M, w \models \alpha$ *iff* $R_\alpha w$, *for any modal constant* α.

A formula A is ***valid in*** M*, denoted* $M \models A$*, if* $M, w \models A$ *for every* $w \in W$*.*

Definition 12 *Given a formula* $A \in MF(\tau)$*, a* τ*-frame* $F = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)} \right\rangle$ *and* $w \in W$ *:*

- *A is* ***valid at*** w ***in*** F*, denoted* $F, w \models A$*, if* $M, w \models A$ *for every model* M *over* F;
- *A is* ***valid in*** F*, denoted* $F \models A$*, if* $F, w \models A$ *for every* $w \in W$*, iff* $M \models A$ *for every model* M *over* F;
- *A is* ***valid,*** *denoted* $\models A$*, if it is valid in every* τ*-frame.*

The following equivalence, hereafter called (COMP), follows immediately from the definitions:

$$[\alpha(\beta_1, \dots, \beta_n)](A_{11}, \dots, A_{1n_1}, \dots, A_{m1}, \dots, A_{mn_m})$$

is equivalent to

$$[\alpha]([\beta_1](A_{11}, \dots, A_{1n_1}), \dots, [\beta_n](A_{m1}, \dots, A_{mn_m})).$$

We note that the basic normal modal logic for the polyadic modal language $\mathcal{L}_\tau$ is axiomatized as the normal modal logic of a standard polyadic modal language, by adding an axiom scheme corresponding to (COMP). For more detail, see [28].

Hereafter, when W is fixed, the complement in W of a subset $X \subseteq W$ will be denoted by $-X$.

Definition 13 *Given a* τ*-frame* $F = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)} \right\rangle$*, every n-ary modal term* $\beta \in MT(\tau)$ *defines two n-ary* ***operators,*** $\langle\beta\rangle$ *and* $[\beta]$*, on* $\mathcal{P}(W)$ *as follows:*

$$[\beta](X_1, \dots, X_n) = \{x \in W | R_\beta x x_1 \dots x_n \textit{ implies } x_1 \in X_1 \textit{ or} \dots \textit{or } x_n \in X_n\},$$

and dually,

$$\langle\beta\rangle(X_1, \dots, X_n) = -[\beta](-X_1, \dots, -X_n),$$

i.e.

$$\langle\beta\rangle(X_1, \dots, X_n) = \{x \in W | R_\beta x x_1 \dots x_n \textit{ for some } x_1 \in X_1, \dots, x_n \in X_n\},$$

In particular, $\langle\beta\rangle = R_\beta$ *for every 0-ary term* β*.*

Note that the operators $\langle\beta\rangle$ and $[\beta]$ are monotone on each of their arguments. Besides, all $\langle\beta\rangle$'s are *normal* and *additive* in sense of [31], and therefore every structure $\left\langle \mathcal{P}(W), \cap, -, \varnothing, \{\langle\alpha\rangle\}_{\alpha\in MT(\tau)}\right\rangle$ is a (complete and atomic) set-theoretic *Boolean algebra with operators* (BAO), also satisfying (COMP), and called here **(polyadic) modal τ-algebra**. In [31] (see also [4]) Boolean algebras with operators are defined as abstract structures and a representation theorem for them, extending Stone representation, has been established. That representation theorem readily extends to the class of polyadic modal τ-algebras which is a variety for any modal similarity type τ.

Sometimes, upon convenience, we will regard the dual operators $\{[\alpha]\}_{\alpha\in MT(\tau)}$ as the basic ones instead.

We can now give an alternative definition of truth of a formula at a state of a model $\left\langle W, \{R_\alpha\}_{\alpha\in MT(\tau)}, V\right\rangle$, by way of extending (in the unique possible way) the valuation $V : VAR \to \mathcal{P}(W)$ to a homomorphism $\overline{V} : MF(\tau) \to \mathcal{P}(W)$ of $MF(\tau)$, regarded as a freely generated algebra over VAR, to $\left\langle \mathcal{P}(W), \cap, -, \varnothing, \{\langle\alpha\rangle\}_{\alpha\in MT(\tau)}\right\rangle$. Then the truth definition for all formulae is uniform:

$M, w \models A$ iff $w \in \overline{V}(A)$.

The equivalence of both definitions is a straightforward exercise and can be found for Boolean algebras with operators e.g. in [4]. The latter definition will be used in further sections, where we will regard formulae as set-theoretic operators.

Definition 14 *A* ***general frame*** *for $\mathcal{L}_\tau$ (**general τ-frame**) is a structure $\left\langle W, \{R_\alpha\}_{\alpha\in MT(\tau)}, \mathbb{W}\right\rangle$ extending a τ-frame with a Boolean algebra of* ***admissible subsets*** *of $\mathcal{P}(W)$, closed under the operators corresponding to the basic modal terms, and therefore under all operators $[\beta]$ (and $\langle\beta\rangle$).*

Thus, $\mathbb{W}$ is a subalgebra of $\left\langle \mathcal{P}(W), \cap, -, \varnothing, \{\langle\alpha\rangle\}_{\alpha\in MT(\tau)}\right\rangle$.

Definition 15 *Given a general τ-frame $\mathfrak{F} = \left\langle W, \{R_\alpha\}_{\alpha\in MT(\tau)}, \mathbb{W}\right\rangle$, a* ***model over*** *$\mathfrak{F}$ is any model over the Kripke τ-frame $\left\langle W, \{R_\alpha\}_{\alpha\in MT(\tau)}\right\rangle$ with valuation of the variables ranging over $\mathbb{W}$.*

Definition 16 *Given a formula $A \in \mathcal{L}_\tau$, a general τ-frame $\mathfrak{F}$, and $w \in W$, we say that A* ***is (locally) valid at*** *w* ***in*** *$\mathfrak{F}$, denoted $\mathfrak{F}, w \models A$, if A is true at w in every model over $\mathfrak{F}$; A* ***is valid in*** *$\mathfrak{F}$, denoted $\mathfrak{F} \models A$, if A is valid in $\mathfrak{F}$ at every $w \in W$, i.e. A is valid in every model over $\mathfrak{F}$.*

Proposition 17 *(See [4]) Local validity in a general τ-frame is preserved under Modus ponens and uniform substitutions. Validity in a general τ-frame is preserved under Modus Ponens, Necessitation, and uniform substitutions.*

General τ-frames and modal τ-algebras are equivalent as semantic structures. For more details on the links and duality between these see [24] or [4]. Hereafter we will deal primarily with general frames.

Kripke τ-frames can be regarded as first-order structures. The associated first-order language with equality and a family of predicates $\{\mathrm{R}_\alpha\}_{\alpha \in TM(\tau)}$, with arities matching those of the respective relations in τ-frames, will be denoted by $\mathcal{L}_\tau^{FO}$. Hereafter we will use the same symbol, R_α, for the predicate R_α in $\mathcal{L}_\tau^{FO}$ and for the relation which interprets it in a given τ-frame. This abuse of notation should not lead to any essential confusion, but will allow us to make smooth transition between syntax and semantics, without being excessively formal.

Definition 18 *Given a modal formula $A \in MF(\tau)$, a formula $\varphi(x)$ of $\mathcal{L}_\tau^{FO}$ is a **local first-order equivalent** of A if for every τ-frame $F = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)} \right\rangle$ and $w \in W$,*

$$F, w \models A \text{ iff } F \Vdash \varphi(w/x),$$

where $F \Vdash \varphi(w/x)$ denotes the first-order truth of $\varphi(x)$ in F under the assignment of w to the variable x.

*The formula A is **locally first-order definable** if it has a local first-order equivalent.*

The **standard translation** ST generalizes the one for monadic languages with the clauses:

- $ST(\sigma) = R_\sigma(x)$ for every modal constant σ;
- $ST([\alpha](A_1, \ldots, A_n) = \forall \overline{y}(R_\alpha x y_1 \ldots y_n \rightarrow \bigvee_{i=1}^n ST(A_i)(y_i/x))$

Again, note that the propositional logical connectives $\wedge, \vee, \rightarrow$, as defined above, have their standard semantic interpretation. Therefore, the purely modal polyadic languages are equally expressive as the traditional ones.

3 Regular and inductive formulae in purely modal polyadic languages

An arbitrary purely modal polyadic language $\mathcal{L}_\tau$ is fixed hereafter.

Definition 19 *An* ***essentially box-formula*** *is a $\mathcal{L}_\tau$-formula of one of the following two types:*

- $B = [\beta](N_1, \ldots, N_m)$ *where β is an m-ary modal term, for $m \geq 1$, and $N_1, \ldots, N_m$ are negative formulae. A formula of this type will be called a* ***headless box.***
- $B = [\beta](p, N_1, \ldots, N_m)$ *where β is an $(m+1)$-ary modal term, for $n \geq 0$, and $N_1, \ldots, N_n$ are negative formulae. A formula of this type will be called a* ***headed box,*** *and the variable p is called the* ***head*** *of the formula. The head need not be the first argument of a headed box, but to simplify the notation we will usually put it in first position.*

  All variables in an essentially box-formula, except for the head of the formula, (if any) are called ***inessential variables*** *in that formula.*

In particular, every formula $[\beta]p$ (including $p \equiv [\iota_1]p$) is a headed box, while every negative formula is a headless box. An example of a headless box, where $\mathbf{1}$ and $\mathbf{2}$ are respectively unary and binary modal terms, is $[\mathbf{2}]([\mathbf{1}]\langle\mathbf{1}\rangle\neg p, \neg[\mathbf{2}](p, q))$, while the formula $[\mathbf{2}]([\mathbf{1}]p, \neg[\mathbf{2}](p, \langle\mathbf{1}\rangle q))$ is not an essentially box-formula, but it is equivalent to the headed box $[\mathbf{2}(\mathbf{1}, \iota_\mathbf{1})](p, \neg[\mathbf{2}](p, \langle\mathbf{1}\rangle q))$.

Note the close analogy between essentially box-formulae and Horn clauses in first-order logic.

Definition 20 *A* ***regular (polyadic) formula*** *is any modal constant σ, or a formula $A = [\alpha](\neg B_1, \ldots, \neg B_n)$ where α is an n-ary modal term and $B_1, \ldots, B_n$ are essentially box-formulae, called the* ***the main components*** *of A.*

The class of regular formulae will be denoted by ***RF****.*

Examples of regular formulae: $\langle\mathbf{2}\rangle([\mathbf{1}]\iota_0, \neg[\mathbf{1}]\iota_0)$, $[\mathbf{1}]\neg p$, $[\mathbf{1}(\mathbf{1})]\neg p$, $[\mathbf{1}]\neg\neg POS$, $[\mathbf{2}](\neg p, \neg\neg POS)$, $[\mathbf{2}](\neg[\mathbf{1}]p, \neg\neg POS)$, where POS is any positive formula. Simple non-examples are $\neg[\mathbf{1}]p$ and $\neg[\mathbf{1}]\neg p$, but they are respectively equivalent to the regular formulae $[\iota_1]\neg[\mathbf{1}]p$ and $[\iota_1]\neg[\iota_1]\neg\neg[\mathbf{1}]\neg p$ (note that $[\iota_1]\neg\neg[\mathbf{1}]\neg p$ is a headless box). A more essential non-example is the formula $[\iota_2](\neg[\mathbf{1}]\langle\mathbf{1}\rangle p, \langle\mathbf{1}\rangle[\mathbf{1}]p)$ which is a purely modal version of McKinsey's formula $[\mathbf{1}]\langle\mathbf{1}\rangle p \to \langle\mathbf{1}\rangle[\mathbf{1}]p$. [4]

Definition 21 *An occurrence of a variable in a regular formula A is* ***essential in*** *A if it is a head of a main component of the formula, otherwise it is* ***inessential in*** *A. A variable in a regular formula A is* ***essential in*** *A if it*

[4] This formula cannot be written as a regular formula even up to semantic equivalence, but the proof of that claim goes beyond the scope of this paper.

has at least one essential occurrence in it, otherwise it is ***inessential in*** A.

A regular formula $A = [\alpha](\neg B_1, \ldots, \neg B_n)$ *is* ***lean*** *if every variable occurring in it is essential in* A.

For example, the variables p and r are essential in $[\mathbf{2}](\neg p, \neg[\mathbf{2}](r, \neg\langle\mathbf{1}\rangle q))$, while q is inessential there.

Definition 22 *A set of essentially box-formulae is:* ***independent*** *if no head of a formula from the set occurs as an inessential variable in any headed box from the set;* ***separated*** *if all headed boxes in the set have different heads;* ***strongly independent*** *if it is independent and separated.*

A headed box $B = [\beta](p, N_1, \ldots, N_m)$ *such that none of* $N_1, \ldots, N_m$ *contains the head* p *(i.e.* $\{B\}$ *is independent) is an* ***essentially positive box of the variable*** p.

For instance:

- the set $\{[\mathbf{2}](\neg[\mathbf{1}]p, q),\ [\mathbf{2}](q, [\mathbf{2}](\neg\langle\mathbf{1}\rangle p, \neg r)),\ [\mathbf{1}][\mathbf{1}]\neg q\}$ is independent but not strongly independent;
- the sets $\{[\mathbf{2}](\neg[\mathbf{1}]q, q)\}$ and $\{[\mathbf{2}](\neg[\mathbf{1}]p, q),\ [\mathbf{2}](p, [\mathbf{2}](\neg\langle\mathbf{1}\rangle r, \neg r))\}$ are not independent but separated;
- the set $\{[\mathbf{2}](\neg[\mathbf{1}]p, q_1),\ [\mathbf{2}](q_2, [\mathbf{2}](\neg\langle\mathbf{1}\rangle p, \neg r)),\ [\mathbf{1}][\mathbf{1}]\neg q_1\}$ is strongly independent;
- the formula $[\mathbf{2}](q, [\mathbf{2}](\neg\langle\mathbf{1}\rangle p, \neg r))$ is an essentially positive box of the variable q, and so is every boxed atom of q.

Definition 23 *A regular formula* $A = [\alpha](\neg B_1, \ldots, \neg B_n)$ *such that the set of essentially box-formulae* $\{B_1, \ldots, B_n\}$ *is independent is a* ***simple regular formula****. In particular, every headed box from* $\{B_1, \ldots, B_n\}$ *is an essentially positive box of its head.*

The class of simple regular formulae will be denoted by ***SRF****.*

For instance, the formula $[\mathbf{3}](\neg[\mathbf{2}](\neg[\mathbf{1}]p, q),\ \neg[\mathbf{2}](q, [\mathbf{2}](\neg\langle\mathbf{1}\rangle p, \neg r)),\ \neg[\mathbf{1}][\mathbf{1}]\neg q)$ is a simple regular formula.

We could also close the class of simple regular formulas under conjunctions, but for technical reasons we prefer to keep it as is.

A lean simple regular formula has the form $A = [\alpha](\neg H_1, \ldots, \neg H_n, P_1, \ldots, P_k)$ where $H_1, \ldots, H_n$ are headed boxes, each containing only its head as a variable, and $P_1, \ldots, P_k$ are positive formulae. After composing constant arguments with the leading modal terms of the headed boxes, it turns into $A = [\alpha](\neg[\beta_1]p_1, \ldots, \neg[\beta_n]p_n, P_1, \ldots, P_k)$. We will show further that, up to frame equivalence, the variables can be assumed different. The following definition considers the particular case when all $\beta_1, \dots, \beta_n$ are just ι_1.

Definition 24 *A lean simple regular formula A in which all headed boxes are just (different) variables will be called a* ***primitive regular formula.***

The primitive regular formulae generalize the "very simple" Sahlqvist formulae in [4].

3.2 Simple regular formulae subsume all polyadic Sahlqvist formulae

Lemma 25 *Every definite Sahlqvist antecedent A is equivalent to a negation of a simple regular formula in which all headed boxes are boxed atoms.*

Proof. Induction on A:

- The cases of A constant, boxed atom or a negative formula are trivial, (note that every constant formula is equivalent to a negation of a positive formula);
- $A = A_1 \wedge A_2$, where $A_1 \equiv \neg B_1$ and $A_2 \equiv \neg B_2$ for some simple regular formulas B_1 and B_2. Then $A \equiv \neg[\iota_2](B_1, B_2)$. After composing the leading terms of B_1 and B_2 with ι_2, this becomes a simple regular formula, since all headed boxes in it are boxed atoms, hence still form an independent set.
- $A = \langle\alpha\rangle(A_1, \dots, A_n)$. Then $A \equiv \neg[\alpha](\neg A_1, \dots, \neg A_n)$ where each $\neg A_1$ is equivalent to a simple regular formula in which all headed boxes are boxed atoms, hence so is $[\alpha](\neg A_1, \dots, \neg A_n)$.

■

Proposition 26 *Every definite SF from dRV is equivalent to a simple regular formula, and hence every SF from dRV is equivalent to a conjunction of simple regular formulas.*

Proof. From Proposition 4 and Lemma 25. Note that if $A \to C$ is a SF and $A \equiv \neg B$ for some simple regular formula B then $A \to C \equiv [\iota_2](B, C)$ is a simple regular formula, and also that applying disjunctions and polyadic boxes to SFs not sharing variables preserves the independence of the essential variables. ■

Actually, **SRF** *properly* extends **dRV**. A simple example is

$$[\mathbf{2}](\neg[\mathbf{2}](\bot, p), \langle\mathbf{2}\rangle(p, \top)),$$

where $\mathbf{2}$ is a binary modality. It defines the frame condition $\forall xyz(R_2xyz \rightarrow \exists uvw(R_2yuv \wedge R_2zvw))$.

3.3 A simple syntactic extension of the classical monadic Sahlqvist formulae

The class of monadic Sahlqvist formulae can be simply extended if the notion of a *box* is generalized by allowing, besides composition of box-modalities from the language, also *tests* (in PDL-style).

Definition 27 *Let $\mathcal{L}$ be a monadic (multi-)modal language and # be a symbol not belonging to $\mathcal{L}$. Then a **box-form of** # in $\mathcal{L}$ is defined recursively as follows:*

- *# is a box-form of #;*
- *If $\mathbf{B}(\#)$ is a box-form of # and $\Box$ is a box-modality in $\mathcal{L}$ then $\Box\mathbf{B}(\#)$ is a box-form of #*;
- *If $\mathbf{B}(\#)$ is a box-form of # and A is a positive formula in $\mathcal{L}$ then $A \rightarrow \mathbf{B}(\#)$ is a box-form of #*;

Thus, box-forms of # are, up to tautological equivalence, of the type

$$\Box_1(A_1 \rightarrow \Box_2(A_2 \rightarrow \cdots \Box_n(A_n \rightarrow \#)\cdots),$$

where $\Box_1, \ldots, \Box_n$ are (compositions of) box-modalities in $\mathcal{L}$, and $A_1, \ldots, A_n$ are positive formulae in $\mathcal{L}$.

Definition 28 *Given a monadic (multi-)modal language $\mathcal{L}$ and a variable p in $\mathcal{L}$, a **box-formula of** p is the result $\mathbf{B}(p)$ of substitution of p for # in any box-form $\mathbf{B}(\#)$ in $\mathcal{L}$.*

Note that every box-formula $\Box_1(A_1 \rightarrow \Box_2(A_2 \rightarrow \cdots \Box_n(A_n \rightarrow p)\cdots)$ is equivalent to $\Box_1(\neg A_1 \vee \Box_2(\neg A_2 \vee \ldots \Box_n(\neg A_n \vee p)\cdots)$ which can be represented as a headed box $[\alpha](\neg A_1, \neg A_2, \ldots, \neg A_n, p)$ with a head p and all other variables inessential there.

Definition 29 ***Simply generalized monadic Sahlqvist formulae*** *are defined by replacing in the definition of classical monadic modal Sahlqvist formulae* 'boxed atoms' *by* 'box-formulae', *and further requiring that the set of all these box-formulae occurring in the construction of the formula, is independent.*

For instance, $\Diamond(\Box(\Box\Diamond q \rightarrow \Box\Box p_1) \wedge \Box\Box(\Diamond\Box q \rightarrow \Box(\Diamond q \rightarrow p_2))) \rightarrow \Diamond(p_1 \wedge \Box(\Diamond p_2 \vee q))$ is not a Sahlqvist formula, but a simply generalized one.

The proof of Lemma 25 and Proposition 26 can now be modified accordingly to obtain:

Proposition 30 *Every simply generalized monadic Sahlqvist formula is equivalent to a conjunction of simple regular formulas.*

It should be noted that this extension of monadic Sahlqvist formulae is only syntactic, because all inessential variables in the box-formulae have only positive occurrences in the simply generalized Sahlqvist formula, and hence can be eliminated by replacement with $\perp$, thus producing an ordinary Sahlqvist formula.

3.4 Inductive polyadic formulae

Let $A = [\alpha](\neg H_1, \ldots, \neg H_n, P_1, \ldots, P_k)$ be a regular formula, where $H_1, \ldots, H_n$ are headed boxes with (not necessarily different) heads respectively $p_1, \ldots, p_n$, and $P_1, \ldots, P_k$ are positive formulae (hence, equivalent to negated headless boxes). In general, such formula need not have the virtues of a Sahlqvist formula. For instance, de Rijke has shown in [37] that $[\iota_2](\neg[\mathbf{2}]([\mathbf{2}](p,p),p), \langle\mathbf{2}\rangle(\langle\mathbf{2}\rangle(p,p),p))$ is not FO definable. An even simpler example is $[\iota_2](\neg[\mathbf{2}](p,p), \langle\mathbf{2}\rangle(p,p))$ which defines the non-elementary frame condition " *For every x the binary relation R_x on the remaining two variables y and z has an unoriented cycle of odd length.*"

Definition 31 *Given a regular formula $A = [\alpha](\neg H_1, \ldots, \neg H_n, P_1, \ldots, P_k)$, the **dependency digraph of** A is a digraph $G = \langle \mathbf{V}_A, \mathbf{E}_A \rangle$ where $\mathbf{V}_A = \{p_1, \ldots, p_n\}$ is the set of heads in A, and $p_i \mathbf{E}_A p_j$ iff p_i occurs as an inessential variable in a formula from $\{H_1, \ldots, H_n\}$ with a head p_j. A digraph is called **acyclic** if it does not contain oriented cycles.*

Definition 32 ***An inductive (polyadic) formula*** *is any regular formula A with an acyclic dependency digraph.*

In particular, in any inductive formula $[\alpha](\neg H_1, \ldots, \neg H_n, P_1, \ldots, P_k)$ all headed boxes $H_1, \ldots, H_n$ are essentially positive boxes of their respective heads.

The class of inductive formulae will be denoted by $\mathbf{I}$. Note that the particular case when there are no arcs in the dependency digraph corresponds to the class **SRF**, so every simple regular formula is inductive.

Example 33 *The formula $[\mathbf{3}](\neg[\mathbf{1}]p, \neg[\mathbf{2}](\neg p, q), \langle\mathbf{1}\rangle[\mathbf{1}]q)$ is an inductive formula but not a simple regular formula.*

The class **I** can be further closed under conjunctions, and then it extends essentially the original class of monadic Sahlqvist formulae. On syntactic level, this can be easily seen from the following example.

Definition 34 ***Generalized monadic Sahlqvist formulae*** *are defined by replacing in the definition of classical monadic Sahlqvist formulae* 'boxed atoms' *by* 'boxed formulae', *and further requiring that the set of all such boxed formulae occurring in the antecedent has an acyclic dependency digraph.*

Generalized monadic Sahlqvist formulae are essentially the restriction of **I** to the monadic (multi)-modal language.

Example 35

(1) The formula

$$D_1 = p \wedge \Box(\Diamond p \to \Box q) \to \Diamond\Box q$$

is not a Sahlqvist formula, nor it is tautologically reducible to one. Furthermore, its local FO correspondent:

$$FO(D_1) = \exists y(Rxy \wedge \forall z(Ryz \to \exists u(Rxu \wedge Rux \wedge Ruz)))$$

is not (syntactically) a Kracht formula (see [33] and [4], Sect. 3.7). On the other hand, written in a purely modal polyadic language, D_1 becomes a generalized monadic Sahlqvist formula, but not a simply generalized one:

$$D_1 = [\iota_3](\neg p, \neg[\alpha(\iota_2(\alpha, \alpha))](\neg p, q), \langle\alpha\rangle[\alpha]q),$$

where α is the modal term corresponding to $\Box$. However, it is not difficult to check that this formula is frame-equivalent (and hence axiomatically equivalent) to the Sahlqvist formula $p \to \Diamond(\Diamond p \vee \Box\bot)$.

(2) The formula

$$D_2 = p \wedge \Box(\Diamond p \to \Box q) \to \Diamond\Box\Box q$$

written in a purely modal polyadic language, is again a generalized monadic Sahlqvist formula:

$$D_2 = [\iota_3](\neg p, \neg[\alpha(\iota_2(\alpha, \alpha))](\neg p, q), \langle\alpha\rangle[\alpha][\alpha]q).$$

Its local FO correspondent:

$$FO(D_2) = \exists y(Rxy \wedge \forall z(R^2yz \to \exists u(Rxu \wedge Rux \wedge Ruz)))$$

is not a Kracht formula either, and moreover, as we will prove in Section 7, D_2 is not *frame-equivalent to any Sahlqvist formula in the basic modal* *language. Still, this formula is frame-equivalent to a Sahlqvist formula in the basic* temporal *language:*

$$D_2^t = p \to FGGP(Fp \wedge Pp).$$

Further we will prove that every inductive formula is locally first-order definable and canonical, thereby extending the Sahlqvist theorem in all previously proved versions.

3.5 Equivalences, pre-processing, and reducibility to inductive formulae

The syntactic definition of the class of inductive formulae, just like that of the Sahlqvist formulae, is rather rigid and sensitive to even very innocuous (e.g. tautological) transformations. For instance, the class **I** misses some quite simple cases of locally first-order definable and canonical formulae, e.g. all those of the type $[\alpha](p \to q) \wedge [\beta](q \to p) \to POS(p, q)$ where α, β are arbitrary unary modal terms and $POS(p, q)$ is any positive formula of p and q. While the dependency graph of such formula contains a cycle $\{p, q\}$, the formula is easily seen to be locally equivalent to the constant formula $POS(\bot, \bot)$.

It is natural, therefore, to attempt extending that class by closing under a suitable equivalence which preserves the important semantic properties of the formulae from **I**, while breaking their syntactic shape. For instance, such is the tautological equivalence, which for a purely modal language $\mathcal{L}_\tau$ should be understood as follows: the formulae from $\mathcal{L}_\tau$ are translated to the traditional polyadic language, i.e. all disjunctions and conjunctions are treated as logical connectives rather than boxes or diamonds, and then tautological equivalence is defined as usual. Moreover, it is decidable whether a modal formula is semantically equivalent to an inductive formula, and therefore the closure of **I** under semantic equivalence produces an even larger decidable class of elementary canonical formulae.

On the other hand, the undecidability of *axiomatic* equivalence to an inductive formula follows by an easy adaptation of a similar result of Chagrov and Zakharyaschev (see [7]) for Sahlqvist formulae. Therefore, the largest decidable extensions by equivalence of **I** lie between semantic and axiomatic equivalences. This issue is explored in more detail in [12] and [13].

Another, related approach to effective extension of the class **I** is by way of a syntactic *pre-processing*, i.e. systematic syntactic transformations of modal formulae to locally equivalent inductive formulae. For instance in [45] a large class of so called *complex Sahlqvist formulae* is introduced and shown to be effectively reducible to inductive formulae by way of non-trivial substitutions, preserving the formula up to local equivalence.

The question of syntactic reducibility to inductive formulae is studied in more detail in [12]. Since this issue is relatively unrelated to the rest of this paper, it will not be discussed here.

By further pre-processing, an inductive formula can, for instance, be made lean by eliminating all inessential variables. Since these only occur positively in the formula, they can be all replaced by $\perp$ and that substitution would preserve the formula up to local equivalence.

Also, the set of essentially positive boxes in an inductive formula can be made separated by means of successive *splittings* of a common head of two essentially positive boxes into two different variables, illustrated by the following example: $[\alpha](\neg[\beta_1]p, \neg[\beta_2]p, P(p, \overline{q}))$ is locally equivalent to $[\alpha](\neg[\beta_1]p_1, \neg[\beta_2]p_2, P(p_1 \vee p_2, \overline{q}))$, where p_1, p_2 are new variables. Thus we obtain the following.

Proposition 36 *Any inductive formula can be converted into a locally equivalent one in the following* standard *form:*

$$A = [\alpha](\neg H_1, \ldots, \neg H_n, P_1, \ldots, P_k),$$

where $\{H_1, \ldots, H_n\}$ is a separated set of essentially positive boxes and $P_1, \ldots, P_k$ are positive formulae.

Furthermore, if the inductive formula is simple regular, after elimination of the inessential variables every headed box H in it becomes a *unary* box over its head, so it can be eventually converted into the following standard form:

$$[\alpha](\neg[\beta_1]q_1, \ldots \neg[\beta_n]q_n, P_1, \ldots P_k)$$

where $\beta_1, \ldots \beta_n$ are unary modal terms, $q_1, \ldots, q_n$ are different propositional variables, and $P_1, \ldots P_k$ are positive formulae. Each of n and k above can be 0, and that standard form may become a constant formula. Note that all primitive regular formulae are in standard form.

Unlike simple regular formulas, not all essentially positive boxes in an inductive formula can be made unary boxes. Still, as we will realize later, these essentially positive boxes can be regarded as *parameterized* unary boxes.

We note that this pre-processing does not affect the (a)cyclicity of the dependency graph of the formula. Hereafter, whenever suitable, we can assume that any inductive formula has been pre-processed to one in a standard form.

Finally, we note that another algorithmic approach has been proposed in [10] where an algorithm, called SQEMA, has been developed, to identify elementary canonical formulae by systematic transformation to so called 'pure formulae' (see Section 10) in suitably extended languages, from which the local first-order equivalent can be readily obtained.

4 Local definability of inductive and regular formulae

In this section we prove the first-order definability part of Sahlqvist's theorem, extended to the class of inductive formulae by adapting and generalizing the method of *minimal valuations* of Sahlqvist - van Benthem (see [3], [4]).

We then show that all regular formulae have equivalents in the extension of first-order logic with least fixed points FO(LFP).

4.1 Local first-order definability of inductive formulae

Theorem 37 *Every inductive formula is locally first-order definable. Moreover, its local first-order equivalent can be computed effectively.*

Proof.

Let $A = [\alpha](\neg H_1, \ldots, \neg H_n, C_1, \ldots, C_k)$ be a pre-processed inductive formula, where $\{H_1, \ldots, H_n\}$ is a separated set of essentially positive boxes and $C_1, \ldots, C_k$ are positive formulae. Let $\overline{\mathbf{q}} = q_1, \ldots, q_n$ be the variables occurring in A, $\overline{\mathbf{Q}} = Q_1, \ldots, Q_n$ be the respective unary predicate variables, and $\overline{\mathbf{y}} = y_1, \ldots, y_{n+k}$ be a string of fresh different individual variables. By $\overline{\mathrm{ST}(A)}$ we denote the second-order closure of $\mathrm{ST}(A)$, which corresponds to validity at a point in a frame. Then

$\overline{\mathrm{ST}(A)} =$

$$\forall \overline{\mathbf{Q}} \forall \overline{\mathbf{y}} \left(R_\alpha x y_1 \ldots y_{n+k} \to (\bigvee_{i=1}^{n} \neg \mathrm{ST}(H_i)(y_i/x) \vee \bigvee_{i=1}^{k} \mathrm{ST}(C_i)(y_{n+i}) \right)$$

$$\equiv \forall \overline{\mathbf{Q}} \forall \overline{\mathbf{y}} \left(R_\alpha x y_1 \ldots y_{n+k} \wedge \bigwedge_{i=1}^{n} \mathrm{ST}(H_i)(y_i/x) \to POS(y_{n+1}, \ldots, y_{n+k}) \right)$$

for some positive first-order formula POS.

First, let us consider the particular case when A is a simple regular formula, so all essentially positive boxes are unary boxes: $H_j = [\beta_j](q_j)$ for some modal term β_j and essential variable q_j . Then

$$\mathrm{ST}(H_j)(y_j/x) \equiv \forall z_j \left(R_{\beta_j} y_j z_j \to Q_j(z_j) \right).$$

Note that, in any τ-frame $\mathbf{F} = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)} \right\rangle$, once the variable y_j is assigned a value u there, the formula above says that the set $R_{\beta_j}(y_j)$ of R_{β_j}-successors of u in $\mathbf{F}$ is included in the interpretation of Q_j in $\mathbf{F}$; in other words, $R_{\beta_j}(y_j)$ is the *minimal interpretation* of Q_j that satisfies that formula in $\mathbf{F}$ for the given value of y_j. With this in mind, we define the so called **minimal valuation** V_m of the propositional variable q_j uniformly in any given τ-frame $\mathbf{F}$ as follows:

$$V_m(q_j) = R_{\beta_j}(y_j).$$

With a slight (but harmless and justified) abuse of notation, in what follows we will treat $V_m(q_j)$ as a unary predicate in the first-order language for τ-frames, and allow ourselves the liberty to substitute it for Q_j in the formula $\mathrm{ST}(A)$. Furthermore, we will use $\mathbf{V}_m(\overline{\mathbf{q}})$ as an abbreviation for the tuple $V_m(q_1), \dots, V_m(q_n)$.

It is now easy to see that for any τ-frame $\mathbf{F}$ and $w \in \mathbf{F}$:

$$\mathbf{F}, w \models \forall \overline{\mathbf{Q}} \mathrm{ST}(A) \;\text{ iff }\; \mathbf{F}, w \models \mathrm{ST}(A)(\mathbf{V}_m(\overline{\mathbf{q}})/\overline{\mathbf{Q}}),$$

hence

$$\mathbf{F}, w \models A \;\text{ iff }\; \mathbf{F}, \mathbf{V}_m, w \models A.$$

Indeed, $\overline{\mathrm{ST}(A)} = \forall \overline{\mathbf{y}} \forall \overline{\mathbf{Q}}(ANT(\overline{\mathbf{Q}}) \to POS(\overline{\mathbf{Q}}))$ where $\overline{\mathbf{Q}}$ is the string of predicates corresponding to all essential variables and $ANT(\overline{\mathbf{Q}}) = R_\alpha x y_1 \dots y_{n+k} \wedge \bigwedge_{i=1}^n \mathrm{ST}(H_i)(y_i/x)$. Now, note again that, once the parameters $x, y_1, \dots, y_{n+k}$ are fixed so that $R_\alpha x y_1 \dots y_{n+k}$ holds, the valuation V_m is *the minimal one* (in set-theoretic sense) which makes each H_i, and hence $ANT(\overline{\mathbf{Q}})$, true. Therefore if $\mathbf{F}, \mathbf{V}_m, w \models A$, where w is the assigned value for x, then $POS(V_m(\overline{\mathbf{q}}))$ must be true in order for $\overline{\mathrm{ST}(A)}$ to hold at w. Now, take any valuation V. If it falsifies any H_i, then ANT is rendered false, so the whole formula is true. Otherwise, $V_m(q) \subseteq V(q)$ for every essential variable q. Then, by monotonicity of positive formulae, $POS(V_m(\overline{\mathbf{q}})) \to POS(V(\overline{\mathbf{q}}))$ is valid, hence $POS(V(\overline{\mathbf{q}}))$ is true, so the formula turns out true again.

Thus, A defines the following local first-order condition on frames, equivalent to $\mathrm{ST}(A)(\mathbf{V}_m(\overline{\mathbf{q}})/\overline{\mathbf{Q}})(w/x)$:

$$FO(A, x) = \forall \overline{\mathbf{y}} \left(R_\alpha x y_1 \dots y_{n+k} \to POS(\mathbf{V}_m(\overline{\mathbf{q}})/\overline{\mathbf{Q}}) \right)$$

such that

$$\mathbf{F}, w \models A \text{ iff } \mathbf{F} \Vdash FO(A, x)(w/x),$$

where $\Vdash$ denotes first-order truth.

Now, the proof for the general case of an inductive formula A essentially repeats in several steps the one above. The key concern again is to define the right minimal valuation. Let G_A be the dependency digraph of A. First, note that since G_A does not contain cycles, it defines a *strict partial ordering* $\prec$ between the vertices: $q_i \prec q_j$ iff there is an arc path leading from q_i to q_j. Consider any linear extension of that partial ordering: $q_1 \prec \cdots \prec q_n$. Following that ordering, a minimal valuation can be defined on the set of essential variables inductively as follows.

Suppose all $\prec$-predecessors (if any) of an essential variable q have already been assigned values. Let the string of these predecessors be $\overline{\mathbf{q}}_q$ and let the string of second-order variables corresponding to them be $\overline{\mathbf{Q}}_q$.

Take any essentially positive box H_j with a head q_j:

$$H_j = [\beta_j](q_j, \neg P_{j1}(q_1, \ldots, q_{j-1}), \ldots, \neg P_{jn_j}(q_1, \ldots, q_{j-1}))$$

where $P_{j1}, \ldots, P_{jn_j}$ are positive, for $j = 1, \ldots n$.

Then:

$$\mathrm{ST}(H_j)(y_j/x) \equiv \forall z_j \forall \overline{\mathbf{u}_j} \left(R_{\beta_j} y_j z_j u_{j1} \ldots u_{jn_j} \wedge \textstyle\bigwedge_{i=1}^{n_j} \mathrm{ST}(P_{ji})(u_{ji}/x) \rightarrow Q_j(z_j) \right).$$

Note that all predicate variables Q_k occurring in any $\mathrm{ST}(P_{ji})(u_{ji}/x)$ above correspond to predecessors of q_j, so they are amongst $\overline{\mathbf{Q}}_{q_j}$ and hence they have already been assigned their minimal values.

Then we put

$$V_m(q_j) = \left\{ z \,\middle|\, \exists \overline{\mathbf{u}_j} (R_{\beta_j} y_j z u_{j1} \ldots u_{jn_j} \wedge \bigwedge_{i=1}^{n_j} \mathrm{ST}(P_{ji})(V_m(\overline{\mathbf{q}}_j)/\overline{\mathbf{Q}}_{q_j})(u_{ji}))) \right\}.$$

In particular, if $\overline{\mathbf{q}}_{q_j}$ is empty, i.e. q_j is $\prec$-minimal, then $V_m(q_j)$ is defined as before.

Now, an inductive argument on $\prec$ proves that V_m has indeed the properties of the minimal valuation needed to prove first-order definability of A as in the case of a simple regular formula. ■

Example 38 *Let us compute the local first-order equivalent for the inductive formula from example 33:*

$$D_3 = [\mathbf{3}](\neg[\mathbf{1}]p, \neg[\mathbf{2}](\neg p, q), \langle\mathbf{1}\rangle[\mathbf{1}]q).$$

Since $p \prec q$, we first compute $V_m(p) = R_1(y_1)$.

Then $V_m(q) = \{z | \exists s(R_2 y_2 s z \wedge R_1 y_1 s)$. Thus, $FO(B)(x) = \forall y_1 y_2 y_3 (R_{\mathbf{3}} x y_1 y_2 y_3 \rightarrow \exists v(R_{\mathbf{1}} y_3 v \wedge \forall w(R_{\mathbf{1}} v w \rightarrow \exists s(R_{\mathbf{2}} y_2 s w \wedge R_{\mathbf{1}} y_1 s))))$.

Remark 39 *Note that in the latter example above, once $V_m(p)$ is determined, then $[\mathbf{2}](\neg p, q)$ can be regarded as a unary box $[\alpha^1(V_m(p))](q)$ where $\alpha^1(V_m(p))$ is a unary parameterized modal term, the relation of which can be accordingly computed: $R_{\alpha^1} xy$ iff $\exists s(R_2 x s y \wedge V_m(p)(s))$. This trick will be essential in the proof of canonicity of inductive formulas.*

4.2 Definability of regular formulae in FOL with least fixpoints

Here we extend the definabilty result for **I** to the class of all regular formulae, by further extending the minimal valuations technique. The minimal valuations now are recursively defined and eventually expressed in a first-order logic with least fixpoints FO_μ. For background on FO_μ see e.g. [15] or [1].

Proposition 40 *Every regular formula has a local correspondent in the first-order logic extended with fixpoint operators FO_μ.*

Proof. Let $A = [\alpha](\neg H_1, \ldots, \neg H_n, C_1, \ldots, C_k)$ be a regular formula, where $\{H_1, \ldots, H_n\}$ is a set of essentially positive boxes and $C_1, \ldots, C_k$ are negations of headless essentially box-formulae, i.e. positive formulae. The only guaranteed effect of pre-processing here is that all inessential variables can be eliminated, i.e. we can assume that A is lean.

We will begin as in the procedure for computing the first-order equivalent of an inductive formula. Let $\overline{\mathbf{q}} = q_1, \ldots, q_n$ be the essential variables occurring in A (not necessarily different), $\overline{\mathbf{Q}} = Q_1, \ldots, Q_n$ be the respective unary predicate variables, and $\overline{\mathbf{y}} = y_1, \ldots, y_{n+k}$ be a string of fresh different individual variables. By $\overline{\mathrm{ST}(A)}$ we denote the universal second-order closure of $\mathrm{ST}(A)$, which corresponds to validity at a point in a frame. Then

$\overline{\mathrm{ST}(A)} =$

$$\forall\overline{\mathbf{Q}}\forall\overline{\mathbf{y}}\left(R_\alpha x y_1 \ldots y_{n+k} \rightarrow (\bigvee_{i=1}^{n} \neg\mathrm{ST}(H_i)(y_i/x) \vee \bigvee_{i=1}^{k} \mathrm{ST}(C_i)(y_{n+i})\right)$$

$\equiv \forall \overline{\mathbf{Q}} \forall \overline{\mathbf{y}} \left((R_\alpha x y_1 \dots y_{n+k} \wedge \bigwedge_{i=1}^n \mathrm{ST}(H_i)(y_i/x)) \rightarrow POS(y_{n+1}, \dots, y_{n+k}) \right)$

for some positive formula POS.

Now, instead of computing the minimal valuations of $q_1, \dots, q_n$ step-by-step explicitly, as in the case of inductive formulas, we write for them a system of *recursive equations* of the type:

$$\begin{cases} \Phi_1(Q_1, \dots, Q_n) \subseteq Q_1, \\ \qquad \dots \\ \Phi_n(Q_1, \dots, Q_n) \subseteq Q_n, \end{cases}$$

where $\Phi_1, \dots, \Phi_n$ are monotone operators, uniformly composed from the headed essentially box-formulae as follows. Let

$$H_j = [\beta_j](q_j, \neg P_{j1}(q_1, \dots, q_n), \dots, \neg P_{jn_j}(q_1, \dots, q_n))$$

where $P_{j1}, \dots, P_{jn_j}$ are positive, for $j = 1, \dots n$. Then:

$$\mathrm{ST}(H_j)(y_j/x) \equiv \forall z_j \left(\exists \overline{\mathbf{u}_j} (R_{\beta_j} y_j z_j u_{j1} \dots u_{jn_j} \wedge \bigwedge_{i=1}^{n_j} \mathrm{ST}(P_{ji})(u_{ji}/x)) \rightarrow Q_j(z_j) \right)$$

and we define

$$\Phi_j(Q_1, \dots, Q_n) = \left\{ z_j \mid \exists \overline{\mathbf{u}_j} \left(R_{\beta_j} y_j z_j u_{j1} \dots u_{jn_j} \wedge \bigwedge_{i=1}^{n_j} \mathrm{ST}(P_{ji})(u_{ji}/x) \right) \right\}.$$

Note that Φ_j is monotone in each $Q_1, \dots, Q_n$ since all P_{ji}'s are positive.

The recursive system above has a least pre-fixed point solution (see [1]) which is also a least fixed point:

$$\mathbf{V}_m(q_1) = \mu Q_1.\Phi_1(Q_1, \dots, Q_n), \dots, \mathbf{V}_m(q_n) = \mu Q_n.\Phi_n(Q_1, \dots, Q_n).$$

Now, the local equivalent in FO_μ of A is (as before):

$$\mathrm{FO}_\mu(\mathrm{A}, \mathrm{x}) = \forall \overline{\mathbf{y}} \left(\mathrm{R}_\alpha \mathrm{x} \mathrm{y}_1 \dots \mathrm{y}_{\mathrm{n+k}} \rightarrow \mathrm{POS}(\mathbf{V}_\mathrm{m}(\overline{\mathbf{q}}) / \overline{\mathbf{Q}}) \right)$$

■

We will illustrate the procedure described above with two well-known examples of non-elementary formulae.

Example 41 *Gödel-Löb formula: $GL = [1]([1]q \rightarrow q) \rightarrow [1]q$.*

(1) Pre processing into regular formula:

$\neg[1](\neg[1]q \vee q) \vee [1]q \equiv [\iota_2](\neg[\alpha](\neg[1]q, q), [1]q)$
where $\alpha = 1 \circ \iota_2$, resp. $xR_\alpha yz$ iff $xR_1 y \wedge y = z$.

(2) Composing the recursive equation(s) for the minimal valuation $\mathbf{V}_m(q)$ for q. The only condition for Q is: $\forall y \forall z(xR_\alpha yz \wedge R(y) \subseteq Q \to Q(z))$, i.e. $\forall y(xR_1 y \wedge R_1(y) \subseteq Q \to Q(y))$.

This can be written as

$$\Phi(Q) \subseteq Q,$$

where

$$\Phi(Q) = \{y \mid xR_1 y \wedge R_1(y) \subseteq Q\}.$$

Note that Φ is a monotone operator, depending on the parameter x. Since $\mathbf{V}_m(q)$ is to be the *minimal* valuation satisfying the equation above, it must be the *least* (pre-)fixed point of Φ. Thus, $\mathbf{V}_m(q) = \mu Q.\Phi(Q)$.

(3) Computing $\mu X.\Phi(X)$:
$\Phi_0 = \varnothing$;
$\Phi_1 = \Phi(\varnothing) = \{y \mid xR_1 y \wedge R_1(y) = \varnothing\}$;
$\Phi_2 = \Phi(\Phi_1) = \{y \mid xR_1 y \wedge R(y) \subseteq \Phi_1\}$
$= \{y \mid xR_1 y \wedge \forall y_1(yR_1 y_1 \to xR_1 y_1 \wedge R_1(y_1) = \varnothing)\}$;
$\Phi_3 = \Phi(\Phi_2) = \{y \mid xR_1 y \wedge \forall y_1 \forall y_2(yR_1 y_1 \to$
$xR_1 y_1 \wedge (y_1 R_1 y_2 \to xR_1 y_2 \wedge R_1(y_2) = \varnothing))\}$;
$\ldots$
$\Phi_{n+1}(X) = \{y \mid xR_1 y \wedge \forall y_1 \ldots \forall y_n(yR_1 y_1 \to xR_1 y_1 \wedge \ldots$
$(y_{n-1} R_1 y_n \to xR_1 y_n \wedge R_1(y_n) = \varnothing) \ldots)\}$,
from which $\mu X.\Phi(X)$ becomes evident.

(4) Finally, computing the FO_μ-equivalent:
$\mathrm{FO}_\mu(GL, x) = \forall u(xR_1 u \to \mu X.\Phi(X)(u)) =$
$\forall u \exists n \geq 0 \forall y_1 \ldots \forall y_n(xR_1 u \wedge (uR_1 y_1 \to xR_1 y_1 \wedge (\ldots(y_{n-1} R_1 y_n \to xR_1 y_n \wedge R_1(y_n) = \varnothing) \ldots))$.

It is now easy to check that $\forall x GL(x)$ is equivalent to transitivity of R_1 and non-existence of infinite R_1-chains.

Example 42 *Segerberg's induction axiom*

$$IND = [2](q \to [1]q) \to (q \to [2]q).^5$$

(1) Pre processing to regular formula
$\neg[2](\neg q \vee [1]q) \vee \neg q \vee [2]q \equiv [\iota_3](\neg[\alpha](\neg q, q), \neg q, [2]q)$
where $\alpha = 2(\iota_2(\iota_1, 1))$, resp. $xR_\alpha yz$ iff $xR_2 y \wedge yR_1 z$.

(2) Composing the recursive equation(s) for $\mathbf{V}_m(q)$. The conditions for Q are:
(a) $\forall y \forall z(xR_\alpha yz \wedge Q(y) \to Q(z))$, i.e. $\forall z(\exists y(xR_\alpha yz \wedge Q(y)) \to Q(z))$, and
(b) $Q(x)$.

These can be written as:

$$\Phi_1(Q) \subseteq Q,$$
$$\Phi_2(Q) \subseteq Q$$

$\Phi(Q) \subseteq Q$ where

$$\Phi_1(Q) = \{z | \exists y(xR_\alpha yz \wedge Q(y))\},$$
$$\Phi_2(Q) = \{z | z = x\}.$$

Both equations refer to the same variable, so they can be put together as

$$\Phi(Q) \subseteq Q,$$

where

$$\begin{aligned}\Phi(Q) &= \{z | z = x \vee \exists y(xR_\alpha yz \wedge Q(y))\} \\ &= \{z | z = x \vee \exists y(xR_2 y \wedge yR_1 z \wedge Q(y))\}.\end{aligned}$$

Φ is a monotone operator, depending on the parameters x, z. Again, Q must be the least pre-fixed point of Φ. Thus, $Q = \mu X.\Phi(X)$

(3) Computing $\mu X.\Phi(X)$: $\Phi_0 = \varnothing$, $\Phi_1 = \Phi(\varnothing) = \{x\}$, $\Phi_2 = \Phi(\Phi_1) = \{x\} \cup \{z | xR_2 x \wedge xR_1 z\}$. If $\neg xR_2 x$, this is the fixpoint: $\mu X.\Phi(X) = \{x\}$, otherwise the unfolding continues:
$\Phi_3 = \Phi(\Phi_2)$
$= \{x\} \cup \{z | xR_2 x \wedge xR_1 z\} \cup \{z | xR_2 x \wedge \exists y_1(xR_2 y_1 \wedge xR_1 y_1 \wedge y_1 R_1 z)\}$,
$\ldots$
$\Phi_{n+2} = \Phi_{n+1} \cup \{z | xR_2 x \wedge \exists y_1 \ldots \exists y_n(xR_2 y_1 \wedge \ldots$
$\wedge xR_2 y_n \wedge xR_1 y_1 \wedge y_1 R_1 y_2 \wedge \ldots \wedge y_n R_1 z)\}$,
from which $\mu X.\Phi(X)$ is evident.

(4) Computing the FO_μ-equivalent:
$\mathrm{FO}_\mu(IND, x) = \forall u(xR_2 u \to \mu X.\Phi(X)(u)) = \forall u(xR_2 u \to xR_2 x \wedge (u = x \vee$
$\exists n \geq 0 \exists y_1 \ldots \exists y_n(xR_2 y_1 \wedge \ldots \wedge xR_2 y_n \wedge xR_1 y_1 \wedge y_1 R_1 y_2 \wedge \ldots \wedge y_n R_1 u)))$.

We note that an algorithm for computing FO_μ-equivalents of classical modal formulae, based on Ackermann's method for second-order quantifier elimination, and in particular covering the two examples above, has been developed in [36] and further extended in [43]. On the other hand, the algorithm SQEMA developed in [10] can be extended with a recursive version of Ackermann's rule to compute the FO_μ-equivalents of all regular formulae. For more details, see [11].

5 Polyadic descriptive frames and their topology

In this section we obtain results about descriptive frames for polyadic modal languages which will be used further. Most of these will be generalizations of known properties of monadic descriptive frames, but we will establish some important relations between them and will present them in a way suitable for purely modal languages.

Every general τ-frame $\mathfrak{F} = \left\langle W, \{R_\alpha\}_{\alpha\in MT(\tau)}, \mathbb{W}\right\rangle$ determines a **topological space** $T(\mathfrak{F})$ with a base of clopen sets $\mathbb{W}$. For detailed study of this topology, its properties and applications in modal logic, see [41] and for topological treatment of Sahlqvist formulae see [42].

Hereafter, a **closed** set in a general τ-frame $\mathfrak{F}= \left\langle W, \{R_\alpha\}_{\alpha\in MT(\tau)}, \mathbb{W}\right\rangle$ will mean a subset of the domain closed with respect to the above mentioned topology, i.e. an intersection of a family of admissible sets. Let $\mathbf{C}(\mathbb{W})$ be the set of all closed subsets of W in $T(\mathfrak{F})$.

5.1 Parametrized modal terms and formulae

We are now going to extend the set of modal terms and the respective class of operators to allow parametrization with closed sets and operators.

Definition 43 *Let $\mathfrak{F}= \left\langle W, \{R_\alpha\}_{\alpha\in MT(\tau)}, \mathbb{W}\right\rangle$ be a general τ-frame. We define the set $PMT(\tau,\mathfrak{F})$ of $\mathfrak{F}$-**parameterized modal terms** and their respective operators on $\mathcal{P}(W)$ by induction as follows:*

PMT1: *$MT(\tau) \subseteq PMT(\tau,\mathfrak{F})$;*
PMT2: *For every $(n+1)$-ary term $\beta \in PMT(\tau,\mathfrak{F})$ and a closed set Z in $T(\mathfrak{F})$, $\beta(Z)$ is an n-ary term in $PMT(\tau,\mathfrak{F})$ such that $\langle\beta(Z)\rangle(X_1,\ldots,X_n) = \langle\beta\rangle(X_1,\ldots,X_n,Z)$.*

Accordingly, we define $R_{\beta(Z)}x, x_1,\ldots,x_n$ iff there exists $x_{n+1} \in Z$ such that $R_\beta x x_1 \ldots x_n x_{n+1}$. Also, note that the dual of $\langle\beta(Z)\rangle$ is $[\beta(Z)]\,(X_1,\ldots,X_n) = [\beta]\,(X_1,\ldots,X_n,-Z)$.

We will further allow the parameter to be taken from any argument by putting

$\langle\beta^{(j)}(Z)\rangle(X_1,\ldots,X_n) = \langle\beta\rangle(X_1,\ldots,X_{j-1},Z,X_{j+1},\ldots,X_n)$ and respectively,

$[\beta^{(j)}(Z)](X_1,\ldots,X_n) = [\beta](X_1,\ldots,X_{j-1},-Z,X_{j+1},\ldots,X_n)$ for $j = 1,\ldots,n$.

Furthermore, the parameters can be represented by formulae, too.

Definition 44 *Given a general τ-frame $\mathfrak{F}$, a $\mathfrak{F}$-**parameterized formula** is a formula in the extended language built over the set of modal terms $PMT(\tau, \mathfrak{F})$.*

*An $\mathfrak{F}$-parameterized formula A is **positive in a variable** p if all occurrences of p in A are positive; A is **positive** if it is positive in every variable occurring in A.*

Definition 45 *Given a general τ-frame $\mathfrak{F}$, an $\mathfrak{F}$-parameterized formula $A(p_1, \ldots, p_n)$ is **closed in** $\mathfrak{F}$ if the operator $\lambda X_1 \ldots X_n.A(X_1, \ldots, X_n)$ in $\mathfrak{F}$ is closed, i.e. $A(X_1, \ldots, X_n)$ is closed whenever $X_1, \ldots, X_n$ are closed in $T(\mathfrak{F})$.*

5.2 Descriptive frames and closed operators in them

Definition 46 *Let $\mathfrak{F}= \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)}, \mathbb{W} \right\rangle$ be a general τ-frame and $\beta \in PMT(\tau, \mathfrak{F})$. The relation R_β is **tight in** $\mathfrak{F}$ if the following condition holds: for any $x, x_1, \ldots, x_n \in W$,*

$R_\beta x, x_1, \ldots, x_n$ iff

$\forall X_1, \ldots, X_n \in \mathbb{W}(x_1 \in X_1, \ldots, x_n \in X_n \Rightarrow x \in \langle\beta\rangle(X_1, \ldots, X_n))$.

Note that this condition is equivalent to: for every $x \in W$,

$R_\beta x, x_1, \ldots, x_n$ iff

$x \in \bigcap\{\langle\beta\rangle(X_1, \ldots, X_n) | X_1, \ldots, X_n \in \mathbb{W} \ \&\ x_1 \in X_1, \ldots, x_n \in X_n\}$.

In particular, every R_β for a 0-ary term β is tight.

Definition 47 *A family of sets $\mathcal{F}$ has the **finite intersection property (FIP)** if the intersection of every finite subfamily of $\mathcal{F}$ is non-empty.*

Definition 48 *A general τ-frame $\left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)}, \mathbb{W} \right\rangle$ is:*

- ***differentiated** if for every $x, y \in W$, if $x \neq y$ then there is $X \in \mathbb{W}$ such that $x \in X$ and $y \notin X$;*
- ***tight** if for every basic modal term β the relation R_β is tight in $\mathfrak{F}$;*
- ***discrete** if $\{w\} \in \mathbb{W}$ for every $w \in W$;*
- ***compact** if every family of admissible sets in $\mathfrak{F}$ with FIP has a non-empty intersection.*
- ***refined** if it is differentiated and tight.*
- ***descriptive** if it is refined and compact.*

We note that:

- The canonical general frame of every normal modal logic in any purely modal polyadic language without nominals or any special inference rules is descriptive.
- the property of being differentiated is expressed by the tightness of R_{ι_1}, and so it becomes redundant. We keep it in the definition mainly to respect the tradition.
- compactness of a general τ-frame $\mathfrak{F}$, as defined above, is equivalent to the standard topological notion of compactness of $T(\mathfrak{F})$, i.e. every family of *closed* sets with the FIP has a non-empty intersection.
- by (a weaker version of) Tychonov's theorem, if $\mathfrak{F}$ is compact then for every $n \in N$, the product space $(T(\mathfrak{F}))^n$ is compact, too.

Hereafter, closedness of Cartesian products of sets will mean closedness in the respective product topology.

It is immediate to see that for any compact and differentiated τ-frame $\mathfrak{F}$, the $T(\mathfrak{F})$ is a compact Hausdorff space with some additional properties, necessary to prove the canonicity of any inductive formula, which will be obtained in the rest of this section.

Proposition 49 *In every discrete frame $\mathfrak{F}$ the topology $T(\mathfrak{F})$ is discrete.*

Proof. Every non-empty set is a union of its singleton subsets, which are open in $T(\mathfrak{F})$, hence every subset of $\mathfrak{F}$ is open. ■

Furthermore, every discrete frame is refined, while the converse need not hold, e.g. canonical general frames are descriptive, but not discrete. In fact, no infinite descriptive frame is discrete.

Lemma 50 *In any differentiated τ-frame $\mathfrak{F}= \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)}, \mathbb{W}\right\rangle$, for any n -ary term $\beta \in PMT(\tau, \mathfrak{F})$, R_β is tight iff for every $x \in W$ the set $R_\beta(x) = \{(x_1, \ldots, x_n) | R_\beta x x_1 \ldots x_n\}$ is closed, i.e. R_β is* ***point-closed.***

Proof. For 0-ary modal terms β each of these conditions is trivially true, so we can assume that $\rho(\beta) > 0$.

First, note that
$\forall X_1, \ldots, X_n \in \mathbb{W}(x_1 \in X_1, \ldots, x_n \in X_n \Rightarrow x \in \langle\beta\rangle(X_1, \ldots, X_n))$ iff
$\forall X_1, \ldots, X_n \in \mathbb{W}(x \in [\beta](-X_1, \ldots, -X_n) \Rightarrow (x_1, \ldots, x_n) \in -(X_1 \times \cdots \times X_n))$.

Therefore, R_β is tight iff for every $x \in W$,
$R_\beta(x) = \bigcap\{-(X_1 \times \cdots \times X_n) | X_1, \ldots, X_n \in \mathbb{W} \;\&\; x \in [\beta](-X_1, \ldots, -X_n)\}$. ■

Definition 51 *A family $\mathcal{F}$ of subsets of a set X is called* ***downwards di-***

rected *if $\mathcal{F}$ contains a subset of the intersection of any two (and hence, of any finite number of) members of $\mathcal{F}$.*

Lemma 52 *If A is a closed set in a τ-frame $\left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)}, \mathbb{W} \right\rangle$ then there exists a downwards directed family $\{A_i : i \in I\}$ of clopen subsets (from $\mathbb{W}$) such that $A = \bigcap_{i \in I} A_i$.*

Proof. Since A is closed, $A = \bigcap_{i \in J} A_i$ for a family of clopen sets $\{A_i : i \in J\}$. Since any finite intersection of clopen sets is clopen, we can close that family under finite intersections. The resulting family $\{A_i : i \in I\}$ is now downwards directed and $A = \bigcap_{i \in I} A_i$. ■

Lemma 53 *If $\mathfrak{F} = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)}, \mathbb{W} \right\rangle$ is a differentiated and compact general τ-frame, then the following are equivalent for any n-ary term $\beta \in PMT(\tau, \mathfrak{F})$:*

(i) R_β is tight.

(ii) (Esakia's lemma) For any downwards directed family $\{X_{1i} \times \cdots \times X_{ni}\}_{i \in I}$ of closed subsets of W^n,

$$\bigcap_{i \in I} \{\langle \beta \rangle (X_{1i}, \dots, X_{ni})\} = \langle \beta \rangle (\bigcap_{i \in I} X_{1i}, \dots, \bigcap_{i \in I} X_{ni}).$$

(iii) For every $x \in W$ the set $R_\beta(x)$ is closed.

Proof. Again, the non-trivial case is $\rho(\beta) > 0$.

(i) $\Rightarrow$ (iii): Lemma 50.

(iii) $\Rightarrow$ (ii): The inclusion $\supseteq$ follows from the monotonicity of $\langle \beta \rangle$.

For the converse inclusion, let $x \in \bigcap_{i \in I} \{\langle \beta \rangle (X_{1i}, \dots, X_{ni})\}$. Then, due to the downwards directedness, the family of closed sets

$$R_\beta(x) \cup \{X_{1i} \times \cdots \times X_{ni}\}_{i \in I}$$

has the FIP, so it has a non-empty intersection, i.e. there is a tuple $(x_1, \dots, x_n)$ such that $R_\beta x x_1 \dots x_n$ and $(x_1, \dots, x_n) \in \bigcap_{i \in I} \{(X_{1i} \times \cdots \times X_{ni}\} = \bigcap_{i \in I} X_{1i} \times \cdots \times \bigcap_{i \in I} X_{ni}$.

Therefore, $x \in \langle \beta \rangle (\bigcap_{i \in I} X_{1i}, \dots, \bigcap_{i \in I} X_{ni})$.

(ii) $\Rightarrow$ (i): The implication from left to right in the tightness condition for R_β holds by definition. For the converse, it suffices to note that by (ii):
$\bigcap \{\langle \beta \rangle (X_1, \dots, X_n) | X_1, \dots, X_n \in \mathbb{W} \ \wedge \ x_1 \in X_1, \dots, x_n \in X_n\}$
$= \langle \beta \rangle (\{x_1\}, \dots, \{x_n\})$. ■

Lemma 54 *In every descriptive τ-frame $\mathfrak{F}$, each of the conditions of Lemma 53 holds for every term $\beta \in PMT(\tau, \mathfrak{F})$.*

Proof. We prove by induction on β that Esakia's lemma holds for every β. For the basic terms tightness holds by definition, and hence the claim holds by Lemma 53. The inductive step for $\beta = \alpha(\alpha_0, \dots, \alpha_m)$ is quite straightforward. Finally, suppose the claim holds for some $(n+1)$-ary term $\beta \in PMT(\tau, \mathfrak{F})$ and let Z be a closed set in $T(\mathfrak{F})$. Then

$\langle \beta(Z) \rangle (\bigcap_{i \in I} X_{1i}, \dots, \bigcap_{i \in I} X_{ni})$

$= \langle \beta \rangle (\bigcap_{i \in I} X_{1i}, \dots, \bigcap_{i \in I} X_{ni}, \bigcap_{i \in I} Z)$

$= \bigcap_{i \in I} \langle \beta \rangle (X_{1i}, \dots, X_{ni}, Z)$

$= \bigcap_{i \in I} \langle \beta(Z) \rangle (X_{1i}, \dots, X_{ni})$. ■

Lemma 55 *In any descriptive τ-frame $\mathfrak{F}$, for every positive $\mathfrak{F}$-parameterized formula $A(p_1, \dots, p_n)$ the corresponding operator in $\mathfrak{F}$ $\lambda X_1 \dots X_n . A(X_1, \dots, X_n)$ satisfies Esakia's lemma: for any downwards directed family of closed sets $\{X_{1i} \times \dots \times X_{ni}\}_{i \in I}$,*

$$\bigcap_{i \in I} \{A(X_{1i}, \dots, X_{ni})\} = A(\bigcap_{i \in I} X_{1i}, \dots, \bigcap_{i \in I} X_{ni}).$$

In particular, every positive $\mathfrak{F}$-parameterized formula $A(p_1, \dots, p_n)$ is closed in $\mathfrak{F}$.

Proof. First, note that every positive $\mathfrak{F}$-parameterized formula A is equivalent to an $\mathfrak{F}$-parameterized formula built from propositional variables and modal constants by applying only polyadic boxes and polyadic diamonds with terms from $PMT(\tau, \mathfrak{F})$.

We shall prove the statement by induction on A, assuming it is constructed as above. For propositional variables and modal constants the claim is trivial. For $\langle \beta \rangle (X_1, \dots, X_n)$ the inductive step is the Esakia's lemma and follows from Lemma 54.

Finally, for $[\beta](X_1, \dots, X_n)$ the inductive step follows from the identity

$$[\beta](\bigcap_{i \in I} X_{1i}, \dots, \bigcap_{i \in I} X_{ni}) = \bigcap_{i_1 \in I, \dots, i_n \in I} [\beta](X_{1i_1}, \dots, X_{ni_n})$$

which easily follows from the definition $[\beta](X_1, \ldots, X_n)$, combined with the equality

$$\bigcap_{i_1 \in I, \ldots, i_n \in I} [\beta](X_{1i_1}, \ldots, X_{ni_n}) = \bigcap_{i \in I} [\beta](X_{1i}, \ldots, X_{ni})$$

which follows from the downward directedness. ■

6 Canonicity of the inductive formulae

6.1 Local d*-persistence via closed valuations*

Definition 56 *A formula $A \in \mathcal{L}_\tau$ is* ***locally d-persistent*** *if for every descriptive general τ-frame $\mathfrak{F} = \langle F, \mathbb{W} \rangle$, where $F = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)} \right\rangle$, and $w \in W$,*

$\mathfrak{F}, w \models A$ *iff* $F, w \models A$.

Theorem 57 *Every inductive formula is locally d-persistent.*

Proof. We will follow the scheme of the proof of canonicity of Sahlqvist formulae presented in [4], to which the reader is referred for those technical details which would not differ in the more general case presented here.

Again, as in the previous proof, let

$$A = [\alpha](\neg H_1, \ldots, \neg H_n, Q_1, \ldots, Q_k)$$

be a pre-processed inductive formula, where

$$H_j = [\beta_j](q_j, \neg P_{j1}(q_1, \ldots, q_{j-1}), \ldots, \neg P_{jn_j}(q_1, \ldots, q_{j-1}))$$

for $P_{j1}, \ldots, P_{jn_j}$ positive, $j = 1, \ldots n$. Let again the dependency digraph of A determine a partial order on the heads, extended to a linear ordering $\prec$.

Take any descriptive general frame $\mathfrak{F} = \langle \mathbf{F}, \mathbb{W} \rangle$ such that $\mathfrak{F} \models A$. As we showed in the proof of theorem 37, if $\mathbf{F}, V_m \models A$ for the minimal valuation V_m, defined as before, then $\mathbf{F}, V \models A$ for *any* valuation V, so it suffices to prove that $\mathbf{F}, V_m \models A$. The problem is that *the minimal valuation need not be admissible* in $\mathfrak{F}$. However, it will suffice to show the following:

(C1) V_m is closed i.e. an intersection of admissible valuations.

(C2) For every closed valuation U in $\mathfrak{F}$ and a positive formula P, $U(P) = \bigcap_{U\subseteq V} V(P)$ where the intersection ranges over all admissible valuations V which extend U.

For (C1), we can restrict our consideration to the variables occurring in A , i.e. the essential variables $q_1, \ldots, q_n$. We shall prove by $\prec$ -induction that every valuation

$$V_m(q_j) = \left\{ z \,\middle|\, \exists\overline{\mathbf{u}_j}(R_{\beta_j} y_j z u_{j1} \ldots u_{jn_j} \wedge \bigwedge_{i=1}^{n_j} \mathrm{ST}(P_{ji})(V_m(\overline{\mathbf{q}}_{q_j})/\overline{\mathbf{Q}}_{q_j})(u_{ji})) \right\}$$

is of the type $R_\beta(y_j)$ for some $\beta \in PMT(\tau,\ \mathfrak{F})$, and hence, by Lemma 54, is closed in $\mathfrak{F}$.

For the $\prec$-minimal variables the claim is immediate, because their respective essentially positive boxes are unary boxes.

Now suppose the claim holds for all predecessors $\overline{\mathbf{q}}_p$ of the variable $p = q_j$, i.e. for every $q_i \in \overline{\mathbf{q}}_p$, $V_m(q_i) = R_{\beta_{q_i}}(y_i)$ for some $\beta_{q_i} \in PMT(\tau, \mathfrak{F})$, and hence is closed.

Let $n_j = n$ and denote $C_i = P_{ji}(V_m(\overline{\mathbf{q}}_p))$ for $i = 1, \ldots, n$. Note that each C_i is closed by the inductive hypothesis and Lemma 55, since P_{ji} is positive.

Consider the unary term $\gamma = \beta(C_n) \ldots (C_2)(C_1) \in PMT(\tau,\ \mathfrak{F})$, i.e. such that $[\gamma](A) = [\beta](A, C_1, \ldots, C_n)$. Then for any $z \in W$, $R_\gamma yz$ holds iff there exist $u_1, \ldots, u_n$ such that $R_\beta yzu_1 \ldots u_n$ and $u_i \in C_i$ for $i = 1, \ldots, n$. Therefore $V_m(p_j) = R_\gamma(y_j)$.

(C2) follows from Lemma 55.

Now, to complete the proof, let us see why $\mathbf{F}$,$V_m \models A$. As in the proof of theorem 37, let $\overline{\mathrm{ST}(A)} = \forall\overline{\mathbf{y}}\forall\overline{\mathbf{Q}}(ANT(\overline{\mathbf{Q}}) \to POS(\overline{\mathbf{Q}}))$. Fix the parameters $\overline{y}$ consistently with ANT (otherwise the formula turns vacuously true) and take any admissible valuation U defined inductively on $\prec$ and extending V_m. It will render ANT true, hence POS true, because $\mathfrak{F} \models A$. Then, by (C2), POS will be true for V_m. ∎

Finally, we note that the local d-persistence and canonicity result for $\mathbf{I}$ *does not* extend to all regular formulae. In particular, both GL and IND are known not to be canonical. In fact, there are regular formulae which are not even frame-complete. An example (see [5]) is the formula $[1]([1]q \leftrightarrow q) \to [1]q$ which is weaker than GL but has the same class of frames, and therefore is incomplete. That formula can be pre-processed into a regular formula, too:

$[1]([1]q \leftrightarrow q) \rightarrow [1]q \equiv \neg[1](([1]q \rightarrow q) \wedge (q \rightarrow [1]q)) \vee [1]q \equiv \neg([1]([1]q \rightarrow q) \wedge [1](q \rightarrow [1]q)) \vee [1]q$

$\equiv \neg[1](\neg[1]q \vee q) \vee \neg[1](\neg q \vee [1]q) \vee [1]q \equiv [\iota_3](\neg[\alpha_1](\neg[1]q, q), \neg[\alpha_2](\neg q, q), [1]q),$

where $\alpha_1 = 1 \circ \iota_2, \alpha_2 = \iota_2(\iota_1, 1)$.

Remark 58 *We emphasize that the minimal valuations for all essential variables in an inductive formula are of the type $R_\beta(y) = \langle \beta^{-1} \rangle \{y\}$, where β is an ordinary modal term in the case of a simple regular formula, and an appropriately parameterized one in the general case of an inductive formula. The closedness of these valuations would therefore follow in any general frame where the singletons are closed sets (incl. descriptive, discrete, and Kripke frames) as soon as the operators $\langle \beta^{-1} \rangle$ are proved to be closed there.*

The following notion is the central one of the present study.

Definition 59 *A formula $A \in \mathcal{L}_\tau$ is an **elementary canonical formula** if it is locally first-order definable and locally d-persistent.*

The class of elementary canonical formulae will be denoted by **ECF**. Some comments:

- the crucial property of an elementary canonical formula is the (local) d-persistence which implies its canonicity, i.e. validity in every canonical frame of a logic containing the formula as an axiom. Therefore, every logic axiomatized with elementary canonical formulae is complete. However, we note that this notion will have to change accordingly for extended modal languages with nominals, or for modal logics in which special additional rules of inference are allowed, that alter the construction of canonical model and the notion of canonicity.
- the (local) first-order definability is nice but not really essential. In fact, one of the ultimate goals of this study is to extend general completeness results to classes of formulae which are not necessarily first-order definable. This issue will be treated in more details in sequels to this paper.
- locality is not essential either, but it is useful and natural, given the local nature of the notion of truth in modal logic. Moreover, as noted in [3], over transitive frames local and global first-order definability coincide.

In this paper, by '*Sahlqvist theorem*' for a set of formulae **S** we will mean the claim that all formulae from **S** are elementary canonical in the respective sense of that term.

Corollary 60 (Sahlqvist theorem for I) *Every inductive formula is elementary and canonical.*

7 The inductive formulae extend essentially the Sahlqvist formulae

As already noted, from syntactical perspective **I** considerably extends **dRV**. It is not clear yet, though, whether every formula from **I** is not *semantically equivalent* to a formula from **dRV**. This question is particularly interesting in the cases of the classical modal and temporal languages. We will show further that, up to axiomatic equivalence (and hence frame-equivalence), and in terms of locally defined first-order properties **I** does not extend semantically **dRV** in the classical *temporal* language. On the other hand, here we will show that **I** extends *essentially* the Sahlqvist formulae in the classical *modal* language (and in fact, in any non-reversive language). More specifically, we will show that the formula D_2, defined in sub-section 3.4, is *not frame-equivalent* (and hence *not semantically equivalent*) to any Sahlqvist formula in the classical modal language. For that we will have to determine a suitable semantic property of the latter set which is not satisfied by D_2.

Let us denote the classical modal language by $\mathcal{L}_m$. We begin by recalling Proposition 26 for the case of $\mathcal{L}_m$: every classical Sahlqvist formula is semantically equivalent to a conjunction of simple regular formulas in $\mathcal{L}_m$. It suffices, therefore to show that D_2 cannot be locally (and hence semantically) equivalent to a conjunction of simple regular formulas.

Next, we note that, as evident from the proof of Theorem 37 and noted in Remark 58, the minimal valuations for all essential variables in a simple regular formula are of the type $R_\beta(y) = \langle \beta^{-1} \rangle \{y\}$, where β is a modal term. All modal terms in $\mathcal{L}_m$ are, up to equivalence of their associated relations, disjunctions of compositions α^n where α is the basic unary modal term corresponding to the $\Box$. Thus, all minimal valuations in a frame $\langle W, R\rangle$ for essential variables of a simple regular formula in $\mathcal{L}_m$ are finite unions of sets of the type $R^n(y)$; the minimal valuations for the inessential variables are either $\varnothing$ or W. This observation prompts the following definitions.

Definition 61 *A general frame $\langle W, R, \mathbb{W}\rangle$ is* ***ample*** *if for every $w \in W$ and $n \in \mathbf{N}$, $R^n(w) = \{u \mid wR^n u\} \in \mathbb{W}$.*

Note that every ample general frame is discrete, for $R^0(w) = \{w\}$.

Definition 62 *A modal formula A is* ***locally a-persistent*** *if for every ample general frame $\mathfrak{F} = \langle F, \mathbb{W}\rangle$, where $F = \langle W, R\rangle$, and $w \in W$,*

$\mathfrak{F}, w \models A$ *iff* $F, w \models A$.

A modal formula A is ***a-persistent*** *if for every ample general frame $\mathfrak{F} = \langle F, \mathbb{W}\rangle$,* *where* $F = \langle W, R \rangle$,

$\mathfrak{F} \models A$ *iff* $F \models A$.

Clearly, local a-persistence implies a-persistence.

Proposition 63 *Every simple regular formula in* $\mathcal{L}_m$ *is locally a-persistent.*

Proof. According to the observations above, the minimal valuations for all variables in a simple regular formula are admissible in every ample general frame. The claim now follows from the fact the truth of an simple regular formula at a state in a Kripke frame under the minimal valuation implies validity at that state. ■

Corollary 64 *Every conjunction of simple regular formulas with pairwise disjoint sets of variables is locally a-persistent.*

Proposition 65 *The formula* $D_2 = p \wedge \Box(\Diamond p \to \Box q) \to \Diamond\Box\Box q$ *is not frame-equivalent to any classical Sahlqvist formula.*

Proof. Suppose the contrary. Then D_2 is frame equivalent to a conjunction of simple regular formulas A, which can be assumed with pairwise disjoint sets of variables (otherwise, we apply suitable substitutions to the conjuncts, which do not affect the frame equivalence). Since both D_2 and A are canonical, they axiomatize the same modal logic, hence they must be valid in the same general frames. So, to reach a contradiction, it suffices to show that D_2 is not a-persistent. To that aim we are going to define an ample general frame $\mathfrak{F}= \langle W, R, \mathbb{W} \rangle$ as follows:

- Let $Y = \{y_0, y_1, \dots\}, Z = \{z_0, z_1, \dots\}, U_n = \{u_{n0}, u_{n1}, \dots\}$, for each $n \in \mathbf{N}$, be pairwise disjoint countably infinite sets. Let $U = \bigcup\{U_n \mid n \in \mathbf{N}\}$ and $x \notin Y \cup Z \cup U$. Put $W = \{x\} \cup Y \cup Z \cup U$.
- R is defined pointwise as follows:
  $R(x) = Y \cup Z \cup U$;
  $R(y_i) = U_i \cup \{x\}$, for each $i \in \mathbf{N}$;
  $R(z_i) = \{z_i\}$;
  $R(u_{ik}) = \{u_{ik}, z_i\}$.
- To define $\mathbb{W}$ we first introduce some terminology and notation:
  For every $I \subseteq \mathbf{N}$ we denote $U_I = \bigcup\{U_i \mid i \in I\}$.
  Two subsets X_1 and X_2 of W will be called **almost equal,** denoted $X_1 \approx_f X_2$, if their symmetric difference is finite. Note that $\approx_f$ is an equivalence relation on $\mathcal{P}(W)$.
  Now, consider the following family of subsets of W :
  $\mathbb{W}_0 = \{Y\} \cup \{U_I \mid I$ is a finite subset of $\mathbf{N}\} \cup$
  $\{Z \cup U_J \mid J$ is a co-finite subset of $\mathbf{N}\} \cup$

$\{Y \cup U_I \mid I$ is a finite subset of $\mathbf{N}\} \cup$
$\{Y \cup Z \cup U_J \mid J$ is a co-finite subset of $\mathbf{N}\}$.

Finally, we define $\mathbb{W}$ to consist of all subsets of W which are almost equal to some set from $\mathbb{W}_0$.

Lemma 66 *$\mathfrak{F} = \langle W, R, \mathbb{W}\rangle$ is an ample general frame.*

Proof. First, note that almost equality in $\mathcal{P}(W)$ is a congruence with respect to (finite) unions and complements. Besides, $\mathbb{W}_0$ contains $\varnothing$ (take $U_\varnothing$) and $Y \cup Z \cup U$ and is closed under finite unions and relative complements in $Y \cup Z \cup U$, hence $\mathbb{W}$ is closed under finite unions and complements (i.e. under all Booleans). It remains to show that $\mathbb{W}$ is closed under the modal operator $\Box$ on $\langle W, R\rangle$.

Recall that $\Box X = \{w \in W \mid R(w) \subseteq X\}$. Hereafter in the proof we agree to denote by I finite subsets of $\mathbf{N}$, and by J co-finite subsets of $\mathbf{N}$.

We consider all cases:

- If $X \approx_f Y$, $X \approx_f U_I$, or $X \approx_f Y \cup U_I$ then $\Box X \approx_f \varnothing$;
- If $X \approx_f Z \cup U_J$ or $X \approx_f Y \cup Z \cup U_J$ then $\Box X \approx_f Y \cup Z \cup U_{J'}$ for some co-finite subset J' of $\mathbf{N}$.

Thus, $\mathbb{W}$ is closed under all operators. An immediate inspection shows that $\mathfrak{F}$ is ample: first, note that it is discrete; also, every $R(w)$ is in $\mathbb{W}$. Further: for any $m \geq 0$, $R^{2m+1}(x) = R(x) = Y \cup Z \cup U$ and $R^{2m}(x) = R^2(x) = \{x\} \cup Z \cup U$; $R(y_i) = \{x\} \cup U_i$, $R^{n+2}(y_i) = R^{n+1}(x)$; $R^{2m+2}(y_i) = Y \cup Z \cup U$, $R^{2m+3}(y_i) = \{x\} \cup Z \cup U$; $R^{m+1}(u_{ik}) = \{u_{ik}, z_i\}$; $R^{m+1}(z_i) = \{z_i\}$. All these sets are in $\mathbb{W}$. ■

Now, to complete the proof of the theorem, it remains to show that $\mathfrak{F} \models D_2$, while $\langle W, R\rangle, x \not\models D_2$.

First, we show $\mathfrak{F}, x \models D_2$. We will reason set-theoretically, rater than semantically, i.e. treating formulae as sets. Suppose $\mathfrak{F}, x \models P \wedge \Box(\Diamond P \to \Box Q)$ for some $P, Q \in \mathbb{W}$. Then $x \in P$, and $R(x) \subseteq \Diamond P \to \Box Q$. Besides, $Y \subseteq \Diamond\{x\} \subseteq \Diamond P$ (since $\{x\} \subseteq P$), and $Y \subseteq R(x)$, so $Y \subseteq \Diamond P \to \Box Q$. Hence, $Y \subseteq \Box Q$, i.e. $R[Y] = \cup\{R(y) \mid y \in Y\} = U \subseteq Q$. Therefore $Q \approx_f Z \cup U_J$ or $Q \approx_f Y \cup Z \cup U_J$ for some co-finite $J \subseteq \mathbf{N}$, hence $Q \cap Z \neq \varnothing$. Let $z \in Q \cap Z$. Then $z \in \Box\Box Q$ (because $R^2(z) = \{z\}$), so $x \in \Diamond\Box\Box Q$, i.e. $\mathfrak{F}, x \models \Diamond\Box\Box Q$. Thus, $\mathfrak{F}, x \models D_2$.

Now, checking that $\mathfrak{F}, y_i \models D_2$: Let $\mathfrak{F}, y_i \models P \wedge \Box(\Diamond P \to \Box Q)$ for some $P, Q \in \mathbb{W}$. Then $R(y_i) \subseteq \Diamond P \to \Box Q$, in particular $x \in \Diamond P \to \Box Q$, but $x \in \Diamond P$ since xRy_i, so $x \in \Box Q$, i.e. $R(x) \subseteq Q$, i.e. $Y \cup Z \cup U \subseteq Q$. But then $\mathfrak{F}, u_{ii} \models \Box\Box Q$, so $\mathfrak{F}, y_i \models \Diamond\Box\Box Q$.

Then, checking $\mathfrak{F}, u_{ik} \models D_2$: Let $\mathfrak{F}, u_{ik} \models P \wedge \Box(\Diamond P \to \Box Q)$ for some $P, Q \in \mathbb{W}$. Then $R(u_{ik}) \subseteq \Diamond P \to \Box Q$, in particular $u_{ik} \in \Diamond P \to \Box Q$, but $u_{ik} \in \Diamond P$ since $u_{ik} R u_{ik}$, so $u_{ik} \in \Box Q$, i.e. $R(u_{ik}) \subseteq Q$, i.e. $z_i \in Q$. But then $\mathfrak{F}, z_i \models \Box\Box Q$, so $\mathfrak{F}, u_{ik} \models \Diamond\Box\Box Q$.

Similarly, $\mathfrak{F}, z_i \models D_2$.

On the other hand, the local first-order equivalent of D_2 :

$$FO(D_2) = \exists y(Rxy \wedge \forall z(R^2yz \to \exists u(Rxu \wedge Rux \wedge Ruz)))$$

fails at x because every successor of x can see in 2 steps an element of Z and no element $z \in Z$ satisfies $\exists u(Rxu \wedge Rux \wedge Ruz)$.

Thus, D_2 is not a-persistent. ■

Corollary 67 *The frame condition defined by the inductive formula D_2 is not definable by any classical Sahlqvist formula.*

8 Elementary canonical formulae in reversive polyadic modal languages

In this section we introduce extensions of basic polyadic modal languages with *reversive* modalities, generalizing the idea of how the temporal language extends the basic modal language (see [28] for more details). We will establish several technical results in such extensions, which will shed extra light on the topological nature of the inductive formulae and will eventually lead to a uniform proof of both parts of Sahlqvist theorem for **I** which will not hinge as directly on the syntactic shape of the formulae, as the first proof given earlier. First, we will prove that all diamond operators in the reversive extension of a polyadic language act as *closed operators* on descriptive frames in the basic language. Then, we will show that all inductive formulae preserve local validity in descriptive frames when extending the range of the valuations from admissible to closed sets. Finally, we will give an effective procedure of transforming inductive formulas in any polyadic language to primitive regular formulae in the reversive extension preserving their important properties, and will thus establish again the Sahlqvist theorem for **I** in arbitrary polyadic languages. At the end of the section we will discuss some consequences and conjectures.

Definition 68 *The* ***reversive extension*** $\mathcal{L}_{\tau r}$ *of a (purely) modal polyadic language* $\mathcal{L}_\tau$ *is the (purely) modal language obtained by adding for every n-ary modal term* $\alpha \in MT(\tau)$ *new distinct* ***inverse terms*** $\alpha^{-1}, \ldots, \alpha^{-n}$*, such that* α^{-i} *and* β^{-j} *are different whenever* α *and* β *are different. The set of terms in* $\mathcal{L}_{\tau r}$ *will be denoted by* $MT(\tau r)$*.*

The semantics of $\mathcal{L}_{\tau r}$ is defined over **reversive extensions of τ-frames**. These are frames for the language of the reversive extension, in which for every $\alpha \in MT(\tau)$ and $k = 1, \ldots, n$ the relation $R_{\alpha^{-k}}$ is defined as follows:

$$xR_{\alpha^{-k}}y_1 \ldots y_k \ldots y_n \text{ iff } y_k R_\alpha y_1 \ldots x \ldots y_n.$$

Alternatively, we can think that $\mathcal{L}_{\tau r}$ is interpreted on standard τ-frames by extending the interpretation of $\mathcal{L}_\tau$ to the inverse terms as above.

Two modal terms in $\mathcal{L}_{\tau r}$ are **semantically equivalent** if they are interpreted in the same relation in every τ-frame. For instance, if α and β are unary terms then $(\alpha(\beta))^{-1}$ and $\beta^{-1}(\alpha^{-1})$ are semantically equivalent.

Note that not every term in a reversive extension has inverses, even up to semantic equivalence, because inverses of composed terms cannot always be expressed in terms of compositions of the inverses of the components. For instance, the inverse δ^{-1} of $\delta = \gamma(\beta, \alpha)$ where α is a unary term and β, γ are binary terms, is not expressible in terms of α, β, γ and their inverses only, unless transposers are allowed in the language.

Still, by iterating the construction of a reversive extension, any (purely) modal polyadic language $\mathcal{L}_\tau$ can be extended to a language $\mathcal{L}_{r(\tau)}$ in which every term has all its inverses. More precisely, $\mathcal{L}_{r(\tau)}$ is obtained by extending the definition of modal terms in $\mathcal{L}_\tau$ with the following clause:

Definition 69 *(MT iv) If* $n > 0$*,* $k \leq n$ *and* α *is an n-ary modal term then* α^{-k} *is an n-ary modal term, too.*

The language $\mathcal{L}_{r(\tau)}$ will be called the **completely reversive extension of** $\mathcal{L}_\tau$. Such languages will be called **reversive languages**.

The notion of frame for $\mathcal{L}_{r(\tau)}$ extends accordingly, via the clause:

$$R_{\alpha^{-k}} = \{(x_0, x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) \subseteq W^{n+1} | \\ (x_k, x_1, \ldots, x_{k-1}, x_0, x_{k+1}, \ldots, x_n) \in R_\alpha\}.$$

The semantics of $\mathcal{L}_{r(\tau)}$ extends accordingly.

Thus, $R_{\alpha^{-k}}$ is obtained from R_α by transposing the 0-th and the k-th arguments. In particular, for a unary term α, $R_{\alpha^{-1}}$ is the usual inverse of α, as expected.

We can relax a bit the notion of a reversive language, by only requiring that with every n-ary modal term α the language contains a term *semantically equivalent* to α^{-k} for each $k \leq n$. Thus, e.g. the classical tense language is regarded as a reversive language.

We note that reversive polyadic languages are closely related to (yet, different from) Venema's *versatile* languages introduced in [48], and in fact have essentially the same expressive power. For more details on the relations and comparison between these, see [28].

The minimal normal modal logic $\mathcal{K}_{r(\tau)}$ of a reversive polyadic language $\mathcal{L}_{r(\tau)}$ is axiomatized over $\mathcal{K}_{r(\tau)}$ by adding the following *axiom schemata for the inverse modalities* :

R1 $A \to [\alpha](\neg B_1, \ldots, \neg B_{k-1}, C_k, \neg B_{k+1}, \ldots, \neg B_n)$,
where $C_k = \left\langle \alpha^{-k} \right\rangle (B_1, \ldots, B_{k-1}, A, B_{k+1}, \ldots, B_n)$.
R2 $[(\alpha^{-k})^{-k}](\ldots, A, \ldots) \leftrightarrow [\alpha](\ldots, A, \ldots)$.

In particular, in a standard temporal language axiom R1 becomes

$$A \to [\alpha] \left\langle \alpha^{-1} \right\rangle A.$$

From these axioms one can easily derive:

R3 $\left\langle \alpha^{-k} \right\rangle (\neg B_1, \ldots, \neg B_{k-1}, D_k, \neg B_{k+1}, \ldots, \neg B_n) \to A$,
where $D_k = [\alpha](B_1, \ldots, B_{k-1}, A, B_{k+1}, \ldots, B_n)$.

8.2 Computing first-order equivalents of inductive formulae in reversive languages: examples

To illustrate some ideas in what follows, we will compute again the first-order equivalents of the examples 35 and 33, this time in purely algebraic manner, and by considering the formulae as set-theoretic operators in the reversive extensions of their languages.

Example 70 *Consider again the inductive formula*

$$D_2 = p \land \Box(\Diamond p \to \Box q) \to \Diamond\Box\Box q.$$

Let w be a state in a Kripke frame F with a domain W. Then the minimal valuation for p at w is $P(w) = \{w\}$, hence the minimal valuation for q is the minimal subset $Q(w)$ of W such that $w \in \Box(\Diamond\{w\} \to \Box Q(w))$ iff $\Diamond^{-1}\{w\} \in \Diamond\{w\} \to \Box Q(w)$ iff $\Diamond^{-1}\{w\} \cap \Diamond\{w\} \subseteq \Box Q(w)$ iff $\Diamond^{-1}(\Diamond^{-1}\{w\} \cap \Diamond\{w\}) \subseteq Q(w)$. Thus, $Q(w) = \Diamond^{-1}(\Diamond^{-1}\{w\} \cap \Diamond\{w\})$ and the (set-theoretic record of the) local first-order equivalent of D_2 at w is

$$w \in \Diamond\Box\Box\Diamond^{-1}(\Diamond^{-1}\{w\} \cap \Diamond\{w\}).$$

Example 71 *Now, consider the inductive formula*

$$D_3 = [\mathbf{3}](\neg[\mathbf{1}]p, \neg[\mathbf{2}](\neg p, q), \langle\mathbf{1}\rangle[\mathbf{1}]q)$$

and again, let w be a state in a Kripke frame F for the respective language, with a domain W. Let $wR_{\mathbf{3}}u_1u_2u_3$. First, the minimal valuation $P = P(w, u_1, u_2, u_3)$ for p at w associated with u_1, u_2, u_3 is determined by the condition $u_1 \in [\mathbf{1}]P$, i.e. $\langle\mathbf{1}^{-1}\rangle\{u_1\} \subseteq P$. Hence,

$$P(w, u_1, u_2, u_3) = \left\langle\mathbf{1}^{-1}\right\rangle\{u_1\}.$$

Then the minimal valuation for q is the minimal subset $Q = Q(w, u_1, u_2, u_3)$ of W such that $u_2 \in [\mathbf{2}](\neg P(w, u_1, u_2, u_3), Q)$, hence

$\langle\mathbf{2}^{-2}\rangle (P(w, u_1, u_2, u_3), \{u_2\}) \subseteq \langle\mathbf{2}^{-2}\rangle (P(w, u_1, u_2, u_3), [\mathbf{2}](\neg P(w, u_1, u_2, u_3), Q))$ $\subseteq Q$ *(using monotonicity of $\langle\mathbf{2}^{-2}\rangle$ and the set-theoretic analog of the reversive axiom R3). Thus,*

$$Q(w, u_1, u_2, u_3) = \left\langle\mathbf{2}^{-2}\right\rangle(\left\langle\mathbf{1}^{-1}\right\rangle\{u_1\}, \{u_2\}).$$

Then, the local first-order equivalent of D_3 at w in set-theoretic terms is

$$w \in [\mathbf{3}]\left(u_1, u_2, \langle\mathbf{1}\rangle[\mathbf{1}]\left\langle\mathbf{2}^{-2}\right\rangle\left(\left\langle\mathbf{1}^{-1}\right\rangle\{u_1\}, \{u_2\}\right)\right).$$

8.3 Closedness of the inverse operators in basic descriptive frames

Here we consider modal formulae as set-theoretical operators over general frames. In particular, we will consider a more general notion of uniform substitution in formulae, viz. substitutions of *set-theoretic operators* for propositional variables. In the particular case where these operators are defined in terms of formulae we talk about *syntactic* substitutions; more specifically, given a modal similarity type τ, a substitution of formulae from $\mathcal{L}_\tau$ (resp. $\mathcal{L}_{\tau r}$) for variables will be called a τ-**substitution** (resp. τr-**substitution**).

Let τ be any modal similarity type, and let $\mathcal{L}_{\tau r}$ be the reversive extension of the language $\mathcal{L}_{\tau}$. We recall that, since every τ-frame uniquely determines its reversive extension, all formulae from $\mathcal{L}_{\tau r}$ can be interpreted in τ-frames. The situation is different if we want to interpret formulae from $\mathcal{L}_{\tau r}$ in general τ-frames because these need not be closed under the inverse modal operators. Still, we can define validity (local or global) of a formula from $\mathcal{L}_{\tau r}$ in a general τ-frame $\mathfrak{F}$ as validity (local or global) in every model over the underlying τ-frame assigning to the variables admissible sets from $\mathfrak{F}$. This will be made precise further.

Theorem 72 *Let α be an n-ary modal term in the language $\mathcal{L}_{\tau}$, $k \leq n$ and consider α^{-k} as a modal term in $\mathcal{L}_{\tau r}$. Let $\mathcal{F} = \left\langle W, \{R_{\alpha}\}_{\alpha \in MT(\tau)}, \mathbb{W} \right\rangle$ be a descriptive τ-frame. Then $\left\langle \alpha^{-k} \right\rangle$ is a closed operator in $T(\mathfrak{F})$.*

Proof. Given a tuple of closed sets $(A_1, \ldots, A_n)$ in $\mathcal{F}$ we must show that $\left\langle \alpha^{-k} \right\rangle (A_1, \ldots, A_n)$ is a closed set in $\mathcal{F}$. For that we will prove the equality

(1) $\left\langle \alpha^{-k} \right\rangle (A_1, \ldots, A_n) = \bigcap \{B \in \mathbb{W} : \left\langle \alpha^{-k} \right\rangle (A_1, \ldots, A_n) \subseteq B\}$.

The inclusion $(\subseteq)$ is trivial.

For $(\supseteq)$, suppose by contraposition that for some $x_0 \in W$ we have

(2) $x_0 \notin \left\langle \alpha^{-k} \right\rangle (A_1, \ldots, A_n)$.

From (2) we obtain

(3) $R_{\alpha}^{-k}(x_0) \cap A_1 \times \ldots \times A_n = \emptyset$.

Now, from (3) we obtain

(4) $\forall y_1 \ldots \forall y_n((y_1, \ldots, y_n) \in A_1 \times \ldots \times A_n \rightarrow (y_1, \ldots, y_n) \notin R_{\alpha}^{-k}(x_0))$,

which is equivalent to

(5) $\forall y_1 \ldots \forall y_n((y_1, \ldots, y_n) \in A_1 \times \ldots \times A_n \rightarrow \neg R_{\alpha} y_k y_1 \ldots x_0 \ldots y_n)$, ($x_0$ is in the k-th position after y_1).

By the tightness condition for R_{α}, $R_{\alpha}(y_k)$ is a closed set, so we have (recall the proof of Lemma 50)

(6) $R_{\alpha}(y_k) = \bigcap \{-(-B_1 \times \ldots \times -B_n) : R_{\alpha}(y_k) \subseteq -(-B_1 \times \ldots \times -B_n), B_i \in \mathbb{W}\}$,

which is equivalent to

(7) $R_{\alpha}(y_k) = \bigcap \{-(-B_1 \times \ldots \times -B_n) : y_k \in [\alpha](B_1, \ldots B_n), B_i \in \mathbb{W}\}$.

Then, from (4) and (7), denoting $\overline{y} = (y_1, \ldots, y_n)$, we obtain

(8) $(\forall \overline{y} \in A_1 \times \ldots \times A_n)(\exists B_1^{\overline{y}} \ldots B_n^{\overline{y}})(y_k \in [\alpha](B_1^{\overline{y}}, \ldots, B_n^{\overline{y}})$ & $(y_1, \ldots, x_0, \ldots, y_n) \notin -(-B_1^{\overline{y}} \times \ldots \times -B_k^{\overline{y}} \times \ldots \times -B_n^{\overline{y}})$.

Equivalently:

(9) $(\forall \overline{y} \in A_1 \times \ldots \times A_n)(\exists B_1^{\overline{y}} \ldots B_n^{\overline{y}} \in \mathbb{W})(y_k \in [\alpha](B_1^{\overline{y}}, \ldots, B_n^{\overline{y}})$ & $y_1 \notin B_1^{\overline{y}}, \ldots, x_0 \notin B_k^{\overline{y}}, \ldots, y_n \notin B_n^{\overline{y}})$.

From (9) we obtain the following inclusion:

(10) $A_1 \times \ldots \times A_n \subseteq \bigcup\{-B_1^{\overline{y}} \times \ldots \times [\alpha](B_1^{\overline{y}}, \ldots, B_n^{\overline{y}}) \times \ldots \times -B_n^{\overline{y}} : \ \overline{y} \in A_1 \times \ldots \times A_n, x_0 \notin B_k^{\overline{y}}, \& B_i^{\overline{y}} \in \mathbb{W}\}$, where $[\alpha](B_1^{\overline{y}}, \ldots, B_n^{\overline{y}})$ is the k-th component of the product.

Note that the set $A_1 \times \ldots \times A_n$ is a closed set because all A_i are closed. The sets in the union from the right hand side of (10) are open because they are products of elements of $\mathbb{W}$ which are clopen. So, (10) says that the closed set $A_1 \times \ldots \times A_n$ is covered by a family of open sets. Since $T(\mathfrak{F})$ is a compact topological space, there exists a finite subcover of $A_1 \times \ldots \times A_n$, hence there exists a finite subset (of indices) $A_1^{'} \times \ldots \times A_n^{'}$ of $A_1 \times \ldots \times A_n$ such that

(11) $A_1 \times \ldots \times A_n \subseteq \bigcup\{-B_1^{\overline{y}} \times \ldots \times [\alpha](B_1^{\overline{y}}, \ldots, B_n^{\overline{y}}) \times \ldots \times -B_n^{\overline{y}} : \ \overline{y} \in A_1^{'} \times \ldots \times A_n^{'}$ & $B_i \in \mathbb{W}\}$.

By the monotonicity and the distributivity of $\left\langle \alpha^{-k} \right\rangle$ over (finite) unions, from (11) we obtain:

(12) $\left\langle \alpha^{-k} \right\rangle (A_1, \ldots, A_n) \subseteq \bigcup\{\left\langle \alpha^{-k} \right\rangle (\neg B_1^{\overline{y}}, \ldots, [\alpha](B_1^{\overline{y}}, \ldots, B_n^{\overline{y}}), \ldots, \neg B_n^{\overline{y}}) : \overline{y} \in A_1^{'} \times \ldots \times A_n^{'}$ & $B_i \in \mathbb{W}\}$.

Applying the inclusion

(13) $\left\langle \alpha^{-k} \right\rangle (\neg B_1, \ldots, [\alpha](B_1, \ldots, B_k, \ldots B_n), \ldots \neg B_n) \subseteq B_k$,

where $[\alpha](\ldots)$ is in the k-th position (axiom R1), we obtain

(14) $\left\langle \alpha^{-k} \right\rangle (A_1, \ldots, A_n) \subseteq \bigcup\{B_k^{\overline{y}} : \overline{y} \in A_1^{'} \times \ldots \times A_n^{'}$ & $B_i \in \mathbb{W}\} = B_0$.

Since B_0 is a finite union of elements from $\mathbb{W}$ it is itself an element of $\mathbb{W}$. But we have that $x_0 \notin B_k^{\overline{y}}$ for all $\overline{y} \in A_1 \times \ldots \times A_n$. From here we obtain that $x_0 \notin B_0$.

(15) Thus we have found $B_0 \in \mathbb{W}$ such that $\left\langle \alpha^{-k} \right\rangle (A_1, \ldots, A_n) \subseteq B_0$ and $x_0 \notin B_0$.

From (15) we obtain

(16) $x_0 \notin \bigcap\{B \in \mathbb{W} : \left\langle \alpha^{-k} \right\rangle (A_1, \ldots, A_n) \subseteq B\}$.

By contraposition we obtain (1), which completes the proof. ■

9 Persistence of inductive formulae in closed extensions of descriptive frames

Recall that the minimal valuations for the variables in inductive formulae need not be admissible in any descriptive frame, but they are *closed*, i.e. intersections of admissible valuations there, and that accounts for the canonicity of inductive formulae. In this section we will revisit and analyze deeper this property of inductive formulae.

9.1 Closed extensions of general frames

Definition 73 *Let* $\mathfrak{F} = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)}, \ \mathbb{W} \right\rangle$ *be a general* τ*-frame. The* ***closed extension*** *of* $\mathfrak{F}$ *is the structure* $\mathbf{C}(\mathfrak{F}) = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)}, \mathbf{C}(\mathbb{W}) \right\rangle$ *where* $\mathbf{C}(\mathbb{W})$ *is the set of all closed sets of the topology* $T(\mathfrak{F})$*.*

Note that $\mathbf{C}(\mathfrak{F})$ is *not* (necessarily) a general τ -frame since $\mathbf{C}(\mathbb{W})$ is not closed (at least) under negations. Nonetheless, we will define local validity of a modal formula from $\mathcal{L}_{\tau r}$ in $\mathbf{C}(\mathfrak{F})$, using the idea described above.

Definition 74 *Given a general* τ*-frame* $\mathfrak{F} = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)}, \mathbb{W} \right\rangle$*, a* ***model over*** $\mathbf{C}(\mathfrak{F})$ *is every Kripke model over* $\left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)} \right\rangle$ *with a valuation of the variables ranging over* $\mathbf{C}(\mathbb{W})$*.*

Definition 75 *Given a formula* $A \in \mathcal{L}_{\tau r}$*, a general* τ*-frame* $\mathfrak{F}$*, and* $w \in W$*, we say that* A ***is (locally) valid at*** w ***in*** $\mathfrak{F}$*, denoted* $\mathfrak{F}, w \vDash A$*, if* A *is true at* w *in every model over* $\mathfrak{F}$*. Respectively, we say that* A ***is (locally) valid at*** w ***in*** $\mathbf{C}(\mathfrak{F})$*, denoted* $\mathbf{C}(\mathfrak{F}), w \vDash A$*, if* A *is true at* w *in every model over* $\mathbf{C}(\mathfrak{F})$*.*

Definition 76 *A substitution* σ *is* ***closed in a general*** τ***-frame*** $\mathfrak{F}$ *if for every variable* p*,* $\sigma(p)$ *is a closed operator in* $T(\mathfrak{F})$*.*

Lemma 77 *Local validity in a closed extension of a general* τ*-frame* $\mathfrak{F}$ *is preserved under Modus ponens and closed substitutions in* $\mathfrak{F}$*.*

Proof. The claim for Modus ponens is straightforward. For the preservation under closed substitutions, let $\mathfrak{F} = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)}, \mathbb{W} \right\rangle$ be a general τ -frame, σ be a closed substitution in $\mathfrak{F}$ and $\mathcal{M} = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)}, V \right\rangle$ be any model over $\mathbf{C}(\mathfrak{F})$. Then V assigns sets from $\mathbf{C}(\mathbb{W})$ to all variables, hence for every variable p, $\sigma(p)$ is a set in $\mathbf{C}(\mathbb{W})$, too, because σ is a closed operator. Therefore, the effect of σ can be simulated by a valuation V_σ in $\mathbf{C}(\mathfrak{F})$, respectively defining a model $\mathcal{M}_\sigma$ over $\mathbf{C}(\mathfrak{F})$, such that $\mathcal{M}, w \models \sigma(A)$ iff $\mathcal{M}_\sigma, w \models A$ for every $w \in W$. Thus, $\mathbf{C}(\mathfrak{F}), w \models A$ implies $\mathbf{C}(\mathfrak{F}), w \models \sigma(A)$. ■

9.2 Closure-persistence of inductive formulae

Definition 78 *A formula $A \in \mathcal{L}_\tau$ is* ***locally closure-persistent*** *if for every descriptive τ-frame $\mathfrak{F} = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)}, \mathbb{W} \right\rangle$ and $w \in W$,*

$$\mathfrak{F}, w \models A \quad \textit{iff} \quad \mathbf{C}(\mathfrak{F}), w \models A.$$

Theorem 79 *Every inductive formula in $\mathcal{L}_\tau$ is locally closure-persistent.* [6]

Proof. Let $A(p_1, \dots, p_n)$ be an inductive formula and $\mathfrak{F} = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)}, \mathbb{W} \right\rangle$ be a descriptive τ-frame. Without loss of generality we can assume that A has already been pre-processed, so all variables in A are essential and all essential variables are different, i.e. $A = [\alpha](\neg H_1, \dots, \neg H_n, Q_1, \dots, Q_l)$ where $H_1, \dots, H_n$ are headed boxes with (different) heads $p_1, \dots, p_n$ and $Q_1, \dots, Q_l$ are positive formulae. Furthermore, we can assume that the dependency graph of A generates the linear ordering $p_1 \prec \cdots \prec p_n$ and that

$$H_1 = [\beta_1](p_1)$$

and

$$H_k = [\beta_k](p_k, \neg P_{k1}(p_1, \dots, p_{k-1}), \dots, \neg P_{kl_k}(p_1, \dots, p_{k-1}))$$

where $P_{k1}, \dots, P_{kl_k}$ are positive, for $k = 2, \dots n$.

The claim of the theorem can be re-phrased as

$$\bigcap \{A(p_1, \dots, p_n) : p_1, \dots, p_n \in \mathbb{W}\}$$

$$= \bigcap \{A(p_1, \dots, p_n) : p_1, \dots, p_n \in \mathbf{C}(\mathbb{W})\}.$$

[6] The referee has noted that the approach followed here is close in spirit to [22], [19] and [20].

We will need the following main lemma:

Lemma 80 *Let k, $1 \le k \le n$, be fixed. Then:*

$$\bigcap\{A(p_1,\ldots,p_n) : p_1 \ldots p_{k-1} \in \mathbf{C}(\mathbb{W}), p_k \ldots p_n \in \mathbb{W}\} =$$

$$\bigcap\{A(p_1,\ldots,p_n) : p_1 \ldots p_k \in \mathbf{C}(\mathbb{W}), p_{k+1} \ldots p_n \in \mathbb{W}\}.$$

Proof. Note that the inclusion $\supseteq$ is straightforward because $\mathbb{W} \subseteq \mathbf{C}(\mathbb{W})$. For the converse inclusion, suppose that for some $x \in W$,

$$x \notin \bigcap\{A(p_1,\ldots,p_n) : p_1 \ldots p_k \in \mathbf{C}(\mathbb{W}), p_{k+1} \ldots p_n \in B(W)\}.$$

Then

$$x \notin A(p_1^*,\ldots,p_k^*, p_{k+1},\ldots,p_n)$$

for some $p_1^*,\ldots,p_k^* \in \mathbf{C}(\mathbb{W})$ and $p_{k+1},\ldots,p_n \in \mathbb{W}$, so

$$x \notin [\alpha](\neg H_1,\ldots,\neg H_n, Q_1,\ldots,Q_l)(p_1^*,\ldots,p_k^*, p_{k+1},\ldots,p_n),$$

i.e. there exist $y_1,\ldots,y_n, z_1,\ldots,z_l$ such that

(1) $R_\alpha x y_1 \ldots y_n z_1 \ldots z_l$,
(2) $y_j \in H_j(p_1^*,\ldots,p_j^*)$ for $1 \le j \le k$,
(3) $y_j \in H_j(p_1^*,\ldots,p_k^*, p_{k+1},\ldots,p_j)$ for $k+1 \le j \le n$,
(4) $z_i \notin Q_i(p_1^*,\ldots,p_k^*, p_{k+1},\ldots,p_n)$ for $1 \le i \le l$.

The formula H_j for $k+1 \le j \le n$ has the following form:

$$[\beta_j](p_j, \neg P_{j1}(p_1^*,\ldots,p_k^*, p_{k+1},\ldots,p_{j-1}), \neg P_{jl_j}(p_1^*,\ldots,p_k^*, p_{k+1},\ldots,p_{j-1})).$$

Here $p_j \in \mathbb{W}$, hence $\neg p_j \in \mathbb{W}$ and consequently $\neg p_j$ is a closed element and can be taken as a parameter. Then the formula H_j can be represented as

$$\neg \langle \beta_j(\neg p_j) \rangle (P_{j1}(p_1^*,\ldots,p_k^*, p_{k+1},\ldots,p_{j-1}), P_{jl_j}(p_1^*,\ldots,p_k^*, p_{k+1},\ldots,p_{j-1})),$$

denoted shortly by $\neg C_j(p_1^*,\ldots,p_k^*, p_{k+1},\ldots,p_j))$. Note that each C_j, $k+1 \le j \le n$, is a positive parameterized formula.

Now (3) has the following equivalent formulation

(3a) $y_j \notin C_j(p_1^*,\ldots,p_k^*, p_{k+1},\ldots,p_j))$, $k+1 \le j \le n$.

Our strategy now is to find an element $p_k \in \mathbb{W}$ to be substituted in the place of p_k^* in (3a), (4) and in (2) for $j = k$.

Since p_k^* is a closed subset of W we have

(5) $p_k^* = \bigcap\{q_i \in \mathbb{W} : i \in I\}$

where $M = \{q_i \in \mathbb{W} : i \in I\}$ is a downwards directed family of clopen sets.

Now substitute p_k^* from (5) in (3a) and (4). By Esakia's lemma we get:

$$C_j(p_1^*, \ldots, p_{k-1}^*, \bigcap\{q_i \in \mathbb{W} : i \in I\}, p_{k+1}, \ldots, p_j) =$$

$$\bigcap\{C_j(p_1^*, \ldots, p_{k-1}^*, q_i, p_{k+1}, \ldots, p_j) : q_i \in \mathbb{W}, i \in I\}, k+1 \leq j \leq n.$$

Then by (3a), for each j such that $k+1 \leq j \leq n$ there exists $s_j \in I$ such that

(3') $y_j \notin C_j(p_1^*, \ldots, p_{k-1}^*, q_{s_j}, p_{k+1}, \ldots, p_j))$.

Analogously, we obtain from (4) that:

(4') For every i such that $1 \leq i \leq l$ there exists $t_i \in I$ such that

$$z_i \notin Q_i(p_1^*, \ldots, p_{k-1}^*, q_{t_i}, p_{k+1}, \ldots, p_n).$$

Now, we define

(6) $p_k = (\bigcap_{j=k+1}^n q_{s_j}) \cap (\bigcap_{i=1}^l q_{t_i})$.

Since all elements in this finite intersection are from M and since M is closed under finite intersections we obtain that

(7) $p_k \in M$ and, consequently, $p_k \in \mathbb{W}$.

By (6) and the monotonicity of the C_j's and Q_i's we obtain from (3') and (4') that

$y_j \notin C_j(p_1^*, \ldots, p_{k-1}^*, p_k, p_{k+1}, \ldots, p_{j-1}))$, $k+1 \leq j \leq n$, or equivalently

(3") $y_j \in H_j(p_1^*, \ldots, p_{k-1}^*, p_k, p_{k+1}, \ldots, p_{j-1}))$, $k+1 \leq j \leq n$, and
(4") $z_i \notin Q_i(p_1^*, \ldots, p_{k-1}^*, q_{t_i}, p_{k+1}, \ldots, p_n)$ for $1 \leq i \leq l$.

It remains to eliminate p_k^* from (2) for $j = k$. Note that

$$H_k = [\beta_k](p_k^*, \neg P_{k1}(p_1^*, \ldots, p_{k-1}), \ldots, \neg P_{kl_k}(p_1^*, \ldots, p_{k-1}^*)).$$

By (7), $p_k \in M$ and hence $p_k^* \subseteq p_k$. Then by the monotonicity of $[\beta_k]$ we obtain from (2) (for $j = k$) that

(2_k) $y_k \in H_k(p_1^*, \ldots, p_{k-1}^*, p_k)$.

From (1), (2) (for the cases $1 \leq j \leq k-1$), (2_k), (3") and (4") we obtain that

$$x \notin A(p_1^*, \ldots, p_{k-1}^*, p_k, \ldots, p_n)$$

for $p_1^*, \ldots, p_{k-1}^* \in \mathbf{C}(\mathbb{W})$ and $p_k, \ldots, p_n \in \mathbb{W}$. Therefore:

$$x \notin \bigcap\{A(p_1, \ldots, p_n) : p_1, \ldots, p_{k-1} \in \mathbf{C}(\mathbb{W}), p_k, \ldots, p_n \in B(W)\},$$

which completes the proof of the lemma. ■

Now, the claim of the theorem follows immediately by applying the lemma consecutively for $k = 1, \ldots, n$. ■

9.3 Transforming an inductive formula into a primitive regular formula

We have proved in [28] that every inductive formula A in a reversive language can be effectively transformed into an *axiomatically equivalent* primitive regular formula $\mathrm{Pr}(A)$. As we will see in the next section, both local first-order definability and local d-persistence of primitive regular formulae are quite easy to establish, which thus yields the Sahlqvist theorem for $\mathbf{I}$ in reversive languages. Moreover, the local first-order equivalent of an inductive formula A can be computed immediately from $\mathrm{Pr}(A)$ as indicated further.

Hereafter, $\mathcal{L}_\tau$ is an arbitrary, possibly non-reversive polyadic modal language.

Here we replace the axiomatic equivalence of A and $\mathrm{Pr}(A)$ by a stronger semantic equivalence, defined in terms of local validity in closed extensions of descriptive frames, applying the results of the previous sections, and thus extending the Sahlqvist theorem to arbitrary polyadic languages.

Hereafter we will denote local equivalence between formulae A and B in a language $\mathcal{L}_\tau$ by $A \approx_\tau^{\mathbf{l}} B$.

Definition 81 *Let $A = A(p_1, \ldots, p_n)$ and $B = B(q_1, \ldots, q_m)$ be formulae in $\mathcal{L}_{\tau r}$. We say that A and B are* ***locally closure-equivalent in $\mathcal{L}_\tau$****, in symbols $A \approx_\tau^{\mathbf{c}} B$, if for any descriptive or Kripke τ-frame $\mathcal{F} = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)}, \mathbb{W} \right\rangle$ and $w \in W$,*

$$\mathbf{C}(\mathcal{F}), w \models A \textit{ iff } \mathbf{C}(\mathcal{F}), w \models B,$$

i.e.

$$\bigcap_{p_1 \dots p_n \in \mathbf{C}(\mathbb{W})} A(p_1, \dots, p_n) = \bigcap_{q_1 \dots q_m \in \mathbf{C}(\mathbb{W})} B(q_1, \dots, q_m).$$

Clearly, $\approx_\tau^{\mathbf{c}}$ is an equivalence relation. Note that the closure equivalence for Kripke frames means simply equivalence with respect to local frame validity.

Lemma 82 *(Monotonicity lemma) Let A, B, C be any modal formulae and $\overline{p} = p^1, \dots p^m$ be a list of positive occurrences of a variable p in a formula A. Denote by $A(Q/\overline{p})$ the result of the uniform substitution of a formula Q for the occurrences $\overline{p}$ in A. Then $\models B \to C$ implies $\models A(B/\overline{p}) \to A(C/\overline{p})$.*

Proof. Easy structural induction on formulae. ■

Lemma 83 *Let α, β be modal terms in $\mathcal{L}_\tau$, $Q_1, \dots, Q_n$, be positive formulae from $\mathcal{L}_{r\tau}$ not containing the variable p, and $P_1, \dots, P_m$ be any formulae from $\mathcal{L}_{r\tau}$ positive in p. Then the formula $A = [\alpha](\neg[\beta](p, \neg Q_1, \dots, \neg Q_n), P_1, \dots, P_m)$ is locally closure-equivalent in $\mathcal{L}_\tau$ and is locally equivalent in $\mathcal{L}_{r\tau}$ to $A_p = [\alpha](\neg p, \sigma_p(P_1), \dots, \sigma_p(P_m))$ where $\sigma_p(p) = \langle \beta^{-1} \rangle (p, Q_1, \dots, Q_n)$ and $\sigma_p(q) = q$ for every $q \neq p$.*

Proof. First, we will prove the local closure-equivalence in $\mathcal{L}_\tau$. We recall the validity of the formulae $[R1]$ and $[R3]$ in $\mathcal{L}_{r\tau}$, listed in Section 8.1.

Let $\mathcal{F} = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)}, \mathbb{W} \right\rangle$ be any descriptive τ-frame and $w \in W$. Suppose $\mathbf{C}(\mathcal{F}), w \models A$ and substitute $\langle \beta^{-1} \rangle (p, Q_1, \dots, Q_n)$ for p in A. Since $Q_1, \dots, Q_n$ are positive, $\langle \beta^{-1} \rangle (p, Q_1, \dots, Q_n)$ is closed, by Theorem 72.

Then, by Lemma 77 we obtain $(*)$

$\mathbf{C}(\mathcal{F}), w \models [\alpha](\neg[\beta](\langle \beta^{-1} \rangle (p, Q_1, \dots, Q_n), \neg Q_1, \dots, \neg Q_n), \sigma_p(P_1), \dots, \sigma_p(P_m))$.

From the validity of R1 we obtain by contraposition:
$\models \neg[\beta](\langle \beta^{-1} \rangle (p, Q_1, \dots, Q_n), \neg Q_1 \dots, \neg Q_n) \to \neg p$.

Now, from $(*)$ by the monotonicity lemma 82 and Modus Ponens we get

$\mathbf{C}(\mathcal{F}), w \models [\alpha](\neg p, \sigma_p(P_1), \dots, \sigma_p(P_m))$.

Conversely, suppose $\mathbf{C}(\mathcal{F}), w \models [\alpha](\neg p, \sigma_p(P_1), \dots, \sigma_p(P_m))$.

Let $Q = \langle \beta^{-1} \rangle ([\beta](p, \neg Q_1, \dots, \neg Q_n), Q_1, \dots, Q_n)$. Since $Q_1, \dots, Q_n$ are closed (being positive), we claim that the formula $[\beta](p, \neg Q_1, \dots, \neg Q_n)$ is closed in $\mathcal{F}$. Indeed, for any closed sets assigned to the variables occurring in $Q_1, \dots, Q_n$, the respective values $\mathbf{Q}_1, \dots, \mathbf{Q}_n$ are closed sets, and therefore the positive $\mathcal{F}$-

parameterized formula $[\beta(\mathbf{Q}_n)\ldots(\mathbf{Q}_1)](p)$ defines a closed operator by Lemma 55. Then, substituting $[\beta](p,\neg Q_1,\ldots,\neg Q_n)$ for p, by Lemma 77 we obtain $\mathbf{C}(\mathcal{F}),w\models[\alpha](\neg[\beta](p,\neg Q_1,\ldots,\neg Q_n),P_1(Q/p),\ldots,P_m(Q/p))$. $\quad(**)$

Then, from the validity of R2, we obtain

$\models\langle\beta^{-1}\rangle\left([\beta](p,\neg Q_1,\ldots,\neg Q_n),Q_1,\ldots,Q_n\right)\to p$, i.e. $\models Q\to p$,

whence, by the Monotonicity Lemma 82

$\models[\alpha](\neg[\beta](p,\neg Q_1,\ldots,\neg Q_n),P_1(Q/p),\ldots,P_m(Q/p))\to$
$[\alpha](\neg[\beta](p,\neg Q_1,\ldots,\neg Q_n),P_1,\ldots,P_m)$,

hence $\mathbf{C}(\mathcal{F}),w\models[\alpha](\neg[\beta](p,\neg Q_1,\ldots,\neg Q_n),P_1,\ldots,P_m)$ by (**).

The case of $\mathcal{F}$ being a Kripke frame is an easy simplification of the argument above.

The argument for local equivalence in $\mathcal{L}_{r\tau}$ is essentially the same, but simpler, because it does not require any restrictions on the substitutions. ∎

The lemma applies likewise when the argument $\neg[\beta](p,\neg Q_1,\ldots,\neg Q_n)$ is not in the first position.

Theorem 84 *Every inductive formula in $\mathcal{L}_\tau$ can be effectively transformed into a primitive regular formula* $\mathrm{Pr}(A)$ *in $\mathcal{L}_{r\tau}$, such that $A\approx^{\mathbf{c}}_{\tau}\mathrm{Pr}(A)$ and $A\approx^{\mathbf{l}}_{r(\tau)}\mathrm{Pr}(A)$*

Proof. Let $A=[\alpha](\neg H_1,\ldots,\neg H_n,P_1,\ldots,P_k)$ be a pre-processed inductive formula with essentially positive boxes $H_1,\ldots,H_n$ and different heads resp. $q_1,\ldots,q_n$.

Let the dependency digraph of A determine a precedence order on these variables, extended to a linear ordering $\prec$ such that $q_1\prec\ldots\prec q_n$. We transform A into $\mathrm{Pr}(A)$ through a sequence of intermediate formulae $A=A_1,\ldots,A_n=\mathrm{Pr}(A)$ obtained by successive replacement of all essentially positive boxes by variables, one by one inductively on $\prec$, using Lemma 83. We have to show that the lemma will remain applicable throughout that process. Indeed, we can show inductively on $j=1,\ldots,n$ that the formula A_j will have the form $[\alpha](\neg q_1,\ldots,\neg q_{j-1},\neg H'_j,\ldots,\neg H'_n,P'_1,\ldots,P'_k)$ where $H'_i=[\beta_i](q_i,\neg Q_{i1}(q_1,\ldots,q_{j-1}),\ldots,\neg Q_{in_i}(q_1,\ldots,q_{j-1}))$ for some positive formulae $Q_{i1},\ldots,Q_{in_i}$. Assuming this, we note that $Q_{j1},\ldots,Q_{jn_j}$ do not contain q_j while all $\neg H'_{j+1},\ldots,\neg H'_n,P'_1,\ldots,P'_k$ are positive in q_j, hence Lemma 83 applies to $\neg H'_j$ in A_j, so $A_{j+1}=[\alpha](\neg q_1,\ldots,\neg q_j,\neg H''_{j+1},\ldots,\neg H''_n,P''_1,\ldots,P''_k)$ which again satisfies the requirements of the lemma.

In the long run we obtain the primitive regular formula

$$\Pr(A) = A_n = [\alpha](\neg q_1, \ldots, \neg q_n, D_1, \ldots, D_k)$$

where $D_1, \ldots, D_k$ are positive formulae. By Lemma 83, $A \approx^{\mathsf{c}}_{\tau} \Pr(A)$. ■

In particular, every formula A in a reversive language can be effectively transformed into a suitably equivalent primitive regular formula $\Pr(A)$ in the same language. Furthermore, note that $\Pr(A)$ has the same variables as the pre-processed inductive formula A.

Example 85 *Consider the inductive formula*

$$A(q_1, q_2) = [\alpha](\neg[\beta]q_1, \neg[\gamma](\neg Q(q_1), q_2), P(q_1, q)).$$

The precedence order is $q_1 \prec q_2$ and we transform $A(q_1, q_2)$ in two steps. The first step is to substitute $\langle \beta^{-1} \rangle q_1$ for q_1, and after that to replace $\neg[\beta] \langle \beta^{-1} \rangle q_1$ by $\neg q_1$. The result is:

$$A_1(q_1, q_2) = [\alpha](\neg q_1, \neg[\gamma](\neg Q(\left\langle \beta^{-1} \right\rangle q_1), q_2), P(\left\langle \beta^{-1} \right\rangle q_1, q_2)).$$

The second step is to substitute $\langle \gamma^{-2} \rangle (Q(\langle \beta^{-1} \rangle q_1), q_2)$ for q_2 (here γ^{-2} is taken because q_2 is the second argument in $\neg[\gamma](\neg Q(\langle \beta^{-1} \rangle q_1), q_2)$), and after that to replace $\neg[\gamma](\neg Q(\langle \beta^{-1} \rangle q_1), \langle \gamma^{-2} \rangle (Q(\langle \beta^{-1} \rangle q_1), q_2))$ by $\neg q_2$. The result is the primitive regular formula

$$\Pr(A)(q_1, q_2) = [\alpha](\neg q_1, \neg q_2, P(\left\langle \beta^{-1} \right\rangle q_1, \left\langle \gamma^{-2} \right\rangle (Q(\left\langle \beta^{-1} \right\rangle q_1), q_2)).$$

10 Inductive formulae and Sahlqvist theorem in languages with nominals

10.1 Adding nominals and universal modality to purely modal languages

Nominals (or, **names** in [18]) are special sort of propositional variables in modal languages which can only be true in a *single* possible world, i.e. their valuations are singletons. Adding nominals extends considerably the expressive power of the modal language, while generally preserving its tractability and other good features (see [4]).

In order for nominals to work well in the language, we need an additional mechanism which allows references (access) to the state named by a nominal from anywhere in the model. Such a mechanism is e.g. the **universal modality** $[\mathsf{u}]$, the semantics of which in a τ-frame $\left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau)} \right\rangle$ is given by $R_{\mathsf{u}} = W^2$.

Given a purely modal polyadic language $\mathcal{L}_\tau$, we denote by $\mathcal{L}_\tau^{\mathbf{n}}$ its extension with countably many nominals $c_1, c_2, \ldots$, and by $\mathcal{L}_\tau^{\mathsf{u},\mathbf{n}}$ the extension of $\mathcal{L}_\tau^{\mathbf{n}}$ with $[\mathsf{u}]$.

Henceforth by 'variable' we will mean an ordinary propositional variable, not a nominal. The definition of formulae extends accordingly, adding the clause that every nominal is a formula, and extending the set of modal terms as described below.

Definition 86 *A formula of $\mathcal{L}_\tau^{\mathsf{u},\mathbf{n}}$ is **pure** if it does not contain propositional variables.*

Now the definition of modal terms in $\mathcal{L}_\tau^{\mathsf{u},\mathbf{n}}$ extends the basic one with the clause:

- Every pure formula is a 0-ary modal term,

i.e. modal terms can be parameterized with pure formulae. That clause essentially does not extend the expressiveness of the purely modal languages, but gives them more flexibility and eventually enables us to extend considerably the set of inductive formulae in languages with nominals at no extra cost.

Further, the definition of a model accounts for the restriction on the nominals: an **$\mathcal{L}_\tau^{\mathsf{u},\mathbf{n}}$-model** is a structure $M = \langle F, V \rangle$ where F is a $\mathcal{L}_\tau$-frame and V is a valuation for the propositional variables and the nominals such that $V(c)$ for any nominal c is a singleton. To simplify notation we shall write $V(c) = w$ instead of $\{w\}$. Then:

$$M, w \models c \text{ iff } V(c) = w.$$

Finally, the standard translation ST extends by

$$ST(c_i) := (x = y_i),$$

where $y_1, y_2, \ldots$ is a string of reserved variables associated with the nominals $c_1, c_2, \ldots$.

Proposition 87 *Every pure formula is locally first-order definable.*

Proof. The pure formula $A(c_1, \ldots, c_n)$, where $c_1, \ldots, c_n$ are all nominals occurring in A, locally determines the condition $FO(A, x) = \forall y_1, \ldots, \forall y_n ST(A)$.

■

For complete axiomatization of the basic normal logic $\mathcal{K}_\tau^{\mathbf{u},\mathbf{n}}$ of $\mathcal{L}_\tau^{\mathbf{u},\mathbf{n}}$ see [28]. In particular, that axiomatization involves an 'unorthodox' rule of inference forcing every state of a model to be named by a nominal. The notion of 'canonical model' changes accordingly, but the respective property of 'discrete canonicity', see Section 11 still implies completeness. The following result (see [28]), in which 'canonical' refers to discrete-canonical, justifies the importance of pure formulae as axioms.

Proposition 88 *Every extension of $\mathcal{K}_\tau^{\mathbf{u},\mathbf{n}}$ axiomatized over $\mathcal{K}_\tau^{\mathbf{u},\mathbf{n}}$ with pure axioms is canonical.*

10.2 Sahlqvist theorem for inductive formulae revisited

Let $\mathcal{L}_{\tau r}^{\mathbf{n}}$ be an extension of $\mathcal{L}_{\tau r}$ with a denumerable set of nominals $Nom(\mathcal{L})$. If $\langle W, \ldots\rangle$ is a frame then $Nom(W)$ denotes the set of all singletons $\{\{x\} : x \in W\}$.

Proposition 89 *Let $B(q_1, \ldots, q_n)$ be a primitive regular formula in $\mathcal{L}_\tau$ and $\mathfrak{F} = \left\langle W, \{R_\alpha\}_{\alpha\in MT(\tau)}, \mathbb{W}\right\rangle$ be a differentiated general τ-frame. Then*

$$\bigcap_{q_1,\ldots,q_n\in\mathbf{C}(\mathbb{W})} B(q_1, \ldots, q_n) = \bigcap_{c_1,\ldots,c_n\in Nom(W)} B(c_1, \ldots, c_n).$$

Proof. Let $B = [\alpha](\neg q_1, \ldots, \neg q_n, D_1, \ldots, D_k)$ where $q_1, \ldots, q_n$ are different variables and $D_1, \ldots, D_k$ are positive formulae.

The inclusion ($\subseteq$) is obvious because all singletons are closed sets in a differentiated frame.

For ($\supseteq$) suppose that for some $x \in W$ we have $x \notin \bigcap_{q_1,\ldots,q_n\in\mathbf{C}(\mathbb{W})} B(q_1, \ldots, q_n)$. Then there exist $Q_1, \ldots, Q_n \in \mathbf{C}(\mathbb{W})$ such that $x \notin B(Q_1, \ldots, Q_n)$. Then there are $y_1, \ldots, y_n, z_1, \ldots, z_k \in W$ such that $R_\alpha x y_1 \ldots y_n z_1 \ldots z_k$, and $y_1 \in Q_1, \ldots, y_n \in Q_n, z_1 \notin D_1(Q_1, \ldots, Q_n), \ldots, z_k \notin D_k(Q_1, \ldots, Q_n)$. As $D_1, \ldots, D_k$ are monotone we get $z_1 \notin D_1(\{y_1\}, \ldots, \{y_n\}), \ldots, z_k \notin D_k(\{y_1\}, \ldots, \{y_n\})$, hence $x \notin B(\{y_1\}, \ldots, \{y_n\})$, so $x \notin \bigcap_{c_1,\ldots,c_n\in Nom(W)} B(c_1, \ldots, c_n)$. ■

Proposition 90 *Let $\mathfrak{F} = \left\langle W, \{R_\alpha\}_{\alpha\in MT(\tau)}, \mathbb{W}\right\rangle$ be a descriptive or Kripke τ-frame and $A(q_1, \ldots, q_n)$ be any inductive formula in $\mathcal{L}_\tau$ with a locally closure-*

equivalent primitive regular formula $\mathrm{Pr}(A)$ *in* $\mathcal{L}_{\tau r}$. *Then*

$$\bigcap_{q_1,\dots,q_n\in\mathbb{W}} A(q_1,\dots,q_n) = \bigcap_{c_1,\dots,c_n\in Nom(W)} \mathrm{Pr}(A)(c_1,\dots,c_n).$$

Proof. By Theorem 79 we have

$$\bigcap_{q_1,\dots,q_n\in\mathbf{C}(\mathbb{W})} A(q_1,\dots,q_n) = \bigcap_{q_1,\dots,q_n\in\mathbb{W}} A(q_1,\dots,q_n).$$

By Theorem 84 we obtain

$$\bigcap_{q_1,\dots,q_n\in\mathbf{C}(\mathbb{W})} A(q_1,\dots,q_n) = \bigcap_{q_1,\dots,q_n\in\mathbf{C}(\mathbb{W})} \mathrm{Pr}(A)(q_1,\dots,q_n).$$

Then by Proposition 89 we obtain the required equality. ■

Thus, putting together Theorem 79 and Proposition 90, we have obtained the following result which can be regarded as a stronger form of the Sahlqvist theorem for inductive polyadic formulae.

Theorem 91 *Every inductive formula in* $\mathcal{L}_\tau$ *is locally closure-persistent and locally closure-equivalent to a pure formula in the reversive extension* $\mathcal{L}^{\mathbf{n}}_{r\tau}$ *of* $\mathcal{L}^{\mathbf{n}}_{\tau}$.

Note that pure formulae, being 0-ary modal terms, trivially are inductive formulae.

Corollary 92 (Sahlqvist theorem for I) *Every inductive formula in* $\mathcal{L}_\tau$ *is elementary canonical. Moreover, its local first-order equivalent can be effectively computed from the formula.*

Proof. By Proposition 90, applied to Kripke frames, A has a local first-order equivalent determined by $\mathrm{Pr}(A)(c_1,\dots,c_n)$. By the same proposition, applied to descriptive frames, the formula $\mathrm{Pr}(A)(c_1,\dots,c_n)$, and hence the formula A itself, is valid at every point w of the underlying Kripke frame of any descriptive frame where A is valid at w. ■

Applying Propositions 89 and 90, we can describe a simple effective procedure for finding a local first-order equivalent of every inductive formula A:

(1) Transform A, considered as a formula in the language $\mathcal{L}_\tau$, into its primitive form $\mathrm{Pr}(A)$ in $\mathcal{L}_{\tau r}$.
(2) Replace all variables of $\mathrm{Pr}(A)$ by different nominals. The result is a pure formula $\mathcal{L}^{\mathbf{n}}_{r\tau}$. That formula encodes the expected local first order condition.

Let us demonstrate the procedure for computing the local first-order equivalent with the example from the previous sections. Let

$$A(q_1, q_2) = [\alpha](\neg[\beta]q_1, \neg[\gamma](\neg Q(q_1), q_2), P(q_1, q_2))$$

with positive P, Q. Then

$$\Pr(A) = [\alpha](\neg q_1, \neg q_2, P'(q_1, q_2))$$

with $P'(q_1, q_2) = P(\langle\beta^{-1}\rangle\, q_1, \langle\gamma^{-2}\rangle\, (Q(\langle\beta^{-1}\rangle\, q_1), q_2))$.

Consider the case $Q(q) = [\beta]q \vee q$, $P(q_1, q_2) = q_1 \wedge \langle\alpha\rangle\, (q_1, q_2)$. Then

$$\mathbf{T}(A) = \Pr(A)(c_1/q_1, c_2/q_2)$$
$$= [\alpha](\neg c_1, \neg c_2, \left\langle\beta^{-1}\right\rangle c_1 \wedge \langle\alpha\rangle\, (\left\langle\beta^{-1}\right\rangle c_1, \left\langle\gamma^{-2}\right\rangle ([\beta]\left\langle\beta^{-1}\right\rangle c_1 \vee \left\langle\beta^{-1}\right\rangle c_1, c_2)))$$

with two different nominals c_1, c_2. The meaning of $\mathbf{T}(A)$ in descriptive and Kripke frames is derived by considering c_1, c_2 ranging over the set of singletons $Nom(W)$ of the frame $\langle W\ , \ldots\rangle$, by the set-theoretic expression $FO(A)(x) := x \in \bigcap_{c_1, c_2 \in Nom(W)} \mathbf{T}(A)$. This is readily translated into a first-order formula as the standard translation $ST(\mathbf{T}(A))$, thus producing the desired first-order local equivalent of A.

11 Discrete-canonical formulae in reversive languages with nominals

Recall that a general frame $\left\langle W, \{R_\alpha\}_{\alpha\in MT(\tau)}, \mathbb{W}\right\rangle$ is discrete if $Nom(W) \subseteq \mathbb{W}$.

11.1 Sahlqvist theorem for inductive formulae in reversive languages with nominals

Definition 93 *[48] A formula $A \in \mathcal{L}_\tau$ is* ***locally di-persistent*** *if for every discrete general τ-frame $\mathfrak{F} = \langle F, \mathbb{W}\rangle$, where $F = \left\langle W, \{R_\alpha\}_{\alpha\in MT(\tau)}\right\rangle$, and $w \in W$,*

$$\mathfrak{F}, w \models A \text{ iff } F, w \models A.$$

Clearly, every pure formula in a language with nominals is locally di-persistent. Also, note that any formula $A \in \mathcal{L}_\tau$ is locally di-persistent in $\mathcal{L}_\tau$ iff it is locally di-persistent in the extension with nominals $\mathcal{L}_\tau^{\mathbf{n}}$ of $\mathcal{L}_\tau$, since the discrete frames in both languages are the same.

Local di-persistence is important because the appropriately modified canonical general frames in languages with nominals or difference operator (see [48], [4]) are discrete, and therefore canonicity in a language with nominals generally requires di-persistency. Thus, we have the following natural modification of the notion of elementary canonical formulae in languages with nominals.

Definition 94 *A formula $A \in \mathcal{L}_\tau^{\mathbf{n}}$ is* ***discrete-canonical*** *if it is locally di-persistent.*

The following is a strengthening of Lachlan's result that every r-persistent modal formula is elementary (see [4], Ex.5.6.3, and also [3], Theorem 8.7 for the localized version). It can also be derived from results in [25].

Proposition 95 *Every locally di-persistent formula in a language with nominals $\mathcal{L}_\tau^{\mathbf{n}}$ is locally first-order definable.*

Proof. We use a variation of van Benthem's argument in [3], Theorem 8.7, proving that a modal formula is locally first-order definable iff its local validity is preserved under ultrapowers. First, note that local non-validity of a modal formula, being an existential second-order property, is preserved under ultraproducts. Therefore, in order to apply Keisler's characterization of first-order definable properties it suffices to show that local validity of locally di-persistent formulae is preserved under ultraproducts. This follows from the fact that local validity of modal formulae is locally preserved in ultraproducts of general frames (see [3], Theorem 4.12 for the classical modal language, routinely generalized to arbitrary polyadic languages) and that any ultraproduct of Kripke frames regarded as general frames is a discrete general frame. ■

Proposition 96 *Every primitive regular formula in $\mathcal{L}_\tau^{\mathbf{n}}$ is locally di-persistent.*

Proof. Follows immediately from Propositions 89 and 49. ■

Corollary 97 *(*Sahlqvist theorem for ***I*** in reversive languages with nominals*)* *Every inductive formula in a reversive language with nominals is discrete-canonical.*

Proof. Let $\mathfrak{F} = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau r)}, \mathbb{W} \right\rangle$ be any discrete general frame in the reversive language, which can be regarded as $\mathcal{L}_{\tau r}$ for a suitable type τ. Suppose $\mathcal{F}, w \models A$. Then $\mathcal{F}, w \models \Pr(A)$ by Theorem 84, whence the claim follows by Proposition 96. ■

A few comments of comparison with similar earlier results by Venema are in order here. Proposition 96 was proved for 'very simple' Sahlqvist formulae (subsumed here by 'primitive regular formulas') in versatile languages with difference operator by Venema in [48]. As a consequence, the respective Sahlqvist theorem was established there[7]. Furthermore, the fact that every Sahlqvist formula in a temporal language is di-persistent is proved in [48], also noted as an exercise in [4].

Another line of comparison and extension of the present results stems from the relationship between discrete frames and *atomic* modal algebras in the respective polyadic languages. Indeed, every discrete general τ-frame is an atomic modal τ-algebra. Conversely, every atomic τ-algebra $\mathfrak{A}$ is isomorphic, by Jónsson-Tarski theorem to a general τ-frame $\mathfrak{F} = \left\langle W, \{R_\alpha\}_{\alpha \in MT(\tau r)}, \mathbb{W} \right\rangle$ which need not be discrete, because two or more states may not be separable by $\mathbb{W}$. However, as proved in [49], if in addition, all operators $\langle \alpha \rangle$ in $\mathfrak{A}$ are *completely additive*, i.e. preserve arbitrary joins, then $\mathfrak{F}$ can be constructed as a discrete frame over the *atom structure* $At\mathfrak{A}$ of $\mathfrak{A}$ (see [49], [50]) which is a Kripke τ-frame based on the set of atoms of $\mathfrak{A}$. In particular, this condition holds if the language is reversive or versatile (see [50] for versatile languages). Venema has proved in [49] that the validity of all Sahlqvist formulae from the class **dRV** is preserved when passing from atomic τ-algebras in versatile languages to their respective atom structures. Since all primitive regular formulae are in **dRV**, this result can be accordingly generalized to all formulae from **I** using the observations above, Theorem 84 and Corollary 97.

11.2 Characterization of the discrete-canonical formulae in reversive languages with nominals

First, note that amongst all discrete general frames over a Kripke τ -frame F there is a *least* one in terms of the family of admissible sets, viz. the one generated by all singletons in F, denoted here by $\mathcal{S}(F)$. Thus, local di-persistence is equivalent to preservation of the local validity from $\mathcal{S}(F)$ to F for every Kripke τ-frame F.

Now, for every formula $A(p_1, \ldots, p_n)$, in a polyadic language with nominals $\mathcal{L}^{\mathbf{n}}_\tau$ we define the set Δ_A of all **pure substitution instances** of A, i.e. all formulae $A(P_1/p_1, \ldots, P_1/p_n)$ where the variables $p_1, \ldots, p_n$ are uniformly substituted by pure formulae $P_1, \ldots, P_n$.

The algebraic analogue of the following observation was proved in [50].

[7] Venema also allows additional 'non-orthodox' rules in the axiomatic system. Since these rules do not affect the discreteness of the canonical general frames, the canonicity of di-persistent formulae still holds if such rules are added to axioms from **I** in an arbitrary polyadic language.

Lemma 98 *For every τ-frame F, $w \in F$ and a formula $A \in \mathcal{L}_\tau^{\mathbf{n}}$,*

$$\mathcal{S}(F), w \vDash A \text{ iff } F, w \vDash \Delta_A,$$

(i.e. $F, w \vDash A'$ for every $A' \in \Delta_A$).

Proof. First, note that every admissible set in $\mathcal{S}(F)$, being constructed from singletons by applying the set theoretic and modal operators in $\mathcal{L}_\tau$, is therefore of the type $P(\{x_1\}, \ldots, \{x_m\})$ for some pure formula $P(c_1, \ldots, c_m)$. Therefore, every valuation in $\mathcal{S}(F)$ can be simulated by an appropriate pure substitution combined with an appropriate valuation of the nominals in F.

Now, suppose $A = A(p_1, \ldots, p_n)$, and let $\mathcal{S}(F), w \vDash A$. Take any $A' \in \Delta_A$. Then $A' = A(P_1/p_1, \ldots, P_1/p_n)$ for some pure formulae $P_1, \ldots, P_n$. Consider any valuation V in F of the nominals occurring in $P_1, \ldots, P_n$. Then $V(P_1), \ldots, V(P_n)$ are admissible sets in $\mathcal{S}(F)$. Let V' be the valuation in $\mathcal{S}(F)$ assigning them to $p_1, \ldots, p_n$ respectively. Then $\mathcal{S}(F), w \vDash_{V'} A$, hence $F, w \vDash_V A'$. Thus, $F, w \vDash A'$.

Conversely, suppose $\mathcal{S}(F), w \not\vDash_V A$ for some valuation V in $\mathcal{S}(F)$. Then, according to the remark above, there is an appropriate pure substitution instance A' of A and a valuation V' of the nominals in F such that $F, w \not\vDash_{V'} A'$. ■

Definition 99 *Formulas A and B from $\mathcal{L}_\tau$ are **locally di-equivalent** if they are valid at the same states in the same discrete general frames for $\mathcal{L}_\tau$.*

Note that local di-equivalence implies local frame equivalence, and for discrete-canonical formulae the latter implies axiomatic equivalence, too.

Proposition 100 *Every locally di-persistent formula A in a language with nominals is locally di-equivalent to a pure formula.*

Proof. Let A be a locally di-persistent, and hence locally first-order definable, formula in $\mathcal{L}_\tau^{\mathsf{u},\mathbf{n}}$ with a local first-order equivalent $\alpha_A(x)$.

Let Γ_A be the set of all local first-order equivalents $\gamma_P(x)$ of pure formulae $P \in \Delta_A$. We will show that $\Gamma_A \Vdash \alpha_A(x)$. Indeed, suppose $F, w \vDash \Gamma_A$. Then $F, w \vDash \Delta_A$, hence $\mathcal{S}(F), w \vDash A$ by Lemma 98. By local di-persistence, it follows that $F, w \vDash A$, hence $F, w \Vdash \alpha_A(x)$.

By compactness, $\Gamma_A^f \Vdash \alpha_A(x)$ for some finite subset $\Gamma_A^f = \{\gamma_{P_1}(x), \ldots, \gamma_{P_k}(x)\}$ of Γ_A, and therefore $\alpha_A(x)$ is equivalent to $\gamma(x) = \gamma_{P_1}(x) \wedge \ldots \wedge \gamma_{P_k}(x)$, hence A is locally frame-equivalent to the pure formula $P = P_1 \wedge \ldots \wedge P_k$ locally corresponding to $\gamma(x)$. In fact, A is locally di-equivalent to P due to the di-persistence of both A and P. ■

The results above can be summarized in the following theorem, characterizing the discrete-canonical formulae in reversive languages with nominals.

Theorem 101 *For every formula A in a reversive language with nominals $\mathcal{L}_\tau^{\mathbf{n}}$ the following are equivalent:*

(1) A is locally di-equivalent to an inductive formula.
(2) A is locally di-equivalent to a primitive regular formula.
(3) A is locally di-persistent.
(4) A is elementary and discrete-canonical.
(5) A is locally di-equivalent to a pure formula.

Proof. (1) implies (2) by Theorem 84. (2) implies (3) by Corollary 97, since local di-equivalence preserves di-persistence. (3) implies (4) by Proposition 95, and also from [25]. (4) implies (5) by Proposition 100. Finally, (5) implies (1) because every pure formula in $\mathcal{L}_\tau^{\mathbf{n}}$ is an inductive formula by definition. ■

We note that not all locally first-order definable and d-persistent formulae fall in the scope of Theorem 101. A counterexample is the formula $(\Box p \to \Box\Box p) \wedge \Box(\Box p \to \Box\Box p) \wedge (\Box\Diamond p \to \Diamond\Box p)$ (see [3], Lemma 7.5), which is easily seen not to be locally di-persistent.

Still, it would be nice if we could accordingly extend Theorem 101 to any reversive language. The only non-trivial implication there is from (5) to (1). However, at present even the question whether every formula in a reversive language with nominals $\mathcal{L}_\tau$, which is locally di-equivalent to a pure formula in $\mathcal{L}_\tau^{\mathbf{n}}$, is locally di-equivalent to a locally d-persistent formula is open to us.

12 Concluding remarks

In this paper we have extended Sahlqvist formulae and Sahlqvist's theorem, both in scope and depth, gradually shifting the focus on the semantic essence of these, captured by the concept of elementary canonical formulae. The best syntactic approximation of this concept so far are the inductive formulae, but the class of elementary canonical formulae still remains largely under-explored. Let us repeat the main problem here: while the syntax is too restrictive and only partly reflects that semantic idea, the latter seems too complex to be tractable[8]. In a series of papers [9–13] related to this study we explore further a hierarchy of natural and important classes of formulae between **I** and **ECF**, trying to bridge the gap between syntax and semantics in quest for deeper understanding of elementary canonical formulae. In particular, in [10,11] we

[8] However, we know of no proof yet that the class of elementary canonical formulae is not decidable.

develop an intermediate, *algorithmic approach* to elementary canonical formulae, suggested by some algorithms for elimination of second-order quantifiers, such as SCAN ([17]) and DLS ([14], [35]). Each of these defines a set of modal formulae for which the algorithm computes successfully a first-order equivalent, and for the case of SCAN, that set has been recently proved in [29] to subsume all classical Sahlqvist formulae. The relationship of these, and other algorithmically defined classes of formulae with **ECF** is explored in [12]. An alternative algebraic approach to some problems considered in the present paper is discussed in [46,47] in which the problem of finding first-order equivalents of modal formulas is reduced to the problem of solving certain equations in modal algebras by means of an algebraic modal generalization of the Ackermann lemma.

Acknowledgments

Most of the work of the first author on this paper was accomplished while he was at the Rand Afrikaans University (now, the University of Johannesburg), and he acknowledges the financial support for this study provided by that university and by the National Research Foundation of South Africa. The second author acknowledges the financial support provided by *Project No 123* with the Bulgarian Ministry of Science and Education. We are indebted to the referee for scrutinizing reading, expert criticism and a number of corrections on both the submitted and revised versions of this paper, which substantially improved its quality. We also thank Sergei Artemov for some useful critical remarks and corrections.

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